Paperclip graphs

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Abstract

By taking a strip of paper and forming an "$\mathcal{S}$" curve with paperclips holding the paper together at two key locations it is possible to link the paperclips by pulling the paper taut. We explore what other configurations of linked paperclips might be possible by looking at "paperclip graphs" which are shown to be bipartite circle graphs.

1 An experiment with paperclips.

To start, go get a strip of paper and two paperclips. We will wait a minute for you to come back...

Take your strip of paper and shape the strip into an $\mathcal{S}$ curve and fix it with two paperclips as shown in Figure 1.

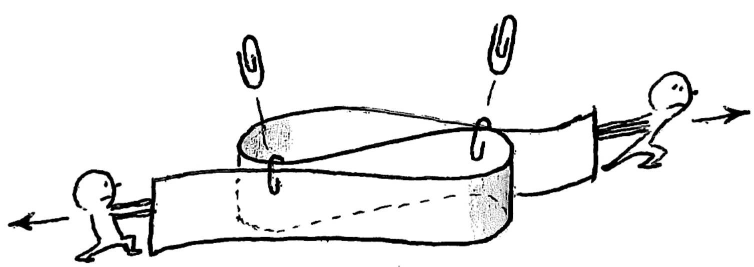


Figure 1: Putting two paperclips on a strip of paper shaped as an $\mathcal{S}$ curve.

What happens when we pull the strip taut? As we begin to pull, the paperclips come closer and closer together. If we keep pulling, presumably they will collide and jam...

We suggest that, before reading on, the readers try this experiment themselves with their strip of paper and paperclips.

Result: At the instant the strip becomes taut, the paperclips pop out (see Figure 2).

And there is more: pick up the paperclips that landed. You will see that they are now *linked.*

This trick seems to go back to Seattle Magician Bill Bowman who used a dollar bill for the strip of paper. The trick was circulated among the magic community in 1954; Martin

Figure 2: Launching the paperclips.

Gardner [3] later popularized it in his book on magic which includes variations involving multiple paperclips, string, and a rubber band.

It is natural to generalize the trick by making the paper more sinuous and adding more paperclips. For example, with three consecutive turns and three paperclips placed as shown in Figure 3, we are tempted to guess that pulling the strip straight will result in a chain of three

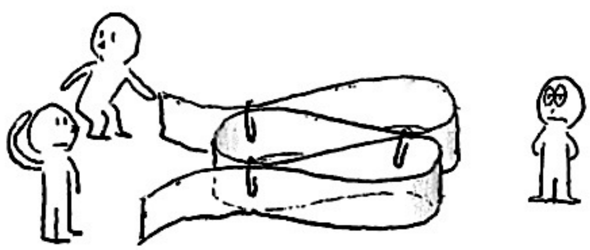


Figure 3: Putting three paperclips on a curve.

linked paperclips: clip-clip-clip. That is what should happen in the idealization of very elastic paperclips and very strong paper. In practice, however, paperclips are somewhat plastic and the paper of limited strength; a sharp concentration of stress attacks the middle paperclip, which ends up twisting it out of shape and often tearing the paper. This physical effect is interesting, but it is a subject for another time. For now, we want to investigate the idealized paperclip linking, where the focus up front is on the topology of the resulting configuration, while the mechanics of deformable media, which matters if you want to (per)form these experiments, fades into the background.

2 Paperclips as arcs.

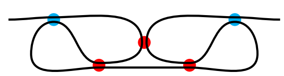
A natural way to represent a configuration of paperclips is as a graph.

Definition. Given a set of linked paperclips we represent the connections by a *paperclip graph* where each paperclip corresponds to a vertex and two vertices are connected if the corresponding paperclips are linked together.

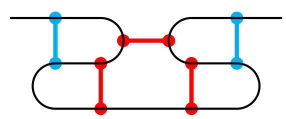
As an example, the result of our earlier experiment with two paperclips is the graph with two vertices joined by an edge (sometimes denoted as P_2, the path on two vertices). Our first step is to take a configuration of paperclips on our strip of paper and find the corresponding paperclip graph. Then we want to understand what is, and is not possible and reverse the procedure.

We can represent our strip of paper by a (noncrossing) curve in the plane where the paperclips correspond to points where the curve passes through the same point. We will assume that the two ends of the curve are in the unbounded region of the plane (so that we can pull the strip taut). Finally, at no point does the curve pass through the same point three or more times (in other words, paperclips only connect two distinct parts of the strip).

