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MAXIMUM CUTS AND QUASIRANDOM GRAPHS

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2.1. INTRODUCTION

In an earlier paper [Chung, Graham, and Wilson (1991)], the authors considered a large class of so-called *quasirandom* graph properties that are mutually equivalent in the sense that any graph satisfying one of the properties must satisfy all of them. Some of these graph properties are the following:

P_1: G has at least $(1 + o(1))n^2/4$ edges and at most $(1 + o(1))n^4/16$ 4-cycles.

$P_2(s)$: For fixed s, each (ordered) graph $M(s)$ on s vertices occurs $(1 + o(1))n^s/2^{\binom{s}{2}}$ times as an induced subgraph of G.

P_3: For any subset S of vertices of G, the number $e(S)$ of edges spanned by S satisfies $e(S) = \frac{1}{4}|S|^2 + o(n^2)$.

$P_3^{(\alpha)}$: For fixed α, $0 < \alpha < 1$, and any subset $S \subseteq V(G)$ with $|S| = \alpha n$,

$$e(S) = \frac{\alpha^2 n^2}{4} + o(n^2).$$

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(In fact, in Chung, Graham, and Wilson (1991) a proof of $P_3 \Leftrightarrow P_3^{(1/2)}$ was given, and $P_3 \Leftrightarrow P_3^{(\alpha)}$ can be proved analogously.)

Other possible candidates for quasirandom graph properties are the following:

Q: For every $S \subseteq V(G)$, the number $e(S, \bar{S})$ of edges between S and $\bar{S}$ satisfies

$$e(S, \bar{S}) = \tfrac{1}{2}|S|\,|\bar{S}| + o(n^2).$$

$Q^{(\alpha)}$: For a fixed α, $0 < \alpha < 1$ and for every $S \subseteq V(G)$ with $|S| = \alpha n$,

$$e(S, \bar{S}) = \tfrac{1}{2}\alpha(1 - \alpha)n^2 + o(n^2).$$

We will use the following convention. Suppose we have two properties $P = P(o(1))$ and $P' = P'(o(1))$, each with occurrences of the asymptotic $o(1)$ notation. By the implication $P \Rightarrow P'$, we mean that for each $\epsilon > 0$ there is a $\delta > 0$ such that if $G(n)$ satisfies $P(\delta)$, then it also satisfies $P'(\epsilon)$, provided $n > n_0(\epsilon)$. For example, $P_1(\delta) \Rightarrow P_3(\epsilon)$ means "If G has at least $(1 + \delta)n^2/4$ edges and at most $(1 + \delta)n^4/16$ 4-cycles, then for any subset S of vertices of G, $|e(S) - \frac{1}{4}|S|^2| < \epsilon n^2$, provided $n > n_0(\epsilon)$."

It is easy to see that $P_3 \Rightarrow Q \Rightarrow Q^{(\alpha)}$. However, there is an obvious obstacle for the reverse implication as illustrated by the following example: Let H have vertex set $A \cup B$, where $|A| = |B| = n/2$, with n even. The edge set of H consists of all pairs in A and a random bipartite graph between A and B. In other words, each $\{a, b\}$, $a \in A$ and $b \in B$, is in $E(H)$ independently with probability $\frac{1}{2}$. It is straightforward to check that almost all H satisfy $Q^{(1/2)}$ but do not satisfy P_3 or $P_3^{(1/2)}$.

Nevertheless, it turns out that the value $\alpha = \frac{1}{2}$ is very special. We will prove the following theorem:

Theorem 2.1. For $\alpha \neq \frac{1}{2}$, $Q^{(\alpha)} \Rightarrow P_3^{(1/2)}$. Therefore $Q^{(\alpha)}$ is a quasirandom property for $\alpha \neq \frac{1}{2}$.

Before proceeding to a proof of the theorem, we need the following (weaker) property (which we term "almost regularity"):

$P_0(\epsilon)$: For all except ϵn vertices, all degrees $\deg(v)$ satisfy $|\deg(v) - n/2| \leq \epsilon n$.

In Section 2.2 we will prove that Q implies P_0. Section 2.3 contains the proof of the main theorem. Section 2.4 includes remarks and further questions.

We note that $Q^{(\alpha)} = Q^{(1-\alpha)}$, so that we can assume without loss of generality that $\alpha < \frac{1}{2}$.

