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On the Limit of a Recurrence Relation

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In this paper we study the asymptotic properties of the sequence of integers $g(n)$, defined by the following recurrence relation:

$$g(n+1) = \left\lfloor \left(1 + \frac{\alpha}{n-a}\right) g(n) \right\rfloor,$$

where $\alpha > 0$ and $\lfloor x \rfloor$ denotes the largest integer not greater than x. For any $\alpha > 0$, the limit $g(n)/n^{\alpha}$ exists. We prove that for $\alpha = 2$, this limit is always rational. For $\alpha = 3$, we give some sufficient conditions which guarantee that the limit is rational.

Keywords: Recurrence relation, rational limit

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1 INTRODUCTION

In this paper we study sequences of integers $\{g(n)\}$ defined by recurrence relations of the following form:

$$g(n+1) = \left\lfloor \left(1 + \frac{\alpha}{n-a}\right) g(n) \right\rfloor, \tag{1}$$

where $\alpha > 0$ and $\lfloor x \rfloor$ denotes the largest integer not greater than x.

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Such recurrences often arise in the study of extremal combinatorial structures, e.g., the Turan number for hypergraphs [1] and the distribution of values of angles determined by coplanar points [2].

In this paper we investigate asymptotic properties of such recurrences, focusing in particular on the cases $\alpha = 2$ and 3. It is not hard to show that $\lim_{n\to\infty} g(n)/n^\alpha$ always exists (see Section 2). If we start with the initial condition $g(b) = c$ (this is no loss of generality), we let $G(\alpha, a, b, c)$ denote that limit. We prove that for $\alpha = 2$, $G(\alpha, a, b, c)$ is always rational. The case of $\alpha = 3$ is more complicated. We provide some sufficient conditions on the initial values so that $G(\alpha, a, b, c)$ is rational. We are not able to prove that $G(\alpha, a, b, c)$ is ever irrational, although we believe this is almost always the case. At the end we present some computational data, as well as a number of conjectures based on this data.

2 ASYMPTOTIC BEHAVIOR

FACT *Given any sequence of integers defined by the recurrence* (1) *with fixed numbers a and α, the limit* $\lim_{n\to\infty}(g(n)/n^\alpha)$ *exists.*

Proof If

$$f(n) = \frac{g(n)}{(n-a)^\alpha},$$

then the recurrence becomes

$$(n+1-a)^\alpha f(n+1) = \left\lfloor \left(1 + \frac{\alpha}{n-a}\right)(n-a)^\alpha f(n) \right\rfloor,$$

That is,

$$f(n+1) = \frac{\left\lfloor (n+1-a)^\alpha \left(\dfrac{1 + (\alpha/(n-a))}{(1 + 1/(n-a))^\alpha}\right) f(n) \right\rfloor}{(n+1-a)^\alpha}.$$

Since $\alpha > 0$, we have

$$1 + \frac{\alpha}{n-a} \le \left(1 + \frac{1}{n-a}\right)^{\alpha}.$$

Thus

$$f(n+1) \le \frac{\lfloor (n+1-a)^{\alpha} f(n) \rfloor}{(n+1-a)^{\alpha}} \le f(n).$$

It is clear that $f(n) \ge 0$. Since $\{f(n)\}$ is a bounded and monotonically decreasing sequence, the limit $\lim_{n\to\infty} f(n)$ exists. It follows immediately that

$$\lim_{n\to\infty} \frac{g(n)}{n^{\alpha}} = \lim_{n\to\infty} f(n) \cdot \left(\frac{n-a}{n}\right)^{\alpha} = \lim_{n\to\infty} f(n).$$

3 THE LIMIT OF $g(n)/n^{\alpha}$ IN THE QUADRATIC CASE

In this section, we determine the limit $G(2,a,b,c)$ where a,b,c are integers. Note that $G(2,a,b,c) = G(2,1,b-a+1,c)$, so without loss of generality, we may assume $a=1$. The recurrence relation (1) becomes

$$g(n+1) = \left\lfloor \left(1 + \frac{2}{n-1}\right) g(n) \right\rfloor = \left\lfloor \frac{n+1}{n-1} g(n) \right\rfloor. \tag{2}$$

It is sometimes convenient to consider the sequence $h(n) = g(n)/\binom{n}{2}$. The corresponding recurrence relation for $h(n)$ is

$$h(n+1) = \frac{\left\lfloor \binom{n+1}{2} h(n) \right\rfloor}{\binom{n+1}{2}}. \tag{3}$$

It is clear that for the sequence with initial value $g(b) = c$, $\lim_{n\to\infty} f(n) = 2G(2,1,b,c)$.

THEOREM 1 *For any rational number p/q where $\gcd(p,q) = 1$, there exist integers b,c such that the limit $2G(2,1,b,c) = p/q$. That is, for the* *sequence determined by the recurrence* (2) *and initial value* $g(b)=c$, $\lim_{n\to\infty} g(n)/\binom{n}{2} = p/q$.

Proof Obviously if $g(b)=0$ for some b, then $G(2,1,b,0)=0$; if $g(b) = \binom{b}{2}$, then $2G(2,1,b,\binom{b}{2}) = 1$. From the recurrence relation (3), one notices that if $h(n)=1+h_1(n)$, then $h(n+1) = 1+\lfloor \binom{n+1}{2} h_1(n)\rfloor / \binom{n+1}{2} = 1+h_1(n+1)$. Without loss of generality, we may assume $0<p/q<1$.