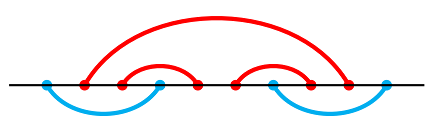
Let us start with an example. Suppose we have taken a strip of paper and connected it in five locations as shown in Figure 4(a). The paperclips can be assigned two different colors; this should be done by following the strip along one side and anytime that we would have to break the paperclip to continue the walk we color it red; otherwise color blue. Now treat both the strip and paperclips as elastic so we can start to pull them apart as shown in Figure 4(b). Continuing this until the paper is made taut, we will get the diagram shown in Figure 4(c) which consists of a straight line (the strip of paper) and then a set of (red) nonintersecting arcs placed on one side and a set of (blue) nonintersecting arcs placed on the other. The arcs connect two points on the strip of paper that correspond to where the paperclips had connected the strip; arcs of the same color cannot intersect as that would lead to a situation where the paper would have to cross itself.



(a) Strip with paperclips.



(b) Starting to pull apart.



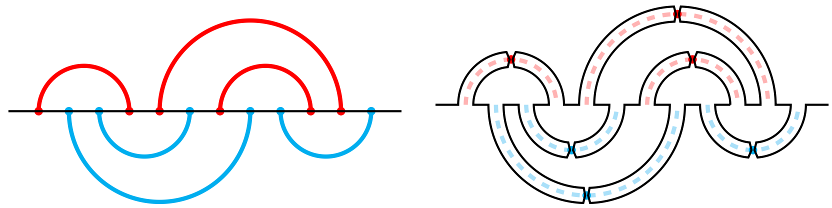
(c) Making the strip taut.

Figure 4: From a strip with paperclips to sets of nonintersecting arcs.

Alternatively, given any collection of nonintersecting arcs on one side of a straight strip and nonintersecting arcs on the other we can find a corresponding way to have a strip of paper with paperclips connecting points on the strip. The key is to treat the paper as elastic and then use the arcs to indicate how to stretch the paper to make the connection. An example of this is shown in Figure 5.

We can now glue the two ends of the paper together to form a circle with red and blue arcs. (We can place all arcs in the interior with the rule that arcs of the same color cannot intersect.) Doing this for Figure 4(c) and Figure 5(a) we get Figure 6. Given a configuration of arcs connecting points on a line, there is a unique circle to which it corresponds. On the other hand, given a circle there are potentially multiple configurations of arcs connecting points to which it corresponds, e.g., by choosing to cut the circle at different points.

The last piece we need is to know when two paperclips (represented by two arcs) will intersect. When represented as a sequence of arcs connecting points on a straight line, this



(a) Configuration of arcs. (b) Stretching the paper.

Figure 5: Going from a set of arcs to a strip with paperclips.

(a) Based on Figure 4(c) (b) Based on Figure 5(a)

Figure 6: Circular representation for arc configurations

will be when the endpoints of the pair of arcs alternate. When represented as arcs connecting points on a circle, this will be when the corresponding arcs intersect. (Recall that arcs of the same color cannot intersect.)

Definition. Given a configuration of arcs connecting distinct points on a circle, a *circle graph* is formed by letting each arc represent a vertex and two vertices are connected by an edge if and only if the corresponding arcs intersect.

We are in the special case when we can color our arcs two colors, red and blue, where no arcs of the same color intersect. That means in the corresponding circle graph that we can color our vertices with two colors so no pair of the same color is joined by an edge. So we have the following.

Observation 1. *Paperclip graphs are bipartite circle graphs.*

As an example, the paperclip graph corresponding to Figure 6(a) is the path graph on five vertices; the paperclip graph corresponding to Figure 6(b) is the cycle graph on six vertices with one additional edge connecting a pair of opposite vertices.

3 Characterization.

Circle graphs have a known characterization through the use of induced subgraphs of locally equivalent graphs.

Definition. Given a graph G with a vertex v the *local complement* $G*v$ is found by replacing the induced subgraph corresponding to the neighbors of v with its complement. The graph H is said to be *locally equivalent* to G if $H = G*v_1*v_2*\cdots*v_k$ for some sequence of vertices $v_1,\ldots,v_k$ (taken in order).

In terms of the representation of intersecting chords the action of $G*v$ is to find the chord represented by v and then reversing the connections on *one* side of the chord. It follows that any graph which is locally equivalent to a circle graph is a circle graph.

Theorem 1 (Bouchet [2]). *A graph is a circle graph if and only if no locally equivalent graph has as an induced subgraph one of the graphs shown in Figure 7.*

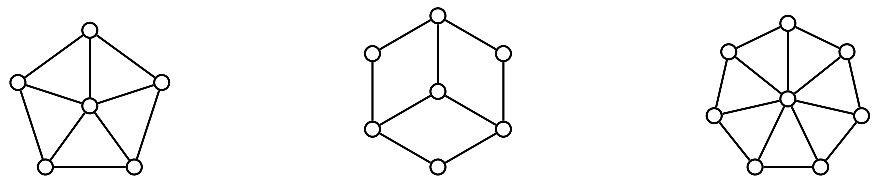


Figure 7: Forbidden induced subgraphs of circle graphs.

