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BOUNDS FOR ARRAYS OF DOTS WITH DISTINCT SLOPES OR LENGTHS

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An $n \times m$ sonar sequence is a subset of the $n \times m$ grid with exactly one point in each column, such that the $\binom{m}{2}$ vectors determined by them are all distinct. We show that for fixed n the maximal m for which a sonar sequence exists satisfies $n - Cn^{11/20} < m < n + 4n^{2/3}$ for all n and $m > n + c \log n \log\log n$ for infinitely many n.

Another problem concerns the maximal number D of points that can be selected from the $n \times m$ grid so that all the $\binom{D}{2}$ vectors have slopes. We prove $n^{1/2} \ll D \ll n^{4/5}$

An $n \times m$ sonar sequence [3] is an array of dots and blanks having n rows and exactly one dot in each of its m columns, subject to the requirement that distinct pairs of dots determine distinct vectors. Any two such vectors must differ in *slope* or in *length*. Whenever p is prime, an example of a $p \times p$ sonar sequence $a_1, a_2, \ldots, a_p$ is given by letting $a_i \equiv i^2 \pmod{p}$, and choosing $1 \le a_i \le p$, so that a_i gives the row coordinate of the dot in the i^{th} column.

Recalling the size of gaps between primes this example shows that in trying to maximize m, it is always possible to achieve $m > n - n^{11/20}$. By Theorem 4, which uses the method of [2], an upper bound for large n is $m < n + 4n^{2/3}$. Theorem 5 uses a new result in [5] to say that for infinitely many n, there exists an $n \times m$ sonar sequence with $m > n + c \log n \log\log\log n$.

Now consider an $n \times n$ array having D dots. Robert Peile raised the question of maximizing D when the vectors determined by different pairs of dots are required to differ in slope. Theorem 1 tells us that $D \le 5n^{4/5}$ for large n. Theorem 2 by algebraic construction shows that $D > (1/2 + o(1))n^{1/2}$. In contrast, random choice in Theorem 3 guarantees $D > \frac{3}{5}n^{1/2}$ for large n.

For comparison let K be the number of dots in $n \times n$ array in which the vectors determined by different pairs of dots differ in length. As proved in [4] an upper bound for K is

$$K < cn(\log n)^{-1/4},$$

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and a lower bound is

$$K > n^{(2-\varepsilon)/3}$$

for any $\varepsilon > 0$ and sufficiently large n.

Theorem 1. *With distinct slopes, an upper bound for the number of dots is $O(n^{4/5})$.*

Proof. Let us consider a set of D points in $n \times n$ array such that all the slopes are distinct. Choose an integer m and write

$$A_d = \{(id, jd) : \ 0 \le i, \ j \le m - 1\},$$

a set of m^2 points. We shall use d as a variable. We cover the $n \times n$ square by translated copies of A_d. Since d^2 copies cover a square of size $dm \times dm$, then

$$d^2 \left(1 + \frac{n}{dm}\right)^2 \le 4n^2/m^2$$

copies cover the whole square if $dm \le n$, that is,

$$d \le [n/m]. \tag{1}$$

If t denotes the number of copies used and $w_1, \dots, w_t$ are the number of points in each, then we have $\sum w_i = D$, hence

$$P = \sum \binom{w_i}{2} \ge \frac{D(D-t)}{2t} \ge D^2/(4t) \ge \frac{1}{16}\frac{D^2m^2}{n^2}$$

if $t \le D/2$, that is, if

$$m^2 \ge 8n^2/D. \tag{2}$$

Each of the P pairs determines a vector whose coordinates are divisible by d. Divide each vector by d; we get a vector with coordinates $< m$, altogether m^2 possibilities, and no two such vectors coincide, because that would mean two parallel vectors in the original system.

We have found $\frac{D^2m^2}{16n^2}$ vectors for every d, and d can run up to $[n/m]$, so we have

$$\frac{D^2m^2}{16n^2}\left[\frac{n}{m}\right] \le m^2,$$

that is,

$$D^2[n/m] \le 16n^2. \tag{3}$$

To optimize this, we select the maximal m allowed by (2):

$$m = 1 + \left[\sqrt{8n^2/D}\right], \quad [n/m] \ge \sqrt{D}/3$$

for large n. (3) yields $D^{5/2} \le 48n^2$, hence

$$D \le 5n^{4/5}.$$

∎

Theorem 2. *It is possible to select $(1/2 + o(1))n^{1/2}$ grid points from an $n \times n$ grid so that all slopes determined by pairs of points are distinct.*

