Final Exam Solutions

(I) $f(x,y) = (x^2 + \cos(y),\ x^2+y^2,\ y^2)$, $g(u,v,w) = u + 3v + 2\cos w$

a) $D(g\circ f)\big|_{(0,0)} = ?$

$$D(g\circ f)\big|_{(0,0)} = Dg\big|_{f(0,0)} \cdot Df\big|_{(0,0)} \quad , f(0,0) = (1,0,0)$$

$$= (1, 3, -2\sin w)_{(1,0,0)} \cdot \begin{pmatrix} 2x & -\sin(y) \\ 2x & 2y \\ 0 & 2y \end{pmatrix}_{(0,0)}$$

$$= [1, 3, 0] \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

(b) 1st order Taylor of g at $\underbrace{(0,0,\pi/2)}_{\vec{x}_0}$

$g(0,0,\pi/2) = 0$

$$\nabla g\big|_{(0,0,\pi/2)} = (1, 3, -2\sin w)_{(0,0,\pi/2)} = (1,3,-2)$$

$$\Rightarrow g(\vec{x}_0 + \vec{h}) = g(\vec{x}_0) + \nabla g\big|_{\vec{x}_0} \cdot \vec{h} + R$$

$$= 0 + (1,3,-2)\cdot(h_1,h_2,h_3) + R$$

$$= h_1 + 3h_2 - 2h_3 + R(\vec{x}_0, \vec{h})$$

(2) $\iint_R e^{(y-3x)} \sin(y-x)\, dx\,dy$

R: Region bounded by $y-x=0$, $y-x=1$
$y-3x=0$, $y-3x=2$

by change of variable:

Ans: Set $u = y-x$
$v = y-3x$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left(\left|\frac{\partial(u,v)}{\partial(x,y)}\right|\right)^{-1} = \frac{1}{\begin{vmatrix} -1 & 1 \\ -3 & 1 \end{vmatrix}} = \left|\frac{1}{2}\right|$$

$$\Rightarrow \iint_R e^{(y-3x)} \sin(y-x)\, dx\,dy = \int_0^2\int_0^1 e^v \sin u \cdot \frac{1}{2}\, du\,dv$$

$$= \frac{1}{2}\int_0^2 (-\cos 1 + 1)e^v dv = \frac{1}{2}(e^2-1)(1-\cos 1)$$

(3)(a) $\int_C f\,ds = ?$, $f(x,y,z) = xyz^2$, $\vec{c}(t) = (1, \cos t, \sin t)$

$0 \le t \le \pi/2$

$$\Rightarrow \int_C f\,ds = \int_0^{\pi/2} f(\vec{c}(t))\,\|c'(t)\|\,dt = \int_0^{\pi/2} \cos t \sin^2 t\, dt = \frac{1}{3}\sin^3 t \Big|_0^{\pi/2} = \frac{1}{3}$$

$f(\vec{c}(t)) \to \cos t \sin^2 t$; $\|c'(t)\| \to \sqrt{0 + \sin^2 t + \cos^2 t} = 1$

(b) $\int_C \vec{F}\cdot d\vec{s} = ?$ $\vec{F}(x,y,z) = (-y, x, e^{-z})$ $\vec{c}(t) = (\cos t, \sin t, t)$
$0 \le t \le 2\pi$

$$\int_C \vec{F}(\vec{c}(t))\cdot\vec{c}'(t)\,dt = \int_0^{2\pi} (-\sin t, \cos t, e^{-t})\cdot(-\sin t, \cos t, 1)\,dt$$

$$= \int_0^{2\pi} 1 + e^{-t}\,dt = 2\pi - e^{-2\pi} + 1$$

(c) $\int_C \vec{F}\cdot d\vec{s} = ?$ $\vec{F}(x,y,z) = (2x + ze^{xz}, 2y, xe^{xz})$, $\vec{c}(t) = (\cos^3 t, \sin^2 t, t/2\pi)$
$0 \le t \le 2\pi$

$\nabla \times \vec{F} = 0 \Rightarrow \vec{F} = \nabla f$ for some f.

