Quasi-Random Tournaments

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ABSTRACT

We introduce a large class of tournament properties, all of which are shared by almost all random tournaments. These properties, which we term "quasi-random," have the property that tournaments possessing any one of the properties must of necessity possess them all. In contrast to random tournaments, however, it is often very easy to verify that a particular family of tournaments satisfies one of the quasi-random properties, thereby giving explicit tournaments with "random-like" behavior. This paper continues an approach initiated in several earlier papers of the authors where analogous results for graphs (with R. M. Wilson) and hypergraphs are proved.

1. INTRODUCTION

A tournament T is a directed graph in which between any two of its nodes v and v', exactly *one* of the directed edges (or *arcs*) (v, v') or (v', v) occurs. Tournaments form perhaps the most widely studied class of directed graphs, and much is known about them (e.g., see [16], [17]). Their name arises from the interpretation of T as representing the outcome of a competition between all pairs of a set of players, with an arc (v, v') indicating that v defeated v'. In trying to understand which properties holds for a "typical" tournament, it has been found useful to introduce the concept of a "random" tournament $T_{1/2}(n)$ on n vertices (e.g., see [2] and [13]). In such a tournament, arcs are chosen independently by flipping a fair coin for each pair $\{v, v'\}$ of vertices to decide which of (v, v') or (v', v) will be an arc (each possibility occurring with probability 1/2). More precisely, this process in-

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duces a probability measure on the space $\mathcal{T}(n)$ of all possible tournaments on n vertices, with each tournament T in $\mathcal{T}(n)$ having probability $2^{-\binom{n}{2}}$. It turns out that there are many properties P that are possessed by the overwhelming majority of tournaments in $\mathcal{T}(n)$ as n becomes large. We can state this more precisely by writing

$$Pr\{T \in \mathcal{T}(n) \mid T \text{ satisfies P}\} \longrightarrow 1 \quad \text{as} \quad n \longrightarrow \infty .$$

We abbreviate this by saying that $T_{1/2}(n)$ almost always has property P. For example, all but $o(n)$ nodes v of $T_{1/2}(n)$ have indegree $(v) := |\{u|(u,v) \text{ is an arc of } T_{1/2}(n)\}|$ and outdegree $(v) := |\{u|(v,u) \text{ is an arc of } T_{1/2}(n)\}|$ that satisfy $|\text{indegree } (v) - \text{outdegree } (v)| = o(n)$.

The main thrust of this paper will be to establish the *equivalence* of a variety of tournament properties, all of which are possessed by almost all $T_{1/2}(n)$, in the following sense: Any family of tournaments satisfying *any one* of the properties must of necessity satisfy *all* the others. We term such properties *quasi-random*. We follow in much the same spirit as in the recent papers [5,6,7,9,18,20,21], in which many properties of quasi-random graphs and hypergraphs, pseudo-random graphs and hypergraphs, and "(p,α)-jumbled" graphs are given.

2. NOTATION AND PRELIMINARIES

A tournament $T = (N, A)$ will consist of a set $N = N(T)$, called the *nodes* of T, and a set $A = A(T)$ of the ordered pairs from N, called the arcs of T. For any two distinct nodes u and v of T, exactly one of the two pairs (u,v) and (v,u) is an arc of T. We use the notation $T(n)$ to indicate the fact that T has n nodes. For $X \subset N$, we let $T[X]$ denote the subtournament of T induced by X, i.e., $T[X] = (X, A(T) \cap X^2)$ where X^2 denotes the set of pairs (x,x') for $x,x' \in X$. We let $\chi_T : N^2 \to \{-1,1\}$ denote the *arc indicator* of T, i.e., for $u,v \in N, u \neq v$,

$$\chi_T(u,v) = \begin{cases} 1, & \text{if } (u,v) \in A; \\ -1, & \text{if } (u,v) \notin A. \end{cases}$$

Define $nd^-(v)$ for a node v of T to be $\{u \mid (u,v) \in A\}$; similarly, define $nd^+(v)$ to be $\{u \mid (v,u) \in A\}$. Further, the *indegree* $d^-(v)$ and *outdegree* $d^+(v)$ of v are defined by

$$d^-(v) := |nd^-(v)|, \qquad d^+(v) := |nd^+(v)|.$$

For $v \in N, X \subset N$, we let $d^-(v,X) := |nd^-(v) \cap X|$, and $d^+(v,X) := |nd^+(v) \cap X|$. Also, for $X, X' \subset N$, define

$$d^-(X,X') := \sum_{v\in X} d^-(v,X'), \qquad d^+(X,X') := \sum_{v\in X} d^+(v,X').$$

An *ordering* of $T = (N,A)$ is a 1-to-1 mapping $\pi: N \to [n] := \{1,2,\ldots,n\}$. An arc (u,v) is said to be *π-increasing* if $\pi(u) < \pi(v)$; otherwise we say that (u,v) is *π-decreasing*. The undirected graph T_π^+ on N is formed by creating for each *π-increasing* arc (u,v) of T (under the ordering π) an (undirected) edge $\{u,v\}$ of T_π^+.