Proof. Let the grid consist of the integer points (x, y), $0 \le x,\ y < n$. Let q be the largest prime power so that $m := q^2 + q + 1 < n$, and let p be the largest prime less than n. The prime number theorem guarantees that $q \sim \sqrt{n}$, $p \sim n$. By a classical result of Singer [1], there exists a perfect difference set $A = \{a_1, a_2, \ldots, a_{q+1}\}$ of residues modulo m, i.e., all the $q^2 + q$ differences $a_i - a_j$, $i \ne j$, are distinct modulo m. Of course, if we translate A by forming $A + d = \{a_i + d \pmod{m} | a_i \in A\}$, then $A + d$ is also a perfect difference set. As d runs from 1 to m the average of $|\{A + d\} \cap [0, p/2)|$ is exactly $(p + 1)(q + 1)/2m$. Therefore for some d at least $p(q + 1)/2m$ of the elements of $A + d$ lie in the interval $[0, p/2)$. Call the set of these elements $B = \{b_1, \ldots, b_t\}$. Thus, $0 \le b_1 < \cdots < b_t < p/2$ where $t \ge \frac{p(q+1)}{2m}$. Since all $b_i - b_j$, $i \ne j$, are distinct (mod m), then all sums $b_i + b_j$, $i \le j$, are distinct (mod m), and consequently distinct as integers. Furthermore, since all $b_i < p/2$ then all sums $b_i + b_j$, $i \le j$, are distinct (mod p). Let $b_i^2 \equiv c_i \pmod{p}$, $1 \le i \le t$. Thus, $b_i^2 = c_i + k_i p$ for integers k_i, where $0 \le c_i < p$.

Now, for our set S of grid points we take

$$S = \{(b_i, c_i) | 1 \le i \le t\}.$$

(Essentially, we have wrapped the parabola $y = x^2$ around a $p \times p$ torus and selected t points (b_i, b_i^2) on it.) To check that S has the distinct slope property, we calculate for $i < j$,

$$\begin{aligned} \frac{c_i - c_j}{b_i - b_j} &= \frac{b_i^2 - k_i p - (b_j^2 - k_j p)}{b_i - b_j} \\ &= \frac{b_i^2 - b_j^2}{b_i - b_j} - p\frac{(k_i - k_j)}{b_i - b_j} \\ &\equiv b_i + b_j \pmod{p} \end{aligned}$$

since p is prime and $b_i - b_j \not\equiv 0 \pmod{p}$. Since by construction all the $b_i + b_j$, $i < j$, are distinct, the S has the distinct slope property. Finally, since $t \ge \frac{p(q+1)}{2m} = (1/2 + o(1))q = (1/2 + o(1))\sqrt{n}$, then we are done. ∎

It may be worth noting, in connection with Theorem 2, that if there are infinitely many primes q such that $q^2 + q + 1$ is a prime (everybody believes this to be so) then there are infinitely many n for which it is possible to select $(1 + o(1))n^{1/2}$ points of an $n \times n$ grid so that different pairs of points determine different slopes.

It would be interesting to know if we could actually get sets of size $n^{1/2+\varepsilon}$ for a fixed $\varepsilon > 0$.

Theorem 3. *For large n suppose an $n \times n$ array has D dots with all slopes distinct. If there are no points at which a new dot can be placed without causing a repeated slope, then $D > \frac{3}{5}n^{1/2}$.*

Proof. We will show that if $D \le \frac{3}{5}n^{1/2}$, then some of the n^2 points will remain not excluded.

Each of the $\binom{D}{2}$ distinct slopes is determined by a coprime pair of integers $\{a, i\}$, and we may suppose $a \leq i$, giving slopes $\frac{a}{i}, \frac{i}{a}, \frac{-a}{i}, \frac{-i}{a}$. A line having one of these slopes, going through one of the D dots, will hit at most $\frac{n}{i}$ other points of the $n \times n$ array. Clearly smaller values of i produce more hits. For each i the number of coprime pairs with $a \leq i$ is $\phi(i)$, therefore the number of hits per dot will be less than $\sum_{i=1}^{x} 4 \cdot \phi(i) \cdot \frac{n}{i}$ provided we choose x large enough to make $\sum_{i=1}^{x} 4\phi(i) > \binom{D}{2}$.

To choose x we rely on the fact from [6] that $\sum_{i=1}^{x} \phi(i) > \frac{3}{10}x^2$ for large x. Accordingly, let $x = \left\lceil \left(\frac{5}{12}\right)^{1/2} D \right\rceil$, and observe that

$$\sum_{i=1}^{x} 4\phi(i) > \frac{12}{10} \cdot \frac{5}{12} D^2 = \frac{D^2}{2} > \binom{D}{2}.$$

Finally, using $D \leq \frac{3}{5}n^{1/2}$, $x < \frac{13}{20}D$, and $\sum_{i=1}^{x} \frac{\phi(i)}{i} < x$, we obtain:

$$D \cdot 4n \sum_{i=1}^{x} \frac{\phi(i)}{i} < \frac{3}{5}n^{1/2} \cdot 4n \cdot \frac{13}{20} \cdot \frac{3}{5}n^{1/2} = \frac{117}{125}n^2.$$

Thus, counting dots plus hits, the total number of points excluded is less than $\frac{3}{5}n^{1/2} + \frac{117}{125}n^2$, which is less than n^2 for large n. ∎

Theorem 4. *If an $n \times m$ sonar sequence exists then $m < n + 5n^{2/3}$.*

Proof. Consider an $n \times m$ sonar sequence. The array of dots and blanks has n rows and m columns, with one dot per column.