Let $\beta=(q-1)/2$. If

$$g(n) = \frac{p}{q}\binom{n}{2} + \frac{\beta}{q}(n-1) + x_n,$$

then

$$\begin{aligned} g(n+1) &= g(n) + \left\lfloor \frac{2}{n-1} g(n) \right\rfloor \\ &= \frac{p}{q}\binom{n}{2} + \frac{\beta}{q}(n-1) + x_n + \left\lfloor \frac{pn+2\beta}{q} + \frac{2x_n}{n-1} \right\rfloor. \end{aligned}$$

Let c_n be the residue class of $pn+2\beta \equiv pn-1 \pmod q$, where $0\le c_n<q$. We denote this writing $c_n=\mathrm{mod}(pn+2\beta, q)$. If $|2x_n/(n-1)| < 1/q$, and $x_n \ge 0$ when $c_n=0$, then

$$\left\lfloor \frac{pn+2\beta}{q} + \frac{2x_n}{n-1} \right\rfloor = \left\lfloor \frac{pn+2\beta}{q} \right\rfloor = \frac{pn+2\beta-c_n}{q}.$$

Therefore

$$\begin{aligned} g(n+1) &= \frac{p}{q}\binom{n}{2} + \frac{\beta}{q}(n-1) + x_n + \frac{pn+2\beta-c_n}{q} \\ &= \frac{p}{q}\binom{n+1}{2} + \frac{\beta}{q}(n) + x_{n+1}, \end{aligned}$$

where

$$x_{n+1} = x_n + \frac{\beta - c_n}{q}.$$

Note that $|x_{n+1}| \le |x_n| + \frac{1}{2}$, and if n is large enough, then $|2x_{n+1}/n| < 1/q$. Thus we can continue the iteration and obtain the following expressions:

$$g(n+2) = \frac{p}{q}\binom{n+2}{2} + \frac{\beta}{q}(n+1) + x_n + \frac{\beta - c_n}{q} + \frac{\beta - c_{n+1}}{q},$$

$$\cdots = \cdots \quad \cdots \quad \cdots$$

$$g(n+q) = \frac{p}{q}\binom{n+q}{2} + \frac{\beta}{q}(n+q-1) + x_n + \sum_{i=n}^{n+q-1} \frac{\beta - c_i}{q}.$$

Since $\gcd(p,q) = 1$, $\beta = (q-1)/2$, and $c_n \equiv pn + 2\beta \equiv pn - 1 \pmod q$, then c_n ranges over all the residue classes of q as n goes over q consecutive integers. Therefore

$$\sum_{i=n}^{n+q-1} c_i = 0 + 1 + \cdots + (q-1) = q(q-1)/2$$

and

$$x_{n+q} - x_n = \sum_{i=n}^{n+q-1} \frac{\beta - c_i}{q} = \beta - (q-1)/2 = 0.$$

This implies that the constant terms $\{x_n\}$ are periodic. In particular, x_n is bounded. Let b be an integer of the form $kq + 1$ which is larger than $2qx_n$ for all x_n, and let

$$g(b) = \frac{p}{q}\binom{b}{2} + \frac{\beta}{q}(b-1).$$

Then we have

$$\frac{g(n)}{\binom{n}{2}} = \frac{p}{q} + \frac{\beta}{q}\frac{n-1}{\binom{n}{2}} + \frac{x_n}{\binom{n}{2}} \to \frac{p}{q} \quad \text{as } n \to \infty.$$

and the theorem is proved.

THEOREM 2 *For any rational number $p/q \in (0,1)$ with $\gcd(p,q) = 1$, there exists an integer $c < \binom{q+1}{2}$ such that for the sequence determined by the recurrence relation (2) with initial value $g(q+1) = c$, the limit $\lim_{n\to\infty} g(n)/\binom{n}{2}$ equals p/q.*

Proof Keep the notation in the proof of Theorem 1. Let

$$g(q+1) = c = \frac{p}{q}\binom{q+1}{2} + \frac{\beta}{q}(q),$$

where $\beta = (q-1)/2$. Note that $x_{q+1} = 0$. We prove that for $i = 1, 2, 3, \ldots, q$,

$$\left\lfloor \frac{p(q+i)+2\beta}{q} + \frac{2x_{g+i}}{q+i-1} \right\rfloor = \left\lfloor \frac{p(q+i)+2\beta}{q} \right\rfloor = \frac{p(q+i)+2\beta - c_i}{q},$$

where $c_i = \mathrm{mod}(p(q+i)+2\beta, q) = \mathrm{mod}(pi-1, q)$. It is enough to show that

$$\left\lfloor \frac{c_i}{q} + \frac{2x_{q+i}}{q+i-1} \right\rfloor = 0,$$

for $i = 1, 2, 3, \ldots, q$.

In the following, we use induction to prove that the following formulas hold for $i = 1, 2, \ldots, q$:

$$0 \le \frac{c_i}{q} + \frac{2x_{q+i}}{q+i-1} < 1$$

and

$$x_{q+i} = \frac{1}{q}\sum_{j=1}^{i-1}(\beta - c_j). \tag{4}$$

For $i = 1$, $x_{q+1} = 0$, and it is obvious that $0 \le c_1/q < 1$.

Suppose that the formulas (4) are valid for $i-1$. We prove that they are valid for i.

Assume that $tq \le ip - 1 < (t+1)q$, where $p-1$, $2p-1, \ldots$, $(l-1)p - 1 \le tq < lp - 1$, $l < i$, and $lp - tq = y$. Thus $c_i = y + (i-l)p \le q-1$. We have

$$\sum_{i=1}^{l-1}(\beta - c_i) = -(l-1)y/2 < 0$$

and

$$\sum_{i=l}^{i-1}(\beta - c_i) = (i-l)\beta - (i-l)y - p(i-l)(i-l-1)/2 > 0.$$

Therefore

$$\begin{aligned}
&\frac{c_i}{q} + \frac{2x_{q+i}}{q+i-1} \\
&\geq \frac{1}{q}\left(y + (i-l)p - \frac{(l-1)y}{q+i-1}\right) \\
&\geq \frac{1}{q}\left(y - \frac{(i-1)y}{q+i-1}\right) \\
&= \frac{y}{q} \cdot \frac{q}{q+i-1} \\
&\geq 0
\end{aligned}$$

and

$$\begin{aligned}
&1 - \left(\frac{c_i}{q} + \frac{2x_{q+i}}{q+i-1}\right) \\
&\geq 1 - \frac{1}{q}\left(y + (i-l)p + \frac{(i-l)(q-1-2y-p(i-l-1))}{q+i-1}\right) \\
&= \frac{1}{q(q+i-1)}((q-y-(i-l)p)(q-1+l) - (i-l)(p-1-y)) \\
&\geq \frac{1}{q(q+i-1)}(q-1-(i-l)p) \\
&> 0,
\end{aligned}$$

where we use the fact that $y + (i-l)p \leq q-1$.