For two nodes $u,v \in N$, the *sameness* set $S(u,v)$ is defined by

$$S(u,v) := \{z \in N \mid \chi_T(u,z) = \chi_T(v,z)\},$$

and we let $s(u,v)$ denote $|S(u,v)|$. Also, let $\bar{S}(u,v) := N \backslash S(u,v)$ and $\bar{s}(u,v) := |\bar{S}(u,v)|$. Thus, $s(u,v) + \bar{s}(u,v) = n$.

If $T' = (N',A')$ is a given tournament (or more generally, a directed graph), we let $N_T^*(T')$ denote the number of labeled occurrences of T' as a subtournament (or subdigraph) of T. In other words,

$$N_T^*(T') := |\{\lambda: N' \longrightarrow N \mid T[\lambda(N')] \cong T'\}|$$

where "$\cong$" denotes the obvious tournament isomorphism.

Finally, we define a structure that will be needed in our discussion. We will call a sequence (v_0,v_1,v_2,v_3) an *even 4-cycle* (denoted by E4C) if

$$\chi_T(v_0,v_1)\chi_T(v_1,v_2)\chi_T(v_2,v_3)\chi_T(v_3,v_0) = 1. \tag{1}$$

We will let $N_T^*(\text{E4C})$ denote the number of (labeled) E4Cs in T.

3. STATEMENTS OF THE MAIN RESULTS

We next consider a set of properties that a tournament $T = T(n)$ might satisfy. Each of the properties will contain occurrences of the asymptotic "little-oh" notation $o(\ )$. However, the dependence of different $o(\ )$'s on the particular property they refer to will ordinarily be suppressed.

Suppose we have two properties P and P', each with occurrences of $o(1)$, so that $P = P(o(1))$, $P' = P'(o(1))$. The implication "$P \Rightarrow P'$" then means that for each $\varepsilon > 0$ there is a $\delta > 0$ so that if $T(n)$ satisfies $P(\delta)$ then it must also satisfy $P'(\varepsilon)$, provided $n > n_0(\varepsilon)$.

It is also possible to consider our various properties as applying to a *family* $\{T(n) \mid n \to \infty\}$ of tournaments. In this case, a condition containing $o(1)$ has the usual meaning as $n \to \infty$. These two interpretations are clearly equivalent, however.

We now list a set of properties for tournaments $T = T(n) = (N, A)$ that are shared by almost all random tournaments $T_{1,2}(n)$.

$P_1(s)$: For all tournaments $T'(s)$ on s nodes,

$$N_T^*(T'(s)) = (1 + o(1))n^s 2^{-\binom{s}{2}}$$

The content of $P_1(s)$ is that all of the $2^{\binom{s}{2}}$ labeled tournaments on s nodes occur asymptotically equally often in T.

P_2: $N_T^*(\text{E4C}) = (1 + o(1))\,(n^4/2)$.
P_3: $\Sigma_{u,v \in N}|s(u,v) - (n/2)| = o(n^3)$.
P_4: $\Sigma_{u,v \in N}||\{w \in N | \chi_T(u,w) = 1 = \chi_T(v,w)\}| - (n/4)| = o(n^3)$.
P_5: For all $X \subset N$, $T' = T[X]$ satisfies

$$\sum_{v \in X} |d_{T'}^+(v) - d_{T'}^-(v)| = o(n^2)\,.$$

In this case we say that T' is *almost balanced.*

P_6: Every subtournament T' of T on $\lfloor n/2 \rfloor$ nodes is almost balanced.
P_7: For every partition of $N = X \cup Y$ with $|X| = \lfloor n/2 \rfloor$, $|Y| = \lceil n/2 \rceil$, we have

$$\sum_{v \in X} |d^+(v,Y) - d^-(v,Y)| = o(n^2)\,.$$

P_8: For all $X, Y \subset N$,

$$\sum_{v \in X} |d^+(v,Y) - d^-(v,Y)| = o(n^2)\,.$$

P_9: For every ordering π of T,

$$|\{(u,v) \in A \,|\, \pi(u) < \pi(v)\}| = (1 + o(1))\frac{n^2}{4}\,.$$

That is, in any ordering of T, asymptotically one half of the arcs are increasing. In an earlier paper [9], the authors with R. M. Wilson have introduced an equivalence class of quasi-random properties for *graphs.* Among these are the graph analogues of $P_1(s)$, P_3, P_4, P_7, and many others. The following properties connect the two classes.

P_{10}: For *every* ordering π, the (undirected) graph T_π^+ is quasi-random.
P_{11}: There exists *some* ordering π so that the graph T_π^+ is quasi-random.

Our main goal will be to show that all these properties are in fact *equivalent.* We will call these (and any other equivalent) properties *quasi-random.*

Occasionally, we will abuse this notation and also refer to tournaments satisfying any one (and therefore all) of these properties as being quasi-random as well.