Here $f = x^2 + y^2 + e^{xz}$

$\Rightarrow \int_C \vec{F} \cdot d\vec{s} = f(\vec{c}(2\pi)) - f(\vec{c}(0))$ by FTLI

$= f(1,0,1) - f(1,0,0) = 1 + e^1 - (1+1) = e - 1$

(4) Green's theorem to evaluate $\int_C \vec{F} \cdot d\vec{s}$, $\vec{F} = (\underbrace{x}_{P}, \underbrace{x+y}_{Q})$

C: circle of radius 2 oriented CCW.

$$\int_C \vec{F} \cdot d\vec{s} = \int_{C^+} P\,dx + Q\,dy = \iint_{\text{disk of rad 2}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy = \iint_{\text{disk of rad 2}} (1-0)\,dx\,dy$$

$= A(\text{Disk of rad 2}) = \pi(2^2) = 4\pi.$

(5)(a) $\iint_S \vec{F} \cdot d\vec{S}$ $\quad \vec{F} = (1,1,2)$, $S: (\cos v \sin u, \sin v \sin u, \cos u)$

$0 \le v \le 2\pi$, $0 \le u \le \pi/4$, outward normal

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/4} (1,1,2) \cdot (\vec{T}_u \times \vec{T}_v)\,du\,dv$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \sin^2 u(\cos v + \sin v) + 2\sin u \cos u\,du\,dv$$

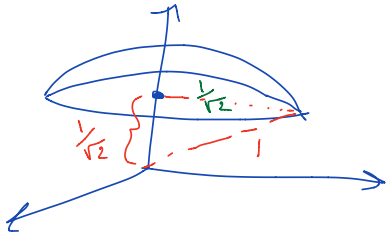
$$= \int_0^{2\pi}\int_0^{\pi/4} \sin 2u\,du\,dv = 2\pi \cdot \frac{1}{2}\left(-\cos\frac{\pi}{2} + \cos 0\right)$$

$$= \pi$$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v \cos u & \sin v \cos u & -\sin u \\ -\sin v \sin u & \cos v \sin u & 0 \end{vmatrix} = (\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u)$$

(b) $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$, $\vec{F} = (x, 0, y e^{z^3})$, S: part of sphere above $z = 1/\sqrt{2}$

outward pointing normal



By Stokes theorem $\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$

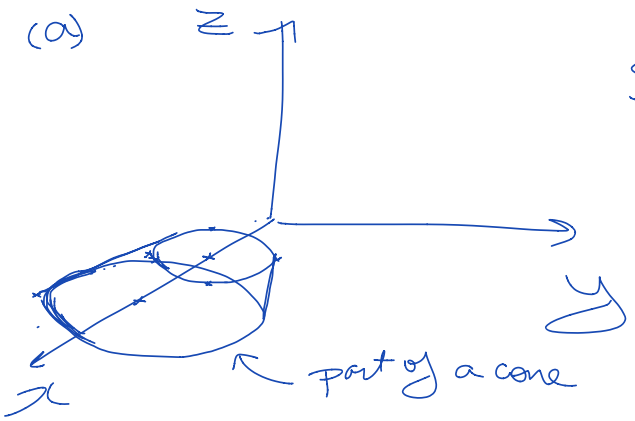
where $\partial S = C = \left(\frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}\right)$ $\quad 0 \le t \le 2\pi$

$$\Rightarrow \int_{\partial S} \vec{F}\cdot d\vec{s} = \int_0^{2\pi} \left(\frac{1}{\sqrt{2}}\cos t, 0, 0\right)\cdot\left(-\frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}\cos t, 0\right) dt$$

$$= \int_0^{2\pi} -\frac{1}{2}\sin t\cos t\, dt = 0$$

(6) S: $x^2 = y^2 + 2z^2$ $\quad 1 \le x \le 2$

(a)