Therefore,

$$0 \leq \frac{c_i}{q} + \frac{2x_{q+i}}{q+i-1} < 1,$$

and consequently,

$$x_{q+i+1} = \frac{1}{q}\sum_{j=1}^{i}(\beta - c_j).$$

This finishes the induction step.

The only thing left to check is that $x_n \geq 0$ when $c_n \equiv pn - 1 \equiv 0 \pmod{q}$. This can be done as follows: When $pn - 1 \equiv 0 \pmod{q}$, $(p-1) + ((n-1)p - 1) = pn - 2 \equiv q - 1 \pmod{q}$. This implies $c_1 + c_{n-1} = q - 1 = 2\beta$. By the same reason, we have $\sum_{i=1}^{n-1}(\beta - c_i) = 0$. Therefore $x_n = x_{q+1} = 0$.

In conclusion, the sequence determined by the recurrence relation (2) and initial value

$$g(q+1) = c = \frac{p}{q}\binom{q+1}{2} + \beta$$

yields the limit $g(n)/\binom{n}{2} \to p/q$ as $n \to \infty$, which proves the theorem.

DEFINITION A value $g(b) = c$ is said to be *reducible* if there exists a preceding value $g(b-1) = c'$ such that

$$c = \left\lfloor \frac{b}{b-2} c' \right\rfloor.$$

Otherwise the initial value $g(b) = c$ is said to be *irreducible*.

THEOREM 3 *For any rational number $p/q \in (0,1)$ where* $\gcd(p,q) \neq 1$, *there exists an integer c such that the initial value $g(q+1) = c$ is irreducible and the limit* $2G(2,1,q+1,c) = p/q$.

Proof There are two cases.

Case 1 $\gcd(p,q) = 2$. We may assume that $p = 2p'$, $q = 2q'$, and $\gcd(p',q') = 1$.

By Theorem 2, we know that the initial value

$$g(q+1) = \frac{p}{q}\binom{q+1}{2} + \frac{q'-1}{2q'}(q) = p'(2q'+1) + q' - 1$$

will yield the limit $g(n)/\binom{n}{2} \to p/q$.

LEMMA 3.1 *Let $p = 2p'$, $q = 2q'$ and* $\gcd(p',q') = 1$. *The initial value*

$$g(q+1) = \frac{p}{q}\binom{q+1}{2} + \frac{q'+1}{2q'}(q) - 1$$

also yields the limit p/q.

Proof of Lemma 3.1 This is the case that

$$g(n) = \frac{p}{q}\binom{n}{2} + \frac{\beta'}{q}(n-1) + x_n,$$
$$g(n+1) = \frac{p}{q}\binom{n+1}{2} + \frac{\beta'}{q}(n) + x_{n+1},$$

where $\beta' = (q+1)/2$, and

$$x_{n+1} = \begin{cases} x_n + \dfrac{\beta' - c_n}{q} & \text{if } c_n \neq 0, \\ x_n + \dfrac{\beta' - c_n}{q} - 1 & \text{if } c_n = 0. \end{cases}$$

where $x_n < 0$ when $c_n = 0$.

In this case,

$$x_{n+q} = \frac{\sum_{i=n}^{n+q-1}(\beta' - c_i)}{q} - 1 = 0 \quad \text{since } \beta' = (q+1)/2.$$

The fact that the numerical conditions are satisfied by the initial value

$$g(q+1) = \frac{p}{q}\binom{q+1}{2} + \frac{q'+1}{2q'}(q) - 1$$

can be checked similarly as in the proof of Theorem 2.

Returning to the proof of Theorem 3, we have

$$g(q) = p'(2q'-1) + q' \ \Rightarrow \ g(q+1) = p'(2q'+1) + q' + 1,$$
$$g(q) = p'(2q'-1) + q' - 1 \ \Rightarrow \ g(q+1) = p'(2q'+1) + q' - 1.$$

Therefore

$$g(q) = \frac{p}{q}\binom{q+1}{2} + \frac{q'+1}{2q'}(q) - 1 = p'(2q'+1) + q'$$

is an irreducible initial value, and it yields the limit p/q.

Case 2 $\gcd(p, q) = k > 2$. We may assume that $p = kp'$, $q = kq'$, and $(p', q') = 1$.

From Theorem 2 and the proof of the Case 1, we know that for an integer c, if

$$m=\frac{p}{q}\binom{q+1}{2}+\frac{q'-1}{2q'}(q)\le c\le\frac{p}{q}\binom{q+1}{2}+\frac{q'+1}{2q'}(q)-1=M,$$

then $g(q+1)=c$ yields the limit $g(n)/\binom{n}{2}\to p/q$. There are k integers between m and M.

Since

$$\begin{aligned}g(q)&=\frac{k}{2}(p'(kq'-1)+q'-1)\;\Rightarrow\\ g(q+1)&=\frac{k}{2}(p'(kq'+1)+q'-1)=m\end{aligned}$$

and

$$\begin{aligned}g(q)&=\frac{k}{2}(p'(kq'-1)+q'+1)-1\;\Rightarrow\\ g(q+1)&=\frac{k}{2}(p'(kq'+1)+q'+1)>M,\end{aligned}$$

there are at most $k-1$ initial values $g(q+1)=c$ that are reducible, where $m\le c\le M$. Therefore there exists an integer $c\in[m,M]$ such that $g(q+1)=c$ is irreducible and yields the limit $g(n)/\binom{n}{2}\to p/q$.

It is easy to check that: (i) for the initial values $g(3)=0,1,2$, the limits of $g(n)/\binom{n}{2}$ are $0,0,\frac{1}{2}$, respectively, and (ii) for any integer b, if $g(b)\le b-2$, then the limit is 0. Combining this with Theorems 2 and 3, we conclude that for any rational number $p/q\in[0,1)$, there is an integer $c<\binom{q+1}{2}$ such that the initial value $g(q+1)=c$ is irreducible and yields the limit p/q. This proves the following theorem.

THEOREM 4 *For the sequences determined by the recurrence relation (2) and the initial values $g(m)=0,1,2,\ldots,\binom{m}{2}-1$, the limits of $g(n)/\binom{n}{2}\,(n\to\infty)$ are exactly the rational numbers*

$$\frac{p}{q},\quad 0\le p<q<m,$$

appearing in the increasing order. In other words, $2G(2,1,m,k)$ is the kth fraction in the ordered list $\{p/q\colon 0\le p<q<m\}$ for $0\le k<\binom{m}{2}$.

