Midterm 2

(1) $\cos(3x) = \dfrac{e^{3ix} + e^{-3ix}}{2}$

$\sin(2x) = \dfrac{e^{2ix} - e^{-2ix}}{2i}$

$\Rightarrow \int \cos(3x)\sin(2x)\,dx$

$= \dfrac{1}{4i}\int (e^{3ix} + e^{-3ix})(e^{2ix} - e^{-2ix})\,dx$

$= \dfrac{1}{4i}\int e^{5ix} - e^{ix} + e^{-ix} - e^{-5ix}\,dx$

$= \dfrac{1}{4i}\left[\dfrac{1}{5i}e^{5ix} - \dfrac{1}{i}e^{ix} - \dfrac{1}{i}e^{-ix} + \dfrac{1}{5i}e^{-5ix} + C\right]$

$= \dfrac{-i}{4}\left[\dfrac{1}{5i}(e^{5ix} + e^{-5ix}) - \dfrac{1}{i}(e^{ix} + e^{-ix}) + C\right]$

$= \dfrac{1}{2}\left[-\dfrac{1}{5}\left(\dfrac{e^{5ix} + e^{-5ix}}{2}\right) + \left(\dfrac{e^{ix} + e^{-ix}}{2}\right) + C\right]$

$\boxed{= -\dfrac{1}{10}\cos(5x) + \tfrac{1}{2}\cos(x) + C}$

② $\int \frac{x^3}{x^2-4}\,dx$ Looks like partial fractions, but we need to use <u>long division</u> first!

$$\begin{array}{r} x \\ x^2-4 \overline{\big) x^3+0x^2+0x+0} \\ \underline{x^3 \qquad -4x} \\ 4x \end{array} \Rightarrow x^3 = (x^2-4)x + 4x$$

So we get $\frac{x^3}{x^2-4} = x + \frac{4x}{x^2-4}$. Now we can do partial fractions:

$$\frac{4x}{(x+2)(x-2)} = \frac{A}{(x+2)} + \frac{B}{(x-2)} \Rightarrow 4x = A(x-2) + B(x+2)$$

@ $x=2 \ldots \quad 8 = 4B \Rightarrow B = 2$

@ $x=-2 \ldots \quad -8 = -4A \Rightarrow A = 2$

So $\int \frac{x^3}{x^2-4}\,dx = \int x + \frac{2}{x+2} + \frac{2}{x-2}\,dx$

$$\boxed{= \frac{x^2}{2} + 2\ln(x+2) + 2\ln(x-2) + C}$$

(3) $$\int_1^\infty \frac{1}{2\sqrt{x}+x^3}\,dx = \lim_{R\to\infty}\int_1^R \frac{1}{2\sqrt{x}+x^3}\,dx$$

Notice that $2\sqrt{x}+x^3 \geq x^3$, $x\geq 0$

$$\Rightarrow \frac{1}{2\sqrt{x}+x^3} \leq \frac{1}{x^3},\quad x\geq 0$$

$$\Rightarrow \int_1^R \frac{1}{2\sqrt{x}+x^3}\,dx \leq \int_1^R \frac{1}{x^3}\,dx, \text{ for all } R\geq 1$$

$$\Rightarrow \lim_{R\to\infty}\int_1^R \frac{1}{2\sqrt{x}+x^3}\,dx \leq \underbrace{\lim_{R\to\infty}\int_1^R \frac{1}{x^3} < \infty}$$

by p-test.

(4) Observe that

$$-1 \leq \cos(n^2) \leq 1 \Rightarrow \frac{-1}{n} \leq \frac{\cos(n^2)}{n} \leq \frac{1}{n}$$

Since $\lim_{n\to\infty}\frac{1}{n} = \lim_{n\to\infty}-\frac{1}{n} = 0$, Squeeze theorem

tells us $\lim_{n\to\infty}\frac{\cos(n^2)}{n} = 0$.

(5) (a) $\sum_{n=5}^{\infty} (7/10)^n = \frac{(7/10)^5}{1-(7/10)}$ since our series is a geometric series and $|7/10| < 1$.

(b) $\sum_{n=5}^{\infty} \frac{(\frac{7}{10})^n}{n^{1/3}} \leq \sum_{n=5}^{\infty} (7/10)^n < \infty$ by above, so our series converges.

(6) $n^3 + 9n \leq n^3 + 9n^3 = 10n^3$

$\Rightarrow \sqrt[3]{n^3+9n} \leq \sqrt[3]{10n^3} = \sqrt[3]{10}\, n$

$\Rightarrow \frac{1}{\sqrt[3]{n^3+9n}} \geq \frac{1}{\sqrt[3]{10}\, n}$

$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{\sqrt[3]{n^3+9n}} \geq \sum_{n=0}^{\infty} \frac{1}{\sqrt[3]{10}\, n} \to \infty$ by p-test,

so our series <u>diverges</u>.