Theorem 1. For $s \geq 4$,

$$P_1(s) \Rightarrow P_2 \Rightarrow P_3 \Rightarrow P_4 \Rightarrow P_5 \Rightarrow P_6 \Rightarrow P_7 \Rightarrow P_8 \Rightarrow P_9 \Rightarrow P_{10} \Rightarrow P_{11} \Rightarrow P_1(s).$$

Before giving a proof of Theorem 1 (which we do in Section 5), we first will describe a simpler equivalence class of properties Q_i that are all implied by quasi-randomness (and consequently shared by almost all $T_{1/2}(n)$) but that are strictly weaker. The proof of their equivalence is a useful warm-up to the techniques needed for Theorem 1.

Q_1: For every tournament $T' = T'(3)$ on 3 nodes,

$$N_T^*(T') = (1 + o(1))\frac{n^3}{8}.$$

Q_2: For the "cyclic" tournament $C_3 = (N, A)$ with $N = \{1, 2, 3\}$ and $A = \{(12), (23), (31)\}$,

$$N_T^*(C_3) \geq (1 + o(1))\frac{n^3}{8}.$$

Q_3: T is almost balanced, i.e.,

$$\sum_{v \in N} |d^+(v) - d^-(v)| = o(n^2).$$

Q_4: For every partition of $N = X \cup Y$,

$$d^+(X, Y) - d^-(X, Y) = o(n^2).$$

Q_5: For every partition of $N = X \cup Y$ with $|X| = \lfloor n/2 \rfloor$, $|Y| = \lceil n/2 \rceil$, we have

$$d^+(X, Y) - d^-(X, Y) = o(n^2).$$

Theorem 2. $Q_1 \Rightarrow Q_2 \Rightarrow Q_3 \Rightarrow Q_4 \Rightarrow Q_5 \Rightarrow Q_1$.

Corollary. $P_i \Rightarrow Q_j$, for $1 \leq i \leq 11$, $1 \leq j \leq 5$.

The following tournament T^* provides an example that satisfies all of the Q_j but none of the P_i:

Example. $T^* = T^*(n) = (N^*, A^*)$, $N^* = X \cup Y \cup Z$, $|X| = |Y| = |Z| = n/3$. Each of the subtournaments $T^*[X]$, $T^*[Y]$, $T^*[Z]$ will be random, say, all equal to some $T_{1/2}(n/3)$. The remaining arcs of T^* are just all pairs $X \times Y$, $Y \times Z$, $Z \times X$ (see Figure 1).

It is easily checked that for almost all choices of $T_{1/2}(n/3)$, T^* satisfies Q_j but not P_i.

4. PROOF OF THEOREM 2

Our proof will follow the outline given in Figure 2.

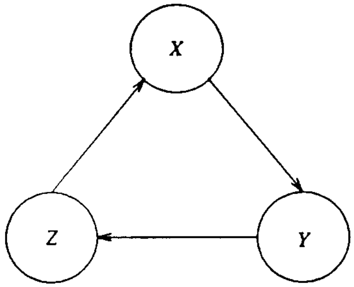
The following observation will be useful:

Fact 1. For any tournament $T = T(n)$,

$$N_T^*(C_3) \le \frac{1}{8} n(n^2 - 1). \tag{2}$$

Proof. Among the eight possible tournaments on three nodes, two are cyclic (C_3 and its complement $\bar{C}_3$), and six are acyclic (denoted by $A(3)$). Each acyclic tournament $A_3 \in A(3)$ contains a unique pair of nodes (z^+, z^-) with outdegree$(z^+) = 2$ and indegree$(z^-) = 2$. Thus, the total number $N_T^*(A(3))$ of acyclic subtournaments of T satisfies

$$\begin{aligned} N_T^*(A(3)) &= 6 \sum_{v \in N} \frac{1}{2} \left\{ \binom{d^+(v)}{2} + \binom{d^-(v)}{2} \right\} \\ &\ge 6n \binom{(n-1)/2}{2} = \frac{3}{4} n(n-1)(n-3) \end{aligned} \tag{3}$$



T^*

FIGURE 1

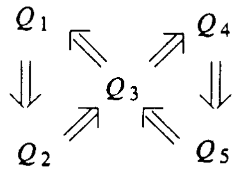


FIGURE 2

since $d^+(v) + d^-(v) = n - 1$ and $\binom{x}{2}$ is convex, and there are $3 \cdot 2 = 6$ ways to order z^+ and z^- in A_3.

Finally, since $N_T^*(C_3) = N_T^*(\overline{C}_3)$ and $N_T^*(T(3)) = n(n-1)(n-2)$, then

$$N_T^*(C_3) \le \frac{1}{2}\left\{n(n-1)(n-2) - \frac{3}{4}n(n-1)(n-3)\right\} = \frac{1}{8}n(n^2-1)$$

as claimed. ▮

Fact 2. $Q_2 \Rightarrow Q_3$.