While only one of the graphs in Figure 7 is bipartite, we still need the full characterization as bipartite graphs are locally equivalent to non-bipartite graphs. As a good exercise to build intuition the reader should show that the central graph in Figure 7 cannot be a paperclip graph.

It is not immediately obvious that there is an efficient way to determine when a graph is a paperclip graph. As an example, try to determine which of the graphs in Figure 8 is not a paperclip graph. Spinrad [4] has found an $O(n^2)$ algorithm to determine if an n-vertex graph is a circle graph (and hence also for paperclip graph) and moreover how to place chords in the circle to produce the graph. It is not known if there is a faster algorithm for determining whether a graph is a paperclip graph (in other words, can the algorithm be sped up if we also assume the graph is bipartite).

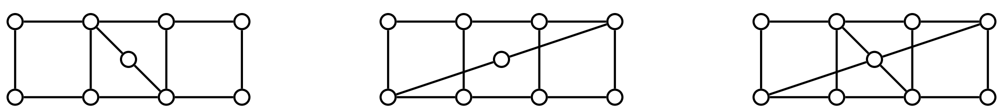


Figure 8: Two of these three graphs are paperclip graphs; one of them is not.

4 There are exponentially many paperclip graphs.

To get an upper bound for the number of paperclip graphs on n vertices, we use the interpretation of using nonintersecting arcs above and below a strip. So start with n paperclips, let i of them correspond to arcs above the strip and $n-i$ below. Then there are $\binom{2n}{2i}$ ways to choose which paperclip endpoints are above the strip (leaving the rest below). The $2i$ endpoints must be connected in the same way as balanced parentheses (the first endpoints of each arc being a "(" and the second being a ")"), the number of ways to do this is the ith Catalan number $C_i = \frac{1}{i+1}\binom{2i}{i}$. Similarly, the number of ways to connect the arcs on the bottom is C_{n-i}. (For more on the Catalan numbers see Stanley [5].) So the total number of possible configurations is

$$\sum_{i=0}^{i=n} \binom{2n}{2i} C_i C_{n-i} \le \sum_{i=0}^{i=n} \frac{2^{2n}}{\sqrt{n}} C_i C_{n-i} = \frac{2^{2n}}{\sqrt{n}} C_{n+1} \le \frac{2^{4n}}{n^2} = \frac{1}{n^2} 16^n.$$

This does not yet prove that there are exponentially many paperclip graphs. This is because there are situations in which exponentially many configurations which were produced in the above argument yield the same graph. For example, there are at least 2^{n-2} ways to produce the path on n vertices, denoted P_n. We see the result is true for P_1 and P_2 where in P_2 we will consider the arc on the right the "active" arc. Now we continue to extend by adding a leaf (see Proposition 2) to the active arc and note that at each step we have one of two possible places to attach, and then the new arc becomes active. Given the final arrangement, we can reconstruct our choices made giving 2^{n-2} configurations.

So we need to find an exponentially large family of graphs which are paperclip. One way to do this is by the following observation.

Observation 2. *If G is a paperclip graph and v is any vertex, then the graph H formed by taking G and adding a new vertex v' that is adjacent to only v is also a paperclip graph.*

If G is a paperclip graph and v is any vertex, then the graph H formed by taking G and adding a new vertex v' which is adjacent to the neighbors of v (i.e., duplicating v) is also a paperclip graph.

In terms of paperclip configurations on a strip of paper, the first one follows by "pinching" the paper together and putting a new paperclip on one side where the paperclip corresponding to v is placed. The second one follows by putting a new paperclip right next to where v is placed.

Corollary 1. *All trees are paperclip graphs.*

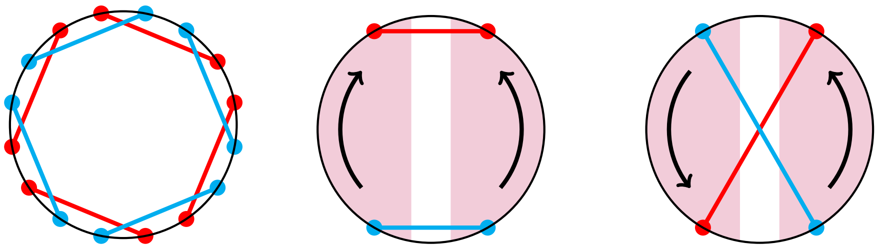
It is known that there are exponentially many trees ($\approx c(2.955765\ldots)^n/n^{3/2}$), which gives an exponential lower bound. We can do better by exhibiting an even larger class of paperclip graphs.

