Name: _____ **PID:** _____

Math 140C: Midterm 2
Foundations of Real Analysis

- You have 1 hour and 20 minutes. No books and notes are allowed.
- You may quote any result stated in the textbook or in class.
- State carefully the hypothesis and conclusion of any result that you use.
- You may not use homework problems (without proof) in your solutions.

1. (10 points) Let X be a measurable space, in which $\mathscr{M}$ is the σ-ring of measurable sets and μ is the measure. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets such that $A_n \in \mathscr{M}$ and $A_{n+1} \subset A_n$, for all $n \geqslant 1$. Assume that $\mu(A_1) < +\infty$ and let $A = \cap_{n=1}^{\infty} A_n$.

(a) (5 points) Prove that $\lim\limits_{n\to\infty} \mu(A_n) = \mu(A)$.

(b) (5 points) Let f be a measurable nonnegative function on X such that $f \in \mathscr{L}(\mu)$ on X. Prove that $\lim\limits_{n\to\infty} \int_{A_n} f \, \mathrm{d}\mu = \int_A f \, \mathrm{d}\mu$.

2. (10 points) Let $\mathscr{E}$ be the family of elementary subsets of $\mathbb{R}$ and $m : \mathscr{E} \to [0, \infty)$ be the Lebesgue set function. For a set $A \subset \mathbb{R}$, we denote by $m^*(A)$ the outer measure of A corresponding to m. For $t \in \mathbb{R}$, we denote by $A + t$ the set $\{x + t | x \in A\}$.

(a) (5 points) Assume that $A \in \mathscr{E}$.
Prove that $m(A + t) = m(A)$, for all $t \in \mathbb{R}$.

(b) (5 points) Assume that $A \subset \mathbb{R}$ is an arbitrary subset.
Prove that $m^*(A + t) = m^*(A)$, for all $t \in \mathbb{R}$.

3. (10 points) Let X be a measurable space, in which $\mathscr{M}$ is the σ-ring of measurable sets and μ is the measure. Consider a function $f : X \to \mathbb{R}$.

(a) (5 points) Assume that the set $\{x \in X |\ f(x) > q\}$ is measurable, for every $q \in \mathbb{Q}$.

Prove that f is measurable.

(b) (5 points) Assume that f is measurable and let $\{f_n : X \to \mathbb{R}\}_{n=1}^{\infty}$ be a sequence of measurable functions.

Prove that the set of points $x \in X$ such that $\lim\limits_{n\to\infty} f_n(x) = f(x)$ is measurable.

4. (10 points) Let X be a measurable space, in which $\mathscr{M}$ is the σ-ring of measurable sets and μ is the measure. Let $f : X \to [0, \infty)$ be a measurable function. For $n \geqslant 1$, let $A_n = \{x \in X |\ n - 1 \leqslant f(x) < n\}$ and $B_n = \{x \in X |\ f(x) \geqslant n\}$.

(a) (3 points) Prove that $f(x) \leqslant \displaystyle\sum_{n=1}^{\infty} n\, \mathbf{1}_{A_n}(x)$, for every $x \in X$.

(b) (3 points) Assume that $\mu(A_n) \leqslant \dfrac{1}{n^3}$, for all $n \geqslant 1$.
Prove that $f \in \mathscr{L}(\mu)$.

(c) (4 points) Assume that $f \in \mathscr{L}(\mu)$.
Prove that the sequence $\{n\, \mu(B_n)\}_{n=1}^{\infty}$ is bounded.

Do not write on this page.

1		out of 10 points
2		out of 10 points
3		out of 10 points
4		out of 10 points
Total		out of 40 points