Proof. Suppose

$$N_T^*(C_3) = (1 + o(1))\frac{n^3}{8}. \tag{4}$$

It follows directly from (3) that if

$$N_T^*(A_3) = \left(\frac{3}{4} + o(1)\right)n^3, \tag{5}$$

then all but $o(n)$ nodes v must satisfy

$$d^+(v) = (1 + o(1))n/2, \qquad d^-(v) = (1 + o(1))n/2, \tag{6}$$

i.e., T is almost balanced. However, since (4) implies (5), then we are done. ▮

Fact 3. $Q_3 \Rightarrow Q_1$.

Proof. Clearly, Q_3 implies all but $o(n)$ nodes v of T must satisfy (6). By (3),

$$\begin{aligned} N_3^*(A_3) &= \frac{1}{2}\sum_{v \in N}\left\{\binom{d^+(v)}{2} + \binom{d^-(v)}{2}\right\} \\ &= (1 + o(1))\frac{n^3}{8}. \end{aligned}$$

This implies $N_3^*(C_3) = (1 + o(1))(n^3/8)$, which in turn implies Q_1. ▮

Fact 4. $Q_3 \Rightarrow Q_4$. Suppose T is almost balanced. Thus, for all but ϵn nodes v we have

$$|d^+(v) - d^-(v)| < \epsilon n .$$

Consider a partition $N = X \cup Y$. Since $d^+(X, X) = d^-(X, X)$, then

$$\begin{aligned} d^+(X,Y) - d^-(X,Y) &= d^+(X,X) + d^+(X,Y) - d^-(X,X) - d^-(X,Y) \\ &= d^+(X,N) - d^-(X,N) \\ &= \sum_{v\in X} (d^+(v,N) - d^-(v,N)) \\ &\le \epsilon n \cdot n + |X| \cdot \epsilon n \qquad \text{by hypothesis} \\ &\le 2\epsilon n^2 . \end{aligned}$$

as required. ∎

Fact 5. $Q_5 \Rightarrow Q_3$. Suppose every partition $N = X \cup Y$ with $|X| = n/2$, $|Y| = n/2$, satisfies

$$d^+(X,Y) - d^-(X,Y) = o(n^2) . \tag{7}$$

(Strictly speaking, we should take $|X| = \lfloor n/2 \rfloor$, $|Y| = \lceil n/2 \rceil$. However, for ease of exposition we shall frequently write $n/2$ when in fact $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$, or even $(n/2) + O(1)$, is the true value. The reader should have no trouble understanding what the actual values should be.) We must show T is almost balanced. Suppose not, i.e., suppose T contains a set $W \subset N$ of $|W| = w = \epsilon n$ nodes satisfying

$$d^+(v) - \frac{n}{2} > \epsilon n, \qquad v \in W. \tag{8}$$

(It is not hard to see that such a set must exist.)

We next want to look at the average behavior of $\Sigma_{v\in X\cup W}(d^+(v) - d^-(v))$ as X ranges over all subsets of $\overline{W} = N\backslash W$ of size $(n/2) - w$. Of course some $X' \subset \overline{W}$ must achieve a value at least as large as the average. Thus, since there are $\binom{n-w}{n/2-w}$ ways to select such X then for some $X' \subset \overline{W}$ of size $(n/2) - w$,

$$d^+(X' \cup W, \overline{X' \cup W}) - d^-(X' \cup W, \overline{X' \cup W})$$

$$= \sum_{v\in X'\cup W} (d^+(v) - d^-(v))$$

$$\geq \frac{1}{\binom{n-w}{n/2-w}} \sum_{\substack{X \subset \overline{W} \\ |X|=(n/2)-w}} \sum_{v \in X \cup W} (d^+(v) - d^-(v))$$

$$\geq \frac{1}{\binom{n-w}{n/2-w}} \binom{n-w-1}{n/2-w-1} (d^+(W,\overline{W}) - d^-(W,\overline{W}))$$

$$= \frac{(n/2)-w}{n-w}(d^+(W,N) - d^-(W,N))$$

$$\geq \frac{(1/2)-\epsilon}{1-\epsilon} \cdot |W| \cdot 2\epsilon n \quad \text{by (8)}$$

$$\geq \tfrac{1}{2}\epsilon^2 n^2 \quad \text{for} \quad 0 < \epsilon \leq \tfrac{1}{3}.$$

However, this contradicts the hypothesis (7), so we conclude T must be almost balanced. ∎

Since the implications $Q_1 \Rightarrow Q_2$ and $Q_4 \Rightarrow Q_5$ are immediate, then Theorem 2 follows from Facts 1–5.

Since it is clear that $P_1(t+1) \Rightarrow P_1(t)$ for any $t \geq 1$ and Q_1 is just $P_1(3)$, then (modulo Theorem 1), the corollary follows.

This completes our discussion of the properties Q_i.