2.2. ALMOST REGULARITY

Lemma 2.1. $Q \Rightarrow P_0$.

Proof. Suppose $P_0(\epsilon)$ is violated. Thus, there is a set $X \subset V = V(G)$, $|X| = \epsilon n$, such that for all $x \in X$, $|\deg(x) - n/2| > \epsilon n$. That is, either $\deg(x) > n/2 + \epsilon n$ or $\deg(x) < n/2 - \epsilon n$. We first consider the case that the subset $X' = \{x \in X\colon \deg(x) > n/2 + \epsilon n\}$ has at least $|X|/2 = \epsilon n/2$ vertices. (The other case can be dealt with in a similar way.) Thus,

$$e(X', \overline{X'}) \geq |X'|\left(\frac{n}{2} + \epsilon n - |X'|\right) \geq |X'| \cdot \frac{n}{2},$$

$$\frac{e(X', \overline{X'})}{|X'|\,|\overline{X'}|} \geq \frac{n/2}{(1 - \epsilon/2)n} \geq \frac{1}{2} + \frac{\epsilon}{4},$$

and so,

$$e(X', \overline{X'}) \geq \frac{1}{2}|X'|\,|\overline{X'}| + \frac{\epsilon^2}{8}\left(1 - \frac{\epsilon}{2}\right)n^2.$$

This contradicts $Q(\delta)$ if $\delta < \epsilon^2/16$ and consequently, Lemma 2.1 is proved. □

Note that for each $S \subset V$, we have

$$2e(S) + e(S, \bar{S}) = \sum_{x \in S} \deg(x). \tag{2.1}$$

Lemma 2.2. $Q \Leftrightarrow P_3$. Thus, Q is a quasirandom property.

Proof. $\Rightarrow$ From Lemma 2.1, we know $Q \Rightarrow P_0$. Using P_0, Q, and (2.1), we have

$$2e(S) = \frac{n}{2}|S| - \frac{1}{2}|S|\,|\bar{S}| + o(n^2)$$

or

$$e(S) = \frac{|S|^2}{4} + o(n^2).$$

Therefore, P_3 holds.

$\Leftarrow$ In Chung, Graham, and Wilson (1991) it is proved that $P_3 \Rightarrow P_0$. Therefore,

$$e(S, \bar{S}) = \frac{n}{2}|S| - 2e(S) + o(n^2)$$
$$= \frac{1}{2}|S|\,|\bar{S}| + o(n^2),$$

and so Q is a quasirandom property. □

Lemma 2.3. For fixed α, $0 < \alpha < 1$,

$$\left(Q^{(\alpha)} \text{ and } P_0\right) \Leftrightarrow P_3^{(\alpha)}.$$

Proof. The proof follows from (2.1) and Lemma 2.2. □

To show $Q^{(\alpha)}$ is quasirandom it is sufficient to show $Q^{(\alpha)} \Leftrightarrow P_0$; this turns out to be true for all $\alpha \in (0, 1)$ except $\alpha = \frac{1}{2}$.

2.3. PROOF OF THE MAIN THEOREM

It suffices to show $Q^{(\alpha)} \Rightarrow P_0$ for $\alpha < \frac{1}{2}$. The proof is more complicated than that of Lemma 2.1. Suppose $Q^{(\alpha)}(\delta)$ holds but $P_0(\epsilon)$ fails, where $\delta < (1 - 2\alpha)\epsilon^2/16$. Let X denote the set of vertices in $V(G)$ with $|\deg(x) - n/2| > \epsilon n$, and suppose $|X| > \epsilon n$. Further, suppose $X' = \{x \in X\colon \deg(x) - n/2 > \epsilon n\}$ has βn vertices with $\beta \geq \epsilon/2$. [The other case that $|\{x \in X\colon \deg(x) - n/2 < \epsilon n\}| \geq \epsilon n/2$ can be dealt with in a similar way and will be omitted.] As in the proof of Lemma 2.1, we have

$$e\left(X', \overline{X'}\right) := p|X'|(n - |X'|) \geq |X'|n/2,$$

where the first equation defines p which implies

$$p \geq \frac{1}{2(1 - \beta)}.$$

We now apply $Q^{(\alpha)}(\delta)$ for two different ways of choosing S:

1. We consider all $S \subseteq V$ containing X'. Let W range over all subsets of $V \setminus X'$ with $|W| = \alpha n - |X'|$. We then have

$$\sum_W e\left(W \cup X', \overline{W \cup X'}\right) \geq e\left(X', \overline{X'}\right)\binom{n - |X'| - 1}{\alpha n - |X'|} + 2e\left(\overline{X'}\right)\binom{n - |X'| - 2}{\alpha n - |X'| - 1}.$$

Thus, there exists a W_0 such that

$$\begin{aligned} e\left(W_0 \cup X', \overline{W_0 \cup X'}\right) &\geq \frac{1}{\binom{n - |X'|}{\alpha n - |X'|}} \sum_W e\left(W \cup X', \overline{W \cup X'}\right) \\ &\geq e\left(X', \overline{X'}\right) \cdot \frac{(1 - \alpha)}{(1 - \beta)} + 2e\left(\overline{X'}\right)\frac{(1 - \alpha)(\alpha - \beta)}{(1 - \beta)^2}. \end{aligned}$$

By our hypothesis, we have

$$\begin{aligned} \tfrac{1}{2}\alpha(1 - \alpha)n^2 + \delta n^2 > e\left(W_0 \cup X', \overline{W_0 \cup X'}\right) \\ \geq (p\beta(1 - \alpha) + \gamma(1 - \alpha)(\alpha - \beta))n^2, \end{aligned}$$

where $e(\overline{X'}) := \gamma\binom{|\overline{X'}|}{2}$.

2. We consider all $S \subseteq V \setminus X'$ with $|S| = \alpha n$. We then have

$$\sum_{S \subseteq V - X'} e(S, \bar{S}) = e\left(X', \overline{X'}\right)\binom{n - |X'| - 1}{\alpha n - 1} + 2e\left(\overline{X'}\right)\binom{n - |X'| - 2}{\alpha n - 1}.$$

Thus, there exists an S_0 such that

$$\begin{aligned} e\left(S_0, \bar{S}_0\right) &\leq \frac{\sum_{S \subset V \setminus X} e(S, \bar{S})}{\binom{n - |X'|}{\alpha n}} \\ &= e\left(X', \overline{X'}\right)\frac{\alpha}{(1 - \beta)} + 2e\left(\overline{X'}\right)\frac{\alpha(1 - \beta - \alpha)}{(1 - \beta)^2}. \end{aligned}$$

From our hypothesis, we obtain

$$\tfrac{1}{2}\alpha(1-\alpha)n^2 - \delta n^2 < e\left(S_0, \bar{S}_0\right) \le (p\beta\alpha + r\alpha(1-\beta-\alpha))n^2.$$

From method 1 we have

$$\gamma < \frac{\frac{1}{2}\alpha(1-\alpha) + \delta - p\beta(1-\alpha)}{(1-\alpha)(\alpha-\beta)},$$

whereas from method 2 we get

$$\gamma > \frac{\frac{1}{2}\alpha(1-\alpha) - \delta - p\beta\alpha}{\alpha(1-\alpha-\beta)}.$$

Together, these imply

$$\begin{aligned}\delta((1-\alpha)(\alpha-\beta) + \alpha(1-\alpha-\beta)) \\ \ge \tfrac{1}{2}\alpha(1-\alpha)((1-\alpha)(\alpha-\beta) - \alpha(1-\beta-\alpha)) \\ + p\beta\alpha(1-\alpha)(1-\alpha-\beta-\alpha+\beta),\end{aligned}$$

which in turn implies

$$\delta(2\alpha(1-\alpha)-\beta) \ge \alpha(1-\alpha)\beta(1-2\alpha)\left(p - \tfrac{1}{2}\right).$$

Since $\beta \ge \epsilon/2$ and $p \ge 1/(2(1-\beta))$, we see that

$$\delta\left(2\alpha(1-\alpha) - \frac{\epsilon}{2}\right) \ge \frac{\alpha(1-\alpha)(1-2\alpha)\epsilon^2}{8(1-\epsilon/2)}.$$

However, this is impossible since we have chosen

$$\delta < (1-2\alpha)\epsilon^2/16.$$

This shows that $Q^{(\alpha)} \Rightarrow P_0$ for $\alpha < \frac{1}{2}$ and the proof of the main theorem is complete. □

2.4. A DIFFERENT PROOF

It may not be immediately obvious why the property $Q^{(\alpha)}$ fails to be quasirandom just for the unique value $\alpha = \frac{1}{2}$. In this section we outline a different proof of this fact that helps to explain the occurrence of this singular value.