4 THE LIMIT OF $g(n)/n^\alpha$ IN THE CUBIC CASE

In this section, we discuss the case that $\alpha = 3$. Similar to the quadratic case, $G(3, a, b, c) = G(3, 2, b-a+2, c)$. Without loss of generality, we may assume $a = 2$. The recurrence relation for the sequence $\{g(n)\}$ becomes

$$g(n+1) = \left\lfloor \left(1 + \frac{3}{n-2}\right) g(n) \right\rfloor = \left\lfloor \frac{n+1}{n-2} g(n) \right\rfloor. \tag{5}$$

Let $h(n) = g(n)/\binom{n}{3}$. The corresponding recurrence relation for $h(n)$ is

$$h(n+1) = \frac{\left\lfloor \binom{n+1}{3} h(n) \right\rfloor}{\binom{n+1}{3}}.$$

For a sequence satisfying recurrence relation (5) and initial value $g(b) = c$, we have $\lim_{n\to\infty} h(n) = 3!G(3, 2, b, c)$.

DEFINITION An ordered pair of integers (p, q) is said to be *balanced* if there exists an integer β such that

$$\sum_{i=1}^{2q} (2\beta - c_i) = 0,$$

where $c_i = \mathrm{mod}(p\binom{n}{2} + 3\beta, q)$. In this case we say that (p, q) is balanced at β.

THEOREM 5 *For any rational number p/q, if the pair (p, q) is balanced at an integer β, then there exist integers b and c such that for the sequence determined by the recurrence* (5) *and initial value $g(b) = c$, $\lim_{n\to\infty} g(n)/\binom{n}{3} = p/q$.*

Proof Without loss of generality, we may assume that $0 < p/q < 1$. If for some integer n,

$$g(n) = \frac{p}{q}\binom{n}{3} + \frac{\beta}{q}(n-2) + x_n,$$

then

$$g(n+1)=\frac{p}{q}\binom{n}{3}+\frac{\beta}{q}(n-2)+x_n+\left\lfloor\frac{p}{q}\binom{n}{2}+\frac{3\beta}{q}+\frac{3x_n}{n-2}\right\rfloor.$$

Let $c_n=\operatorname{mod}(p\binom{n}{2}+3\beta,q)$. If $|3x_n/(n-2)|<1/q$, and $x_n\geq 0$ whenever $c_n=0$, then

$$\begin{aligned}g(n+1)&=\frac{p}{q}\binom{n}{3}+\frac{\beta}{q}(n-2)+x_n+\frac{p\binom{n}{2}+3\beta-c_n}{q}\\&=\frac{p}{q}\binom{n+1}{3}+\frac{\beta}{q}(n-1)+x_{n+1},\end{aligned}$$

where

$$x_{n+1}=x_n+\frac{2\beta-c_n}{q}.$$

Note that $|x_{n+1}|\leq|x_n|+2$. If n is large enough so that $|(3x_{n+1}+6q)/(n-1)|<1/q$, then we may repeat the iteration $2q$ steps, and obtain the following expression:

$$\begin{aligned}g(n+2)&=\frac{p}{q}\binom{n+2}{3}+\frac{\beta}{q}(n)+x_n+\frac{2\beta-c_n}{q}+\frac{2\beta-c_{n+1}}{q},\\ \cdots&=\cdots\quad\cdots\quad\cdots\\ g(n+2q)&=\frac{p}{q}\binom{n+2q}{3}+\frac{\beta}{q}(n+2q-2)+x_n+\sum_{i=n}^{n+2q-1}\frac{2\beta-c_i}{q}.\end{aligned}$$

Since (p,q) is balanced at β, $\sum_{i=n}^{n+2q-1}(2\beta-c_i)=0$, and hence $x_{n+2q}=x_n$. Therefore x_n is periodic. In particular, x_n is bounded. Choose b to be an integer such that $b>3qx_n+6q$ for all x_n, and let

$$c=\frac{p}{q}\binom{b}{3}+\frac{\beta}{q}(b-2)+x_n,$$

where x_n is the smallest positive number such that c is an integer and $x_i\geq 0$ whenever $c_i=0$.

The sequence $\{g(n)\}$ determined by the recurrence relation (5) and the initial value $g(b)=c$ yields the limit $g(n)/\binom{n}{3} \to p/q$ as $n\to\infty$, as required.

THEOREM 6 *For any pair of integers (p,q), let $i=\gcd(q,6)$. Then the pair (tp,tq) is balanced for some integer t if and only if the pair (ip,iq) is balanced.*

Proof It is clear that if (p,q) is balanced at β, then (tp,tq) is balanced at $t\beta$.

Suppose that (tp,tq) is balanced at β, where $t\geq i=\gcd(q,6)$. Let $e_n=\operatorname{mod}(p\binom{n}{2},q)$. Then $te_n=\operatorname{mod}(tp\binom{n}{2},tq)$. Assume that $\beta=kt+\gamma$, where $0\leq\gamma<t$. If $d_n=\operatorname{mod}(e_n+k,q)$, then $td_n+\gamma=\operatorname{mod}(tp\binom{n}{2}+3\beta,tq)$.

Assuming that (tp,tq) is balanced at β, we have

$$\sum_{i=0}^{2q-1}(td_i+\gamma)=4q\beta$$

or

$$t\sum_{i=0}^{2q-1}d_i+2q\gamma=4q\beta. \tag{6}$$

We discuss the different cases according to the value of $\gcd(q,6)$.

(i) $\gcd(q,6)=1$. In this case, $\sum_{i=0}^{2q-1}d_i\equiv p\binom{2q}{3}\equiv 0 \pmod q$, and $d_{i+q}=d_i$ as $e_{i+q}=e_i$. Therefore $2q$ divides $\sum_{i=0}^{2q-1}d_i$ and

$$t\left(\frac{\sum_{i=0}^{2q-1}d_i}{2q}\right)+\gamma=2\beta.$$

It follows that $2\beta\equiv\gamma \pmod t$. Since $\gamma\equiv 3\beta \pmod t$, we then have $\gamma=0$ and $\beta=kt/3$. Thus

$$t\left(\frac{\sum_{i=0}^{2q-1}d_i}{2q}\right)=\frac{2kt}{3},$$

which implies that $k=3\ell$ for some integer ℓ. Note that since $d_i = \mathrm{mod}(e_i + k, q) = \mathrm{mod}(p\binom{i}{2} + 3\ell, q)$, then the above formula can be written as

$$\sum_{i=0}^{2q-1} \mathrm{mod}\left(p\binom{i}{2} + 3\ell, q\right) = 4q\ell.$$

That is, (p,q) is balanced at ℓ.