5. PROOF OF THEOREM 1

The proof of Theorem 1 will be accomplished by establishing a sequence of Facts. A flowchart for the various implications is shown in Figure 3. The symbolism

$$P_i \overset{(Fk)}{\Longrightarrow} P_j$$

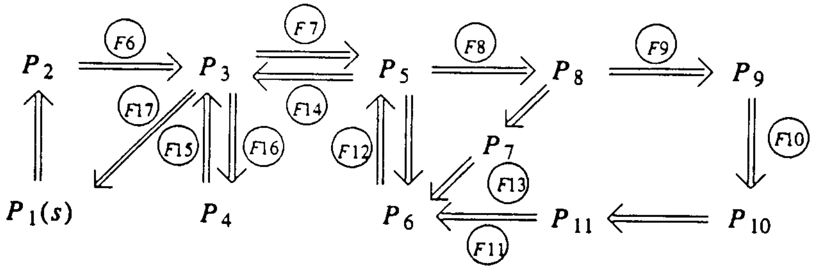


FIGURE 3

indicates that P_i will be shown to imply P_j in Fact k. The proofs of unlabeled implications are immediate and are omitted.

Fact 6. $P_2 \Rightarrow P_3$.

Proof. Suppose

$$N_T^*(E4C) = (1 + o(1))\frac{n^4}{2}. \tag{9}$$

We observe that

$$\begin{aligned} N_T^*(E4C) &= \sum_{u,v \in N} \{s(u,v)_{(2)} + \bar{s}(u,v)_{(2)}\} \\ &\geq 2 \cdot \frac{n^3}{2}\left(\frac{n}{2} - 1\right) = (1 + o(1))\frac{n^4}{2} \end{aligned} \tag{10}$$

by the Cauchy–Schwarz inequality, where $m_{(2)}$ denotes $m(m - 1)$. Hence, by (9) all but $o(n^2)$ pairs u, v in N satisfy

$$s(u,v) = (1 + o(1))\frac{n}{2}, \quad \bar{s}(u,v) = (1 + o(1))\frac{n}{2}.$$

Thus,

$$\sum_{u,v \in N} \left| s(u,v) - \frac{n}{2} \right| = o(n^3),$$

which is P_3. ∎

Fact 7. $P_3 \Rightarrow P_5$.

Proof. Suppose all but $o(n^2)$ pairs u, v in N satisfy

$$s(u,v) = (1 + o(1))\frac{n}{2}. \tag{11}$$

For $X \subset N$, where $x := |X|$, we have

$$\sum_{z \in \bar{X}} (d^+(z,X) + d^-(z,X)) = x(n - x). \tag{12}$$

On the other hand, for $T' = T[X]$, we have

$$\begin{aligned}
\sum_{u,v\in X} s(u,v) &= \sum_{v\in X} (d^+_{T'}(v)_{(2)} + d^-_{T'}(v)_{(2)}) + \sum_{u\in \bar{X}} (d^+(u,X)_{(2)} + d^-(u,X)_{(2)}) \\
&= \sum_{v\in X} (d^+_{T'}(v)^2 + d^-_{T'}(v)^2) - \sum_{v\in X} (d^+_{T'}(v) + d^-_{T'}(v)) \\
&\quad + \sum_{u\in \bar{X}} (d^+(u,X)^2 + d^-_T(u,X)^2) - \sum_{u\in \bar{X}} (d^+(u,X) + d^-_T(u,X)) \\
&= \frac{1}{2} \sum_{v\in X} \{(d^+_{T'}(v) + d^-_{T'}(v))^2 + (d^+_{T'}(v) - d^-_{T'}(v))^2\} \\
&\quad + \frac{1}{2} \sum_{u\in \bar{X}} \{(d^+(u,X) + d^-(u,X))^2 + (d^+(u,X) - d^-(u,X))^2\} \\
&\quad - x(x-1) - x(n-x) \qquad \text{by (12)} \\
&\geq \frac{1}{2} \sum_{v\in X} \{(x-1)^2 + (d^+_{T'}(v) - d^-_{T'}(v))^2\} \\
&\quad + \frac{1}{2} x^2(n-x) - x(x-1) - x(n-x) \\
&= \sum_{v\in X} (d^+_{T'}(v) - d^-_{T'}(v))^2 + (1+o(1)) \frac{x^2 n}{2}. \qquad (13)
\end{aligned}$$

However, by (11) we have

$$\sum_{u,v\in X} s(u,v) = (1+o(1))\frac{x^2 n}{2} + o(n^3). \qquad (14)$$

Combining (13) and (14), we obtain

$$\sum_{v\in X} (d^+_{T'}(v) - d^-_{T'}(v))^2 = o(n^3),$$

so by Cauchy–Schwarz we get

$$\sum_{v\in X} |d^+_{T'}(v) - d^-_{T'}(v)| \leq \left(x \sum_{v\in X} (d^+_{T'}(v) - d^-_{T'}(v))^2\right)^{1/2} = o(n^2),$$

which proves P_5. ∎

Fact 8. $P_5 \Rightarrow P_8$.