$$S: \left(r, r\cos\theta, \frac{r}{\sqrt{2}}\sin\theta\right)$$

$1 \le r \le 2$

$0 \le \theta \le 2\pi$

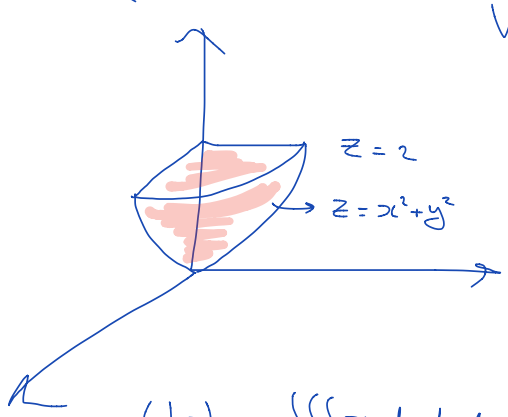
(b) $\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_1^2 \vec{F}\cdot(\vec{T}_r \times \vec{T}_\theta)\, dr\, d\theta$, $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & \cos\theta & \frac{1}{\sqrt{2}}\sin\theta \\ 0 & -r\sin\theta & \frac{r}{\sqrt{2}}\cos\theta \end{vmatrix} = \frac{r}{\sqrt{2}}\vec{i} - \frac{r}{\sqrt{2}}\cos\theta\,\vec{j} - r\sin\theta\,\vec{k}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_1^2 \left(r\vec{i} + r\cos\theta\,\vec{j} + \frac{r}{\sqrt{2}}\sin\theta\,\vec{k}\right)\cdot\left(\frac{r}{\sqrt{2}}\vec{i} - \frac{r}{\sqrt{2}}\cos\theta\,\vec{j} - r\sin\theta\,\vec{k}\right) dr\, d\theta$$

$$= \int_0^{2\pi}\int_1^2 \left(\frac{r^2}{\sqrt{2}} - \frac{r^2}{\sqrt{2}}\right) dr\, d\theta = 0$$

(7) (a)



W: Region above 1st quadrant, below $z=2$ with $z=x^2+y^2$

(b) $\iiint_W x\,dx\,dy\,dz$ using cylindrical coordinates

$W: 0 \le \theta \le \pi/2,\ 0 \le r \le \sqrt{2},\ r^2 \le z \le 2$

$$\iiint_W x\,dx\,dy\,dz = \int_0^{\pi/2}\int_0^{\sqrt{2}}\int_{r^2}^{2} r^2\cos\theta\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{\sqrt{2}} (2-r^2)r^2\cos\theta\,dr\,d\theta = (\sin\tfrac{\pi}{2} - \sin 0)\cdot\left(\frac{2r^3}{3} - \frac{r^5}{5}\right)\Big|_{r=0}^{r=r_2}$$

$$= \frac{4\sqrt{2}}{3} - \frac{4\sqrt{2}}{5} = \frac{8\sqrt{2}}{15}$$

(c) $\iint_{\partial W}\left(\frac{x^2}{4}\vec{i} + \frac{xy}{2}\vec{j} + \vec{k}\right)\cdot d\vec{S} = \iiint_W (\nabla\cdot\vec{F})\,dV$

By Gauss's div. Theorem

$$= \iiint_W \left(\frac{x}{2} + \frac{x}{2} + 0\right) dV$$

$$= \frac{8\sqrt{2}}{15} \text{ by (b)}$$

(8) W: region enclosed by S where S: $(a\cos\theta\sin\phi, b\sin\theta\sin\phi, c\cos\phi)$

$0 \le \phi \le \pi$
$0 \le \theta \le 2\pi$

$V(W) = \frac{4}{3}\pi abc$

Evaluate $\iint_S \vec{F}\cdot d\vec{S}$, $\vec{F} = (x+yz^3, y+\sin(z^2), -z+x^{72})$

By Gauss's div thm

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_W \overbrace{(1+1-1)}^{\nabla\cdot\vec{F}} dV = V(W) = \frac{4}{3}\pi abc$$

(9) $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j} + 0\vec{k}$

$\vec{C}: (2\cos t - \cos 2t,\ 2\cos t\cdot\cos 2t,\ 2\sin t - \sin 2t)$

Show that $\int_C \vec{F}\cdot d\vec{s} = 0$

Ans: $\vec{C}$ is closed so by stokes theorem

$$\int_C \vec{F}\cdot d\vec{s} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S} \quad \text{where } S \text{ is enclosed by } C.$$

However: (1) $\vec{F}$ only has $\vec{i}$ & $\vec{j}$ components that are fcns of x & y $\Rightarrow$ $\nabla\times\vec{F}$ only has a $\vec{k}$ component.

(2) C lies in the plane $x=y$ so does S, so $d\vec{S}$ has no $\vec{k}$ component

$$\Rightarrow \iint_S \nabla\times\vec{F}\cdot d\vec{S} = 0 = \int_C \vec{F}\cdot d\vec{s}$$