(ii) $\gcd(q,6)=3$. In this case, $\sum_{i=0}^{q-1} d_i \equiv p\binom{q}{3} \pmod q$. Hence $\sum_{i=0}^{q-1} d_i$ is a multiple of $q/3$, and $\sum_{i=0}^{2q-1} d_i = 2\sum_{i=0}^{q-1} d_i$. The formula (6) can be written as

$$t\left(\frac{\sum_{i=0}^{q-1} d_i}{q/3}\right) + 3\gamma = 6\beta.$$

It follows that $6\beta \equiv 3\gamma \pmod t$. Combined with $3\beta \equiv \gamma \pmod t$, we have $\gamma = 0$ and $a = kt/3$. Substituting $a = kt/3$ into the formula (6), we obtain

$$\sum_{i=0}^{2q-1} 3d_i = 4qk.$$

That is,

$$\sum_{i=0}^{2q-1} \mathrm{mod}\left(3p\binom{i}{2} + 3k, 3q\right) = 4qk.$$

This proves $(3p, 3q)$ is balanced at k.

(iii) $\gcd(q,6)=2$. In this case $\sum_{i=0}^{2q-1} d_i \equiv p\binom{2q}{3} \equiv 0 \pmod q$. Thus

$$t\left(\frac{\sum_{i=0}^{2q-1} d_i}{q}\right) + 2\gamma = 4\beta.$$

It follows that $4\beta \equiv 2\gamma \pmod t$. Combined with $3\beta \equiv \gamma \pmod t$, we have $2\gamma \equiv 0 \pmod t$. If $\gamma = 0$, then $\beta = kt/3$ and

$$t\sum_{i-0}^{2q-1} d_i = 4q \cdot \frac{kt}{3}.$$

This implies that $k = 3l$ for some integer l, and (p,q) is balanced at l.

If $\gamma = t/2$, then $3\beta = kt + t/2$; hence the formula (6) becomes

$$2\sum_{i=0}^{2q-1} d_i + 2q = 4q \cdot \frac{2k+1}{3}.$$

It follows that 3 divides $2k+1$. Write $k = 3l+1$ for some integer l. We have

$$\sum_{i=0}^{2q-1} (2d_i + 1) = 2q \cdot 2(2l+1).$$

Note that $2d_i + 1 = \mathrm{mod}(2p\binom{i}{2} + 2k, 2q) + 1 = \mathrm{mod}(2p\binom{i}{2} + 2(3l+1) + 1, 2q) = \mathrm{mod}(2p\binom{i}{2} + 3(2l+1), 2q)$. This implies that $(2p, 2q)$ is balanced at $2l+1$.

(iv) $\gcd(q, 6) = 6$. Again, $\sum_{i=0}^{2q-1} d_i$ is a multiple of $q/3$. Thus the formula (6) can be written as

$$t\left(\frac{\sum_{i=0}^{2q-1} d_i}{q/3}\right) + 6\gamma = 12\beta.$$

It follows that $12\beta \equiv 6\gamma \pmod{t}$. Combined with $3\beta \equiv \gamma \pmod{t}$, we have $2\gamma \equiv 0 \pmod{t}$. If $\gamma = 0$ and $\beta = kt/3$, then

$$3\sum_{i=0}^{2q-1} d_i = 4qk.$$

Note that $3d_i = \mathrm{mod}(3p\binom{i}{2} + 3k, 3q)$. This implies that $(3p, 3q)$ is balanced at k. If $\gamma = t/2$ and $\beta = kt + t/2$, then the formula (6) becomes

$$\begin{aligned}
& t\sum_{i=0}^{2q-1} d_i + 2q \cdot \frac{t}{2} = 4q \cdot \frac{(2k+1)t}{6} \\
\Rightarrow\ & 2\sum_{i=0}^{2q-1} d_i + 2q = 4q \cdot \frac{2k+1}{3} \\
\Rightarrow\ & \sum_{i=0}^{2q-1} \mathrm{mod}\left(2p\binom{i}{2} + 2k + 1, 2q\right) = 4q \cdot \frac{2k+1}{3} \\
\Rightarrow\ & \sum_{i=0}^{2q-1} \mathrm{mod}\left(6p\binom{i}{2} + 3(2k+1), 6q\right) = 4q(2k+1),
\end{aligned}$$

i.e., $(6p, 6q)$ is balanced at $2k+1$.

TABLE I Balanced pairs for small initial values

Initial value	*Limit* $H(b,c)$	*Balanced pair*
$g(4)=1$	4/23	(4, 23) is balanced at 4
$g(4)=2$	6/13	(6, 13) is balanced at 4
$g(4)=3$	2/3	(6, 9) is balanced at 1
$g(5)=1$	6/109	(6, 109) is balanced at 28
$g(5)=2$	4/23	(4, 23) is balanced at 4
$g(5)=3$	4/15	(12, 45) is balanced at 14
$g(5)=5$	6/13	(6, 13) is balanced at 4
$g(5)=6$	4/7	(4, 7) is balanced at 2
$g(5)=7$	2/3	(6, 9) is balanced at 1
$g(5)=9$	6/7	(6, 7) is balanced at 1
$g(6)=7$	15/47	(15, 47) is balanced at 14
$g(6)=9$	9/22	(18, 44) is balanced at 9
$g(6)=11$	11/21	(33, 63) is balanced at 10
$g(6)=15$	3/5	(3, 5) is balanced at 1
$g(6)=17$	4/5	(4, 5) is balanced at 1
$g(6)=19$	10/11	(10, 11) is balanced at 2

Example 1 Denote by $H(b,c)$ the limit of $g(n)/\binom{n}{3}$ where $\{g(n)\}$ is determined by the recurrence relation (5) and the initial value $g(b)=c$. Table I gives some balanced pairs for small initial values.