Proof. Suppose every subtournament T' of T is almost balanced. Thus,

$$\sum_{v\in N'} |d^+_{T'}(v) - d^-_{T'}(v)| \leq \epsilon n^2. \qquad (15)$$

However, for $X, Y \subset N$ we have

$$\begin{aligned}\sum_{x\in X} |d^+(x,Y) - d^-(x,Y)| &\le \sum_{x\in X} \{|d^+(x,X\cup Y) - d^-(x,X\cup Y)| \\ &\quad + |d^+(x,X) - d^-(x,X)| \\ &\quad + |d^+(x,X\cap Y) - d^-(x,X\cap Y)|\} \\ &\le 2\epsilon n^2 + \sum_{x\in X} |d^+(x,X\cap Y) - d^-(x,X\cap Y)| \\ &\le 3\epsilon n^2 + \sum_{x\in X\setminus Y} \{|d^+(x,X) - d^-(x,X)| \\ &\quad + |d^+(x,X\setminus Y) - d^-(x,X\setminus Y)|\} \le 5\epsilon n^2 \qquad (16)\end{aligned}$$

which implies P_8. ∎

Fact 9. $P_8 \Rightarrow P_9$.

Proof. Suppose for $X, Y \subset N$, $X \cap Y = \emptyset$, we have

$$\sum_{x\in X} |d^+(x,Y) - d^-(x,Y)| < \epsilon^2 n^2. \qquad (17)$$

Let π be an arbitrary ordering of T. We will show that the difference in absolute value between the numbers of π-increasing and π-decreasing arcs of T is bounded by $5\epsilon n$. Let us partition the set of integers $\{1, 2, \ldots, n\}$ into $t = \lceil 1/\epsilon \rceil$ blocks of consecutive integers, with each block of length $\lfloor n/t \rfloor$ or $\lceil n/t \rceil$. Denote these blocks by $I_1, I_2, \ldots, I_t$ where $a \in I_i, b \in I_j$ with $i < j$ implies $a < b$. The number of arcs (u, v) with $\pi(u)$ and $\pi(v)$ belonging to the same I_j is at most ϵn^2.

We now apply the hypothesis (17) for various choices of X and Y. Namely, for $i = 1, \ldots, t$, define

$$X_i := \{u \in N \mid \pi(u) \in I_j \text{ for some } j > i\},$$

and

$$Y_i := \{v \in N \mid \pi(v) \in I_j \text{ for some } j > i\},$$

A moment's reflection shows that the absolute value of the difference in the numbers of π-increasing and π-decreasing arcs is bounded above by

$$\varepsilon n^2 + \sum_i \sum_{x \in I_i} (|d^+(x, X_i) - d^-(x, X_i)| +$$

$$|d^+(x, Y_i) - d^-(x, Y_i)| < \epsilon n^2 + \sum_i 2\epsilon^2 n^2 \leq 5\epsilon n^2$$

and we are done. ∎

Fact 10. $P_9 \Rightarrow P_{10}$.

Proof. Suppose for every ordering of π of T, the number $e^+(\pi)$ of π-increasing arcs of T, and the number $e^-(\pi)$ of π-decreasing arcs of T, satisfy

$$e^+(\pi) - e^-(\pi) \mid < \epsilon n^2. \tag{18}$$

Let π_0 be an arbitrary fixed ordering of T, and let $G := T^+_{\pi_0}$. To show that G is quasi-random, it is enough to prove (see [9]) that any set S of $n/2$ vertices of G has

$$\left| e(S) - \frac{n^2}{16} \right| < \epsilon n^2.$$

Define two orderings π_1 and π_2 as follows:

$$\pi_1 \colon S \longrightarrow \left\{1, 2, \ldots, \frac{n}{2}\right\}, \qquad \pi_1 \colon N \backslash S \longrightarrow \left\{\frac{n}{2} + 1, \ldots, n\right\}$$

so that for $u, v \in S, \pi_1(u) < \pi_1(v)$ iff $\pi_0(u) < \pi_0(v)$ and for $u', v' \in N \backslash S$, $\pi_1(u') < \pi_1(v')$ iff $\pi_0(u') < \pi_0(v')$;

$$\pi_2 \colon S \longrightarrow \left\{\frac{n}{2} + 1, \ldots, n\right\}, \qquad \pi_2 \colon N \backslash S \longrightarrow \left\{1, 2, \ldots, \frac{n}{2}\right\}$$

so that for $u, v \in S, \pi_2(u) < \pi_2(v)$ iff $\pi_0(u) < \pi_0(v)$ and for $u', v' \in N \backslash S$, $\pi_2(u') < \pi_2(v')$ iff $\pi_0(u') > \pi_0(v')$.

By (18), we have

$$\begin{aligned} |e^+(\pi_1) - e^-(\pi_1)| &< \epsilon n^2, \\ |e^+(\pi_2) - e^-(\pi_2)| &< \epsilon n^2. \end{aligned} \tag{19}$$

However, the number of *edges* spanned by S minus the number of *non-edges* spanned by S is just given by

$$\frac{1}{2}((e^+(\pi_1) - e^-(\pi_1)) + (e^+(\pi_2) - e^-(\pi_2)) < \epsilon n^2.$$

Thus,

$$\left| e(S) - \frac{n^2}{16} \right| < \epsilon n^2$$

and P_{10} is proved. ∎

Fact 11. $P_{11} \Rightarrow P_6$.