To begin with, we define for integers r and t, with $3 \leq r < t/2$, the matrix $M = M_{r,t} = (M(I, e))$ where I ranges over all $\binom{[t]}{r}$, the set of r-element subsets of $[t] := \{1, 2, \ldots, t\}$, e ranges over $\binom{[t]}{2}$, and

$$M(I, e) = \begin{cases} 1, & \text{if } |e \cap I| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We can think of forming a complete graph K_t on $[t]$, and for each complete bipartite graph $K(I, \bar{I})$ (on vertex sets I and $\bar{I} := [t] \setminus I$) and each edge e, letting $M(I, e)$ indicate which e are *edges* of $K(I, \bar{I})$. A related, but somewhat more complicated, matrix $M^* = (M^*(e, I))$ is given by

$$M^*(e, I) = \begin{cases} -(r-1)(r(t-2r) + 2(r-1)), & \text{if } |e \cap I| = 0, \\ (r-1)(t-r-1)(t-2r), & \text{if } |e \cap I| = 1, \\ -(t-r-1)((t-r)(t-2r) \\ \quad -2(t-r-1)), & \text{if } |e \cap I| = 2, \end{cases}$$

where, as in M, $I \in \binom{[t]}{r}$ and $e \in \binom{[t]}{2}$. In particular, M is $\binom{t}{r}$ by $\binom{t}{2}$ and M^* is $\binom{t}{2}$ by $\binom{t}{r}$.

The two matrices M and M^* are related by the following:

Fact.

$$M^*M = 2(t-2r)\frac{(t-2)!}{(r-2)!(t-r-2)!}\,\mathrm{Id}_{\binom{t}{2}},$$

where $\mathrm{Id}_{\binom{t}{2}}$ is the identity matrix of size $\binom{t}{2}$. This follows by direct computation using the definitions of M and M^*. Thus, M^* is a (scalar multiple of a) left inverse of M and it follows, in particular, that M has full rank; that is, rank equal to $\binom{t}{2}$. We remark that for $t = 2r$, the matrix $M_{r,t} = M_{r,2r}$ only has rank $\binom{2r-1}{2}$. This turns out to be the underlying reason for the special behavior of the value $\alpha = \frac{1}{2}$.

Now, consider the property $Q^{(\alpha)}(\epsilon)$ for $\epsilon > 0$, $\alpha < \frac{1}{2}$:

$Q^{(\alpha)}(\epsilon)$: If $S \subset V(G)$ with $|S - \alpha n| < \epsilon n$, then $|e(S, \bar{S}) - \frac{1}{2}\alpha(1 - \alpha)n^2| < \epsilon n^2$, $n > n_0(\epsilon)$, where, as usual, we assume G is a graph on n vertices.

We want to apply $Q^{(\alpha)}(\alpha)$ to $G = G(n)$ in the following way. Let t be large (but fixed) and assume for ease of exposition that $n = tm$ for some integer m. Partition the vertex set V of G into disjoint sets $C_1, C_2, \ldots, C_t$, each of size m, and define

$$\rho_{ij} := \frac{1}{m^2} e(C_i, C_j), \qquad 1 \le i < j \le t.$$

We can associate with this construction a *weighted* complete graph K_t on $[t]$ with the edge $e = \{i, j\}$ of K_t receiving the weight $\rho(e) = \rho_{ij}$.

We now fix r with $3 \le r < t/2$, so that $\beta := r/t$ is close to α (we will be more precise later). We will actually first apply $Q^{(\beta)}(\epsilon)$ to G. This then implies that for each $I \subset \binom{[t]}{r}$, if we form $S = \bigcup_{i \in I} C_i$, then the number $e(S, \bar{S}) = c(I)$ of crossing edges, which is just

$$c(I) = \sum_{\substack{i \in I \\ j \in \bar{I}}} e(C_i, C_j) = m^2 \sum_{\substack{i \in I \\ j \in \bar{I}}} \rho_{ij},$$

satisfies

$$m^2 M \bar{\rho} = \bar{c}, \tag{2.2}$$

Bwhere $\bar{\rho} = [\rho(e)]_{e \in \binom{[t]}{2}}$ and $\bar{c} = [c(I)]_{I \in \binom{[t]}{r}}$ are column vectors. By $Q^{(\beta)}(\epsilon)$, we know

$$c(I) = \left(\tfrac{1}{2}\beta(1 - \beta) + \epsilon(I)\right)n^2, \tag{2.3}$$

where $|\epsilon(I)| < \epsilon$, $I \in \binom{[t]}{r}$.