We do not currently know the values of $H(5,4)$, or $H(5,8)$. It seems that the limits are not given by any balanced pairs.

DEFINITION 2 An ordered pair of integers (p,q) is said to be *almost balanced* if there exists an integer β such that

$$\sum_{i=0}^{2q-1} \frac{2\beta - c_i}{q} = \#\{c_i \mid 0 \le i < 2q, c_i = 0\},$$

where $c_i = \operatorname{mod}(p\binom{i}{2} + 3\beta, q)$.

COROLLARY 6.1 *If a pair of integers (p,q) is almost balanced, then there exist integer b and c such that $H(b,c)=p/q$.*

Proof From the proof of Theorem 5, the formula

$$x_{n+1} = x_n + \frac{2\beta - c_n}{q}$$

is valid if $|3x_n/(n-2)| \leq 1/q$ and $x_n \geq 0$ whenever $c_n = 0$. On the other hand, if $c_n = 0$ and $x_n < 0$, then

$$x_{n+1} = x_n + \frac{2\beta - c_n}{q} - 1.$$

If (p, q) is almost balanced at β, then we can choose the initial value

$$g(n) = \frac{p}{q}\binom{n}{3} + \frac{\beta}{q}(n-2) + x_n$$

such that $x_n \ll n$ and $x_i < 0$ whenever $c_i = 0$. Thus

$$x_{n+1} = \begin{cases} x_n + \dfrac{2\beta - c_n}{q} & \text{if } c_n \neq 0, \\ x_n + \dfrac{2\beta - c_n}{q} - 1 & \text{if } c_n = 0. \end{cases}$$

Therefore,

$$x_{n+2q} - x_n = \sum_{i=n}^{n+2q-1} \frac{2\beta - c_i}{q} - \#\{c_i \mid n \leq i < n+2q, c_i = 0\} = 0.$$

Thus, again $\{x_n\}$ is periodic and the general form of $g(m)$ is

$$g(m) = \frac{p}{q}\binom{m}{3} + \frac{\beta}{q}(m-2) + x_m.$$

Therefore $g(m)/\binom{m}{3} \to p/q$.

Example 2 Let $p = 6$ and $q = 47$. The pair $(6, 47)$ is not balanced at any number. However, it is almost balanced at $\beta = 11$. One can check that $c_{23} = c_{70} = c_{25} = c_{72} = 0$. By taking the initial value

$$g(493) = 2534050 = \frac{6}{47}\binom{493}{3} + \frac{11}{47}(491) - 1,$$

we get the limit $6/47$.

CONJECTURE We conjecture that for a pair of integers (p, q), if there exists an integer β such that $\sum_{i=0}^{2q-1}(2\beta - c_i)/q$ is an integer and

$$0 \leq \sum_{i=0}^{2q-1} \frac{2\beta - c_i}{q} \leq \#\{c_i \mid 0 \leq i < 2q, c_i = 0\},$$

then there exist integers b and c such that $H(b, c) = p/q$.

In the following, we give some bounds to the initial value $g(b)=c$ in order that $H(b,c)$ is not equal to a given rational number p/q.

THEOREM 7 *For the sequence $\{g(n)\}$ determined by the recurrence relation* (5) *and initial value $g(b)=c$, if*

$$g(b) < \frac{p}{q}\binom{b}{3} + \frac{b-2}{3q}$$

or

$$g(b) > \frac{p}{q}\binom{b}{3} + (b-2),$$

then $H(b,c) \neq p/q$.

Proof Write $g(b) = (p/q)\binom{b}{3} + x_b$. If $x_b < (b-2)/3q$, then

$$\begin{aligned} g(b+1) &= g(b) + \left\lfloor \frac{3}{b-2} g(b) \right\rfloor \\ &= \frac{p}{q}\binom{b}{3} + x_b + \left\lfloor \frac{p}{q}\binom{b}{2} + \frac{3x_b}{b-2} \right\rfloor \\ &\leq \frac{p}{q}\binom{b+1}{3} + x_b - \frac{e_b}{q}, \end{aligned}$$

where $e_b = \mathrm{mod}(p\binom{b}{2}, q)$. Note that $x_{b+1} = x_b - e_b/q < x_b < (b+1-2)/(3q)$. Therefore, we can repeat this process, and get

$$x_{b+2q} \leq x_b - \frac{\sum_{i=1}^{2q} e_i}{q} \leq x_b - C,$$

and, in general,

$$x_{n+2q} \leq x_n - \frac{\sum_{i=1}^{2q} e_i}{q} \leq x_n - C,$$

where C is the constant $\sum_{i=1}^{2q} e_i/q > 0$. Since x_b is a fixed number, there exists an integer m such that $x_m < 0$, and $g(m) < p\binom{m}{3}/q$. Thus $h(m) = g(m)/\binom{m}{3} < p/q$. Note that $h(n)$ is a monotonically non-increasing sequence, so that the limit of $h(n)$ cannot be equal to p/q.

On the other hand, if for some initial value b,

$$g(b) > \frac{p}{q}\binom{b}{3} + (b-2),$$

we can write $g(b)$ as

$$g(b) = \frac{p}{q}\binom{b}{3} + a\binom{b-1}{2} + x_b,$$

where $a(b-1)/2 \geq 1$ and $x_b = 0$.

For any $g(n)$ of the form

$$g(n) = \frac{p}{q}\binom{n}{3} + a\binom{n-1}{2} + x_n,$$

with $a(n-1)/2 \geq 1$ and $x_n \geq 0$, we have

$$\begin{aligned} g(n+1) &= \frac{p}{q}\binom{n}{3} + a\binom{n-1}{2} + x_n + \left\lfloor \frac{p}{q}\binom{n}{2} + \frac{3a(n-1)}{2} + \frac{3x_n}{n-2} \right\rfloor \\ &\geq \frac{p}{q}\binom{n}{3} + a\binom{n-1}{2} + x_n + \frac{p}{q}\binom{n}{2} + \frac{3a(n-1)}{2} - 1 \\ &= \frac{p}{q}\binom{n+1}{3} + a\binom{n}{2} + \frac{a}{2}(n-1) + x_n. \end{aligned}$$