Proof. It suffices to show that if $G = T_\pi^+$ is a quasi-random graph for some ordering π then T is a quasi-random tournament. Let $X \subset N = \{1, 2, \dots, n\}$ have size $n/2$. Since G is quasi-random (see [9]), then for any $\epsilon > 0$ we have, for $n > n_0(\epsilon)$,

$$\sum_{x \in Y} |\deg(x, Y) - \overline{\deg}(x, Y)| < \epsilon^2 n,$$

for any $Y \subset N$, where $\deg(x, Y)$ denotes $|\{y \in Y | \{x, y\} \text{ is an edge of } G\}|$ and $\overline{\deg}(x, Y) := |Y| - \deg(x, Y)$.

Let $t = \lceil 1/\epsilon \rceil$. Partition N into blocks $I_1, I_2, \dots, I_t$ of consecutive integers where $|I_i| = n/t$ and for $i < j$, all elements of I_i are less than any element of I_j. Define $X_i := X \cap I_i, Y_i := \bigcup_{j>i} X_j, Z_i := \bigcup_{j<i} X_j$, for $1 \le i \le t$. By the definition of T_π^+,

$$\begin{aligned} d^+(x, Y_i) - d^-(x, Y_i) &= \deg(x, Y_i) - \overline{\deg}(x, Y_i), \\ d^+(x, Z_i) - d^-(x, Z_i) &= \overline{\deg}(x, Z_i) - \deg(x, Z_i). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{x \in X} |d^+(x, X) - d^-(x, X)| &\le \sum_{i=1}^{t} \left\{ \sum_{x \in X_i} |d^+(x, X_i) - d^-(x, X_i)| \right. \\ &\quad + \sum_{x \in X_i} |d^+(x, Y_i) - d^-(x, Y_i)| \\ &\quad \left. + \sum_{x \in X_i} |d^+(x, Z_i) - d^-(x, Z_i)| \right\} \\ &\le \sum_{i=1}^{t} \left\{ \left(\frac{n}{t}\right)^2 + \sum_{x \in X_i} |\deg(x, Y_i) - \overline{\deg}(x, Y_i)| \right. \\ &\quad \left. + \sum_{x \in X_i} |\deg(x, Z_i) - \overline{\deg}(x, Z_i)| \right\} \\ &< t\left\{ \left(\frac{n}{t}\right)^2 + \epsilon^2 n^2 + \epsilon^2 n^2 \right\} \le 5\epsilon n^2 \end{aligned}$$

which implies P_6. ∎

Fact 12. $P_6 \Rightarrow P_5$.

Proof. Suppose every subtournament $T' = (N', A')$ of T on $n/2$ nodes is almost balanced. Thus,

$$\sum_{v \in N'} \left| d^+_{T'}(v) - \frac{n}{4} \right| < \epsilon^2 n^2. \tag{20}$$

Let $T'' = T[X]$ be a subtournament of T on $m = |X|$ nodes. We consider the case that $m \geq n/2$ (the argument for the other case $m < n/2$ is very similar and is omitted). Suppose P_5 fails. Then for some set $W \subset X$ with $|W| = w = 10\epsilon n$ we have

$$d^+_{T''}(v) - \frac{m}{2} > 100\epsilon n$$

for all $v \in W$ (again, the other cases are similar). By a standard averaging argument (as used in the proof of Fact 5), we can find a set X' of $n/2$ nodes with $W \subset X' \subset X$ so that

$$\begin{aligned}
\sum_{v \in W} d^+(v, X') &\geq \frac{1}{\binom{m-w}{m/2-w}} \sum_{\substack{X'' \supseteq W \\ |X''|=n/2}} \sum_{v \in W} d^+(v, X'') \\
&\geq \frac{1}{\binom{m-w}{n/2-w}} \cdot w\left(\frac{m}{2} + 100\epsilon n\right)\binom{m-w-1}{n/2-w-1} \\
&\geq \frac{(n/2)-w}{m-w} \cdot wm\left(\frac{1}{2} + 100\epsilon\right) \geq (1 + 10\epsilon)\frac{wn}{4}.
\end{aligned}$$

Thus,

$$\sum_{v \in W} \left| d^+(v, X') - \frac{n}{4} \right| \geq \frac{10\epsilon w n}{4} \geq 10\epsilon^2 n^2$$

which contradicts (20). This proves P_5. ∎

Fact 13. $P_7 \Rightarrow P_6$.