Now, we invert (2.2) by left-multiplying by M^* to get

$$m^2 M^* M \bar{\rho} = 2(t - 2r)\frac{(t-2)!}{(r-2)!(t-r-2)!}\bar{\rho} = M^* \bar{c}. \tag{2.4}$$

However, direct computation shows that

$$M^*\bar{1} = \frac{2(t-2r)(t-2)!}{r(t-r)(r-2)!(t-r-2)!}\bar{1}, \tag{2.5}$$

where $\bar{1}$ denotes a column vector of all 1s. Thus, we obtain from (2.3), (2.4), and (2.5),

$$\left|\rho(e) - \tfrac{1}{2}\beta(1-\beta)n^2\right| < \epsilon n^2 \tag{2.6}$$

for each $e \in \binom{[t]}{2}$ and $n > n_0(\epsilon)$. This means that all the "edge densities" ρ_{ij} between the various clusters C_i and C_j in G are very close to what is expected. Of course, to apply $Q^{(\alpha)}$ rather than $Q^{(\beta)}$, we choose a sufficiently close rational approximation $\beta = r/t$ to α. It then finally follows that any $n/2$ points of G span $\frac{1}{8}\alpha(1-\alpha)n^2 + o(n^2)$ edges, which in turn implies quasirandomness [see Chung, Graham, and Wilson (1991)]. This argument works for $\alpha \neq \frac{1}{2}$ and fails for $\alpha = \frac{1}{2}$ precisely because the matrix $M_{r,t}$ has full rank $\binom{t}{2}$ for $2 \le r \le t-2$, $r \neq t/2$, but only has rank $\binom{t-1}{2}$ when $r = t/2$ (which corresponds to $\alpha = \frac{1}{2}$).

2.5. CONCLUDING REMARKS

In a recent paper, Chung, Graham, and Wilson (1991) considered various quasirandom properties for tournaments T. Recall that a tournament $T = (N, A)$ is given by a set N of *nodes* together with a set A of ordered pairs of nodes, called *arcs*, such that for all $u, v \in N$, $u \neq v$, either $(u, v) \in A$ or $(v, u) \in A$ (and not both).

One property considered in Chung and Graham (1991a) is the following (where we take $N = [n]$):

$X(\alpha)$: For fixed α, $0 < \alpha < 1$, if $S \subset N$ with $|S| = (1 + o(1))\alpha n$, then

$$\sum_{x \in S} \left| \left|\{u \in \bar{S} \mid (x,u) \in A\}\right| - \left|\{v \in \bar{S} \mid (v,x) \in A\}\right| \right| = o(n^2).$$

In other words, almost all nodes of S have about $\frac{1}{2}|\bar{S}|$ arcs going to $\bar{S}$ and about $\frac{1}{2}|\bar{S}|$ arcs coming from $\bar{S}$.

It turns out that for *all* α, $0 < \alpha < 1$, $X(\alpha)$ is a quasirandom property for tournaments. This holds, in spite of the following result relating quasirandom graphs and quasirandom tournaments. For a tournament $T = T(n) = (N, A)$, let π: $N \to [n]$ be an arbitrary ordering of N and let $G = T_\pi^+ = (N, E)$ be the ("increasing arc") graph on N given by

$$e = \{i, j\} \in E \quad \text{iff } \pi(i) < \pi(j) \text{ and } (i, j) \in A.$$

Theorem [Chung and Graham (1991b)]. T is a quasirandom tournament iff T_π^+ is a quasirandom graph.

We conclude by noting that a number of results have recently been proved that focus on the equivalence of various "random-like" properties of graphs and hypergraphs. Some of these are included in the papers of Rödl (1986), Thomason (1987a, b), Frankl, Rödl, and Wilson (1988), Haviland and Thomason (1991), Spencer and Tetali (1991), Graham and Spencer (1971), Bollobás and Thomason (1981), and Chung and Graham (1990, 1991a). We believe that this work has only scratched the surface of this fascinating topic, and that many more such results will be forthcoming in the near future.

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