That is,

$$x_{n+1} \geq x_n + \frac{a}{2}(n-1).$$

Iterating t times, we have

$$x_{n+t} - x_n \geq \frac{a}{2}\binom{t-1}{2} + t\left(\frac{a(n-1)}{2} - 1\right) \geq \frac{a}{2}\binom{t-1}{2}.$$

Now taking $n = t = b$, we have

$$\begin{aligned} g(2b) &\geq \frac{p}{q}\binom{2b}{3} + a\binom{2b-1}{2} + x_n + \frac{a}{2}\binom{b-1}{2} \\ &\geq \frac{p'}{q'}\binom{2b}{3} + a\binom{2b-1}{2} + x_n, \end{aligned}$$

where

$$\frac{p'}{q'} = \frac{p}{q} + \frac{3a}{8b} \cdot \frac{b-2}{2b-1}.$$

When we keep iterating, it is always true that

$$g(m) \geq \frac{p'}{q'}\binom{m}{3} + a\binom{m-1}{2}.$$

Thus the limit $H(b, c)$ is at least p'/q' which is larger than p/q.

Using Theorem 7, we can show that certain numbers will not appear as a limit $H(b, c)$ for any choice of integers b, c. As an example, we show that $\frac{1}{4}$ cannot be a value of $H(b, c)$.

Example 3 For any integers b and c, $H(b, c) \neq \frac{1}{4}$.

Proof Assume to the contrary that there exists initial value $g(n)$, $(n > 40)$ such that $H(n, g(n)) = \frac{1}{4}$. By Theorem 7, the term $g(n)$ can be written as

$$g(n) = \frac{1}{4}\binom{n}{3} + \frac{\beta}{4}(n-2) + x_n,$$

where $x_n \in [1, 2)$, and $0 < \beta < 4$. Consequently, we have

$$g(n+1) = \frac{1}{4}\binom{n}{3} + \frac{\beta}{4}(n-2) + x_n + \left\lfloor \frac{1}{4}\binom{n}{2} + \frac{3\beta}{4} + \frac{3x_n}{n-2} \right\rfloor.$$

Case 1 If $\beta \geq 1$, then

$$g(n+1) \geq \frac{1}{4}\binom{n}{3} + \frac{\beta}{4}(n-2) + x_n + \left\lfloor \frac{1}{4}\binom{n}{2} + \frac{3\beta}{4} \right\rfloor.$$

Let $t = \lfloor 3\beta \rfloor$, $\gamma = 3\beta - t$, and $c_n = \mathrm{mod}(\binom{n}{2} + t, 4)$. Then

$$\begin{aligned} g(n+1) &\geq \frac{1}{4}\binom{n}{3} + \frac{\beta}{4}(n-2) + x_n + \left\lfloor \frac{\binom{n}{2} + 3\beta - \gamma}{4} \right\rfloor \\ &= \frac{1}{4}\binom{n+1}{3} + \frac{\beta}{4}(n-1) + x_{n+1}, \end{aligned}$$

where

$$x_{n+1} = x_n + \frac{2\beta - \gamma - c_n}{4}.$$

Note that for any value of t, $c_{n+8} = c_n$. Furthermore, in any period of length 8, the values of c_n are $\{0, 0, 1, 1, 2, 2, 3, 3\}$. Our choice of x_n will guarantee that $x_{n+1} \geq 0$. Continuing the iteration, we obtain

$$\begin{aligned} x_{n+8} - x_n &\geq \sum_{i=n}^{n+7} \frac{2\beta - \gamma - c_i}{4} \\ &= 2(2\beta - \gamma) - 3. \end{aligned}$$

If $1 \leq \beta < \frac{4}{3}$, then $\gamma = 3\beta - 3$, and $2(2\beta - \gamma) - 3 = 3 - 2\beta \geq 3 - 2 \cdot \frac{4}{3} = \frac{1}{3}$. If $\beta \geq \frac{4}{3}$, then $\gamma < 1$, and $2(2\beta - \gamma) - 3 > 2(2\beta - 1) - 3 \geq \frac{1}{3}$. Hence, in the case $\beta \geq 1$,

$$x_{n+8} - x_n \geq \frac{1}{3} \Rightarrow x_{n+8k} - x_n \geq \frac{k}{3}.$$

Now

$$g(n + 8k) \geq \frac{1}{4}\binom{n + 8k}{3} + \frac{\beta}{4}(n + 8k - 2) + \frac{k}{3} + x_n.$$

Letting $k = n$, we have

$$g(9n) \geq \frac{1}{4}\binom{9n}{3} + \left(\frac{\beta}{4} + \frac{1}{27}\right)(9n - 2) + x_n,$$

and therefore

$$g(9^m n) \geq \frac{1}{4}\binom{9^m n}{3} + \left(\frac{\beta}{4} + \frac{m}{27}\right)(9^m n - 2) + x_n.$$

As m increases, the linear term $\beta/4 + m/27$ will be larger than 1. By Theorem 7, $\frac{1}{4}$ will not be the limit of $g(n)/\binom{n}{3}$.

Case 2 If $\beta < 1$, then

$$g(n + 1) = \frac{1}{4}\binom{n + 1}{3} + \frac{\beta}{4}(n - 1) + x_{n+1},$$

where

$$x_{n+1} = x_n + \frac{2\beta - \gamma - c_n}{4},$$

and $x_{n+1} \le x_n + \frac{1}{2}$. Thus $3x_{n+1}/(n-1) < \frac{1}{4}$, and

$$\begin{aligned} g(n+2) &= g(n+1) + \left\lfloor \frac{1}{4}\binom{n+1}{2} + \frac{3\beta}{4} + \frac{3x_{n+1}}{n-1} \right\rfloor \\ &\le g(n+1) + \left\lfloor \frac{1}{4}\binom{n+1}{2} + \frac{3\beta}{4} \right\rfloor \\ &= \frac{1}{4}\binom{n+2}{3} + \frac{\beta}{4}(n) + x_{n+1} + \frac{2\beta - \gamma - c_{n+1}}{4}. \end{aligned}$$

That is,

$$x_{n+2} \le x_n + \frac{2\beta - \gamma - c_n}{4} + \frac{2\beta - \gamma - c_{n+1}}{4}.$$