Let copies of an $R \times R$ window be translated so that each dot or blank sits in R^2 windows. Thus the number of windows used will be $W = (n + R - 1)(m + R - 1)$. The average number of dots per window is $A = \frac{R^2 m}{W}$.

If the i^{th} window has w_i dots, then the number of occurrences of a pair of dots in a window is

$$\sum_{i=1}^{W} \frac{w_i(w_i - 1)}{2}.$$

Since A is the average,

$$\frac{WA(A-1)}{2} \leq \sum_{i=1}^{W} \frac{w_i(w_i - 1)}{2}.$$

Because pairs of dots determine distinct vectors, we can get an upper bound on the actual number of occurrences of a pair of dots in a window by counting all the possible patterns of two dots in a window, as follows. There are R^2 places to put the first dot, then, allowing only one per column, there are $R(R-1)$ places to put the second dot. That counts each twice, so the number of possible patterns of two dots in a window is $\frac{R^2 R(R-1)}{2}$.

Putting these restrictions together we have:

$$\frac{WA(A-1)}{2} \leq \sum_{i=1}^{W} \frac{w_i(w_i-1)}{2} \leq \frac{R^2(R^2-R)}{2},$$

$$m(A-1) \leq R^2 - R,$$

$$\frac{mR^2m}{W} \leq R^2 + m - R,$$

$$m \leq \frac{W}{m} + \frac{W}{R^2} - \frac{W}{Rm}.$$

Expanding this we choose the integer R such that $n^{2/3} \leq R < n^{2/3} + 1$, so that $W < nm + n^{2/3}m + n^{5/3} + n^{4/3}$, and $R^2 \geq n^{4/3}$. From these we can see that $m < n + n^{2/3} + \frac{n^{5/3}+n^{4/3}}{m} + \frac{m}{n^{1/3}} + \frac{m}{n^{2/3}} + n^{1/3} + 1$. Finally because $n < m < 2n$ we have $m < n + 4n^{2/3} + 4n^{1/3} + 1$. Thus the proof is complete that for large n, $m < n + 5n^{2/3}$. ∎

Comment. More careful computation shows that actually $m < n + 3n^{2/3} + 2n^{1/3} + 9$ for all n.

Theorem 5. *For some constant $c > 0$ there are infinitely many integers n such that an $n \times m$ sonar sequence exists with $n > n + c \log n \log\log\log n$.*

Proof. In [5] S. W. Graham and C. J. Ringrose prove that, for infinitely many primes p, the least quadratic non-residue between 1 and p is larger than $L = c \log p \log\log\log p$.

If p is any odd prime, a $p \times p$ sonar sequence $a_1, a_2, \ldots, a_p$ can be obtained by letting $a_i \equiv bi^2 \pmod p$ where $b \not\equiv 0 \pmod p$, and choosing $1 \leq a_i \leq p$. To verify that distinct pairs of dots determine distinct vectors notice modulo p that $a_{i+k} - a_i \equiv a_{j+k} - a_j$ only if $2ik \equiv 2jk$, which happens only when $k \equiv 0$ or $i \equiv j$.

When p is such that all the numbers between 1 and L are quadratic residues, and b is a non-residue, we find $L < a_i \leq p$ for each i from 1 to p. Thus with $m = p$, and $n < p - L$, this will be an $n \times m$ sonar sequence with $m > n + c \log n \log\log\log n$. ∎

Open Problem Does an $n \times n$ array with n dots exist for every n, in which distinct pairs of dots determine vectors which differ in slope or in length?

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References

[1] J. Singer: A Theorem in Finite Projective Geometry and Some Applications to Number Theory, *Trans. Amer. Math. Soc.*, **43** (1938), 377–385.

[2] P. Erdős, and P. Turán: On a Problem of Sidon in Additive Number Theory and Some Related Problems, *J. London Math. Soc.*, **16** (1941), 212–215.

[3] S. W. Golomb, and H. Taylor: Two-dimensional Synchronization Patterns for Minimum Ambiguity, *IEEE Trans. Inform. Theory* **IT-28** (1982), 600–604.

[4] P. ERDŐS, and R. K. GUY: Distinct Distances Between Lattice Points, *Elemente Der Mathematik*, **25**, (1970), 121–123.

[5] S. W. GRAHAM, and C. J. RINGROSE: Lower Bounds for Least Quadratic Non-residues, to appear in *Number Theory at Allerton Park*, Proceedings of a conference in honor of Paul T. Bateman, Allerton Park, 1989, Birkhauser Verlag, Boston.

[6] G. H. HARDY, and E. M. WRIGHT: *An Introduction to the Theory of Numbers*, 3rd Edition, 1954, Oxford.

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