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A Remark on a Paper of Erdös and Nathanson

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A set A of integers is said to be an *asymptotic basis of order* h if every sufficiently large integer can be represented as a sum of h (not necessarily distinct) elements of A. In a recent paper [EN], Erdös and Nathanson prove the following interesting result.

Theorem 1. *Let A be an asymptotic basis of order h, and let $f(n)$ denote the number of pairwise disjoint representations of n as a sum of h elements of A. Suppose $t \geq 2$ and and $c > \log^{-1}(t^h/(t^h-1))$. Then, if $f(n) \geq c \log n$ for all sufficiently large n, then A can be partitioned into the disjoint union of t sets, each of which is an asymptotic basis of order h.*

A critical component in their proof is the following combinatorial result.

Theorem 2 [EN]. *Suppose $S(n)$ is a set of disjoint h-element subsets of $\omega = \{1,2,3,\ldots\}$ such that for some $c > \log^{-1}(t^h/(t^h-1))$, we have $|S(n)| \geq c \log n$ for all sufficiently large n. There there exists a partition of $\omega = C_1 \cup \cdots \cup C_t$ such that $S(n)$ contains h-element subsets of each C_i, $1 \leq i \leq t$, for all sufficiently large n.*

Erdös and Nathanson raise the question as to what extent the size condition on $f(n)$ in Theorem 1 can be relaxed without affecting the validity of the conclusion. In particular, they suggest that theorem could even hold under the much weaker assumption that $\lim_{n\to\infty} f(n) = \infty$. This question is still not resolved. However, it would follow if the corresponding assumption, namely, $\lim_{n\to\infty} |S(n)| = \infty$, were enough to guarantee the validity of Theorem 2. Our purpose in this note is to point out that this is *not* the case, and in fact, the growth restriction they give for $|S(n)|$ in Theorem 2 is (up to a constant factor) best possible. For ease of exposition, we restrict our arguments to the simplest case, namely, $t = k = 2$.

Theorem. *For each n, there exists a set $S'(n)$ of mutually disjoint pairs of integers so that:*

(i) $|S'(n)| > c \log n$ *for any* $c < 1/\log 2$ *as* $n \to \infty$,

(ii) *for any partition of $\omega = C_1 \cup C_2$, infinitely many $S'(n)$ have either no pair from C_1 or no pair from C_2.*

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Proof. The whole proof is based on the following simple idea. For a (rapidly) increasing sequence of integers $N \to \infty$, we will form many *perfect matchings* on $[2N] := \{1, 2, \ldots, 2N\}$ i.e., sets $M = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_N, y_N\}\}$ where all the entries in these N pairs are distinct and between 1 and $2N$, inclusive. The plan will be to choose as few perfect matchings M as possible so that *any* N-element set $X \subset [2N]$ is "split" by one of M's, i.e., X hits each of the N pairs of M. This implies that for any partition of $\omega = C_1 \cup C_2$, some C_i has $|C_i \cap [2N]| \le N$, and therefore, this C_i has no pairs in at least one of the perfect matchings M. One trivial way to accomplish this is to choose *all possible* perfect matchings on $[2N]$. However, since there are $\frac{(2N)!}{2^N N!} \sim \left(\frac{2N}{e}\right)^N \sqrt{2}$ such perfect matchings then this construction only yields families $S'(n)$ with $|S'(n)| = (1+o(1))\,\frac{\log n}{\log\log n}$. To obtain the claimed result, we have to be more careful in forming our perfect matchings. To do this, we will choose them randomly.

More precisely, we select t perfect matchings M_i, $1 \le i \le t$, independently and uniformly at random. For a fixed N-element set $X \subset [2N]$, let us call M_i "X-bad" if it does not split X. A simple calculation shows that the probability of *not* splitting X is $1 - 2^N \Big/ \binom{2N}{N}$. Thus, the probability that *all* the M_i are X-bad is $\left(1 - 2^N \Big/ \binom{2N}{N}\right)^t$. Since there are just $\binom{2N}{N}$ different X's to consider then if we have

$$\text{(1)} \qquad \binom{2N}{N}\left(1 - \frac{2^N}{\binom{2N}{N}}\right)^t < 1$$

then with positive probability, for any N-set $X \subset [2N]$, at least one M_i is not X-bad. In particular, if t is chosen to satisfy (1), then there is *some* choice of perfect matchings M_i, $1 \le i \le t$, so that any N-set $X \subset [2N]$ is split by one of the M_i. Finally, we form our desired $S'(n)$'s by placing these M_i consecutively for each N, for a sequence of N's rapidly tending to infinity.

An easy calculation shows that

$$t > \frac{2^N \sqrt{N}\,\log 4}{\pi}$$

is enough for (1) to hold. Inverting, we find that (i) holds. Of course, (ii) holds by the choice of the various $M_i = M_i(N)$, and the theorem is proved. ∎

We point out that similar arguments can be used to prove analogous results for general h and t. Our result shows that the combinatorial approach used by Erdös and Nathanson cannot be pushed much further in trying to prove the conjecture mentioned earlier, namely that $\lim_{n\to\infty} f(n) = \infty$ implies that A can be decomposed into t disjoint asymptotic bases of order h. It would be interesting in this case, however, to determine the largest value α (in place of $1/\log 2$) for which the theorem is valid. By Theorem 1, and (i), it follows that

$$1/\log 2 \leq \alpha < 1/\log 4/3 .$$

References

[EN] P. Erdös and M. B. Nathanson, Partitions of bases into disjoint unions of bases, J. Num. Th. 29 (1988), 1-9.