Theorem 2 (Wessel and Pöschel [6])**.** *All outerplanar graphs are circle graphs.*

We give a quick sketch of the proof. We note that any induced subgraph of a circle graph is still a circle graph (i.e., the intersections between the remaining arcs are preserved). Since every outerplanar graph is a subgraph of a cycle with noncrossing interior chords, it suffices to show how to construct those graphs. This is shown in Figure 9. Namely, place alternating arcs to form a cycle. To add in chords we find the corresponding arcs and the situation is as shown in Figure 9(b), we now "reverse one side" and the effect is to introduce exactly one new crossing in the diagram (and hence one new edge in the graph, the desired chord). Now repeat as needed.

We now have that all bipartite outerplanar graphs are paperclip graphs. This includes all trees and so should be a larger family, and indeed we have the following.

Theorem 3 (Bodirsky et al. [1])**.** *The number of bipartite outerplanar graphs on n vertices is $\approx c(4.57715\ldots)^n/n^{5/2}$.*



(a) The graph C_8 (b) Place to add a chord (c) Reversing one side

Figure 9: Forming cycles and adding chords.

If we let paper_n denote the number of paperclip graphs on n vertices, then we have shown $4.57715\ldots \le \left(\text{paper}_n\right)^{1/n} \le 16$ for large n. Both bounds leave a lot to be desired, for the lower bound there are many paperclip graphs which are unaccounted for (e.g., all complete bipartite graphs are paperclip) and the upper bound massively overcounts.

Problem 1. *Improve the upper and/or lower bounds for* $\left(\text{paper}_n\right)^{1/n}$*.*

Problem 2. *Determine if* $\lim_{n\to\infty} \left(\text{paper}_n\right)^{1/n}$ *exists.*

5 The folding number.

All known results about (bipartite) circle graphs become results about paperclip graphs. However, there is one aspect of paperclip graphs that does not generalize to all circle graphs, and that is how it can be related to our starting characterization which was a strip of paper connected at various points by paperclips.

When we are constructing a configuration, let us keep track of the "folds", these are the points when we reverse the direction of the strip of paper. So for example in Figure 2 there are two folds while in Figure 3 there are three folds.

Definition. Given a paperclip graph G, the *folding number of* G, denoted $\text{fold}(G)$, is the minimum number of folds needed for a construction of G involving a strip of paper and paperclips.

Because there are multiple ways to represent a graph in terms of intersecting arcs on a circle (as seen earlier by the argument for paths), and multiple ways to go from arcs on a circle to a configuration of arcs on a line (by choosing where to cut the circle), the same graph can be represented in many different configurations of a strip of paper with paperclips. So in general determining $\text{fold}(G)$ for a paperclip graph is a nontrivial problem. As an example, in Figure 10 are two different representations of the graph P_4 with different folding numbers. This shows that $\text{fold}(P_4) \le 2$, and in fact $\text{fold}(P_4) = 2$.

Problem 3. *Given a graph* G*, or more generally a graph family such as* P_n*, determine* $\text{fold}(G)$*.*

Another avenue of exploration is to determine the family of graphs which can be determined with k folds.

Figure 10: Different representations of P_4 with differing number of folds.

Observation 3. *The graphs which can be formed with exactly one fold are the empty graphs (graphs with no edges).*

When there are only two folds, then by examining how the paperclips are placed along the "middle" strip of paper we get the following.

Proposition 1. *The graphs which can be formed with exactly two folds are the bipartite graphs with vertex sets* $a_1, \ldots, a_p, b_1, \ldots, b_q$ *where* a_i *is adjacent to* $\{b_1, \ldots, b_{t_i}\}$ *and* $t_1 \geq t_2 \geq \cdots \geq t_p$.

An example of this is shown in Figure 11. The general argument follows by examining a similar generic picture.

Figure 11: An example of a graph that can be formed with two folds.

Problem 4. *Determine the graphs which can be formed with k folds for additional small values of k (in other words, graphs with* $\mathrm{fold}(G) \leq k$*).*

Even with three folds the situation gets more interesting because of the variety of ways to make three folds and how the corresponding substrips of paper can interact with linking by paperclips.

Acknowledgment.

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References

[1] Manuel Bodirsky, Éric Fusy, Mihyun Kang, and Stefan Vigerske, Enumeration and asymptotic properties of unlabeled outerplanar graphs, *Electronic Journal of Combinatorics* **17** (2007), #R66, 24 pp.

[2] André Bouchet, Circle graph obstructions, *Journal of Combinatorial Theory, Series B* **60** (1994), 107–144.

[3] Martin Gardner, *Encyclopedia of Impromptu Magic*, Magic Inc., Chicago, Ill (1978), 445–448.

[4] Jeremy Spinrad, Recognition of circle graphs, *Journal of Algorithms* **16** (1994), 264–282.

[5] Richard Stanley, *Catalan Numbers*, Cambridge University Press, 2015.

[6] W. Wessel and R. Pöschel, On circle graphs, in *Graphs, Hypergraphs, and Applications*, H. Sachs (ed.), 1985, 207–210.