Repeating this procedure, and noting that the inequality $3x_{n+i}/(n+i-2) < \frac{1}{4}$ holds for $i = 1, 2, \ldots, 8$, we have

$$\begin{aligned} x_{n+8} - x_n &\le \sum_{i=n}^{n+7} \frac{2\beta - \gamma - e_i}{4} \\ &= 2(2\beta - \gamma) - 3. \end{aligned}$$

If $\beta \le \frac{2}{3}$, then $2(2\beta - \gamma) - 3 \le 4\beta - 3 \le -\frac{1}{3}$. If $\frac{2}{3} < \beta < 1$, then $\gamma = 3\beta - 2$, and $2(2\beta - \gamma) - 3 = 1 - 2\beta \le -\frac{1}{3}$. Hence in the case that $\beta < 1$, we always have

$$x_{n+8} - x_n \le -\tfrac{1}{3}.$$

Now

$$\begin{aligned} g(n+8k) &= \frac{1}{4}\binom{n+8k}{3} + \frac{\beta}{4}(n+8k-2) + x_{n+8k} \\ &\le \frac{1}{4}\binom{n+8k}{3} + \frac{\beta}{4}(n+8k-2) - \frac{k}{3} + x_n. \end{aligned}$$

Letting $k = n$, we have

$$g(9n) \leq \frac{1}{4}\binom{9n}{3} + \left(\frac{\beta}{4} - \frac{1}{27}\right)(9n-2) + x_n$$

and, therefore,

$$g(9^m n) \leq \frac{1}{4}\binom{9^m n}{3} + \left(\frac{\beta}{4} - \frac{m}{27}\right)(9^m n-2) + x_n.$$

As m increases, the linear term $\beta/4 - m/27$ will drop below 0. By Theorem 7, $\frac{1}{4}$ cannot be the limit of $g(n)/\binom{n}{3}$. This prove our claim.

By the same argument, one can show that there are no integers b and c such that $H(b,c) = \frac{3}{4}$.

5 QUESTIONS AND CONJECTURES

In this section, we first list some data about the balanced pairs and the almost balanced pairs (Tables II and III). They were obtained with the help of the Maple symbolic computation package.

These data lead us to the following conjectures.

CONJECTURE 1 For any prime k, the pair $(k, k+1)$ is a balanced pair.

CONJECTURE 2 For any rational number of the form $p/2^i \neq \frac{1}{2}$, p odd, there are no integers b and c such that $H(b,c) = p/2^i$.

Clearly, there are many more questions than answers at this point. For example, find integers b and c so that $H(b,c)$ is irrational. We believe good candidates are $H(5,4)$ and $H(5,8)$. Can we characterize those b and c for which $H(b,c)$ is rational? Conceivably, $H(b,c)$ is irrational for almost all pairs (b,c). Can $H(b,c)$ be rational without the existence of balanced or almost balanced pairs? What values of p/q

TABLE II Balanced pairs for $0 < p < q < 20$, $\gcd(p, q) = 1$

p/q	*Balanced pair*	β	p/q	*Balanced pair*	β	p/q	*Balanced pair*	β
1/2	(2, 4)	1	1/3	(3, 9)	2	2/3	(6, 9)	1
2/5	(2, 5)	1	3/5	(3, 5)	1	4/5	(4, 5)	1
1/6	(6, 36)	7	5/6	(30, 36)	11	2/7	(2, 7)	2
4/7	(4, 7)	1 or 2	6/7	(6, 7)	1 or 2	4/9	(12, 27)	5
5/9	(15, 27)	4	7/9	(21, 27)	8	8/9	(24, 27)	7
1/10	(2, 20)	5	9/10	(18, 20)	5	1/11	(1, 11)	3
2/11	(2, 11)	3	3/11	(3, 11)	3	5/11	(5, 11)	2
6/11	(6, 11)	2	7/11	(7, 11)	3	10/11	(10, 11)	2
2/13	(2, 13)	3	4/13	(4, 13)	3	5/13	(5, 13)	3
6/13	(6, 13)	4	7/13	(7, 13)	3	10/13	(10, 13)	2
11/13	(11, 13)	3	12/13	(12, 13)	4	5/14	(10, 28)	7
9/14	(18, 28)	7	2/15	(6, 45)	10	4/15	(12, 45)	8
7/15	(21, 45)	11 or 14	14/15	(42, 45)	10 or 13	2/17	(2, 17)	4
3/17	(3, 17)	5	8/17	(8, 17)	5	10/17	(10, 17)	3
11/17	(11, 17)	3 or 4	14/17	(14, 17)	4	15/17	(15, 17)	4
16/17	(16, 17)	4	1/18	(6, 108)	25	17/18	(102, 108)	29
2/19	(2, 19)	5	3/19	(3, 19)	4	6/19	(6, 19)	5
9/19	(9, 19)	5	11/19	(11, 19)	5	15/19	(15, 19)	5
16/19	(16, 19)	4	18/19	(18, 19)	5			

TABLE III Almost balanced pairs for $0 < p < q < 30$

(p, q)	β	(p, q)	β	(p, q)	β
(8, 11)	3	(9, 13)	4	(6, 17)	5
(9, 17)	5	(13, 17)	4	(8, 19)	5
(10, 19)	6	(14, 19)	5	(16, 19)*	6
(11, 23)	6	(13, 23)	5	(14, 23)	7
(20, 23)	6	(12, 25)	6	(16, 25)	8
(24, 25)	7	(7, 29)	9	(9, 29)	8
(11, 29)*	9	(13, 29)*	8	(15, 29)	8
(20, 29)*	9	(22, 29)	8	(24, 29)	7
(26, 29)	8	(27, 29)*	9		

* The pair is also balanced.

do not occur as values of $H(b, c)$? Must any such forbidden value have $q = 2^t$ for some t?

Of course, all of these questions (and more) can be raised for larger values of α, and in particular, $\alpha = 4, 5, \ldots$ These recurrences are all special cases of the more general recurrence

$$f(n+1) = \frac{\lfloor p(n)f(n) \rfloor}{p(n)},$$

where $p(n)$ is some function (e.g., a polynomial) which tends to infinity as n tends to infinity (e.g., $\alpha = 3$, $a = 2$ in the recurrence (1) corresponds to the choice $p(n) = \binom{n+1}{3}$). What is true in this more general situation?

Acknowledgments

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