Proof. Suppose for any partition of $N = X \cup Y$ into two almost equal parts, we have

$$\sum_{v \in X} |d^+(v, Y) - d^-(v, Y)| < \epsilon n^2. \tag{21}$$

Since $P_7 \Rightarrow Q_5 \Rightarrow Q_3$, T is almost balanced, and so all but $o(n)$ nodes of T satisfy

$$|d^+(v) - d^-(v)| \leq \epsilon n\,.$$

Let $X \subset N$ with $|X| = n/2$, and consider the subtournament $T' = T[X]$. Then

$$\sum_{v \in X} |d^+_{T'}(v) - d^-_{T'}(v)| \leq \sum_{v \in X} (|d^+(v) - d^-(v)| + |d^+(v,Y) - d^-(v,Y|) \leq 3\epsilon n^2$$

which implies P_6. ▮

Fact 14. $P_5 \Rightarrow P_3$.

Proof. By P_5, for any $\epsilon > 0$, if $n > n_0(\epsilon)$ and $X \subset N$ then $T' = T[X]$ satisfies

$$\sum_{v \in X} |d^+_{T'}(v) - d^-_{T'}(v)| < \epsilon n^2\,.$$

Note that

$$\begin{aligned} s(u,v) &= d^+(u, nd^+(v)) + d^-(u, nd^-(v))\,, \\ \bar{s}(u,v) &= d^+(u, nd^-(v)) + d^-(u, nd^+(v))\,. \end{aligned} \tag{22}$$

Thus,

$$\begin{aligned} \sum_{u,v \in N} |s(u,v) - \bar{s}(u,v)| &= \sum_{v} \sum_{u} |d^+(u, nd^+(v)) + d^-(u, nd^-(v)) \\ &\qquad - d^+(u, nd^-(v)) - d^-(u, nd^+(v))| \\ &\leq \sum_{v} \sum_{u} \{|d^+(u, nd^+(v)) - d^-(u, nd^+(v))| \\ &\qquad + |d^+(u, nd^-(v)) - d^-(u, nd^-(v)|\} \\ &\leq \sum_{v} \left\{ \sum_{u \in nd^+(v)} (2|d^+(u, nd^+(v)) - d^-(u, nd^+(v))| \right. \\ &\qquad + |d^+(u) - d^-(u)|) \\ &\qquad + \sum_{u \in nd^-(v)} (2|d^+(u, nd^-(v)) - d^-(u, nd^-(v))| \\ &\qquad \left. + |d^+(u) - d^-(u)|) \right\} \leq \sum_{v} 10\epsilon n^2 \leq 10\epsilon n^3 \end{aligned}$$

by the hypothesis P_5. Since this clearly implies P_3, we are done. ▮

Fact 15. $P_4 \Rightarrow P_3$.

Proof. Suppose P_4 holds. First, we will show that T is almost balanced. By P_4,

$$\sum_{u,v} |nd^+(u) \cap nd^+(v)| \le (1 + o(1))\frac{n^3}{4}.$$

But also,

$$\begin{aligned}\sum_{u,v} |nd^+(u) \cap nd^+(v)| &\ge \sum_w |nd^-(w)|_{(2)} \\ &\ge \frac{1}{n}\left(\sum_w |nd^-(w)|\right)^2 \\ &\quad - \sum_w |nd^-(w)| \text{ by Cauchy–Schwarz} \\ &= \frac{1}{n}\binom{n}{2}^2 - \binom{n}{2} = (1 + o(1))\frac{n^3}{4}.\end{aligned}$$

Thereore, almost all w must satisfy

$$|nd^-(w)| = (1 + o(1))\frac{n}{2}, \qquad |nd^+(w)| = (1 + o(1))\frac{n}{2}.$$

Since P_4 implies that almost all pairs u, v have

$$|nd^+(u) \cap nd^+(v)| = (1 + o(1))\frac{n}{4}$$

then

$$\begin{aligned}|nd^+(u) \cap nd^-(v)| &= |nd^+(u)| - |nd^+(u) \cap nd^+(v)| \\ &= (1 + o(1))\frac{n}{4}\end{aligned}$$

and

$$\begin{aligned}|nd^-(u) \cap nd^-(v)| &= |nd^-(u)| - |nd^+(u) \cap nd^-(v)| \\ &= (1 + o(1))\frac{n}{4}.\end{aligned}$$

Therefore,

$$s(u,v) = |nd^+(u) \cap nd^+(v)| + |nd^-(u) \cap nd^-(v)|$$
$$= (1 + o(1))\frac{n}{2}$$

for almost all pairs u, v. This implies

$$\sum_{u,v} \left| s(u,v) - \frac{n}{2} \right| = o(n^3)$$

which is just P_3. ∎

Fact 16. $P_3 \Rightarrow P_4$.

Proof. By P_3, all but εn^2 pairs of nodes u, v satisfy

$$\left| s(u,v) - \frac{n}{2} \right| < \varepsilon n ,$$
$$\left| \bar{s}(u,v) - \frac{n}{2} \right| < \varepsilon n .$$

Furthermore, P_3 implies T is almost balanced since we have already shown $P_3 \Rightarrow P_5$, for example. Thus, for all but εn nodes v, we have

$$\left| d^+(v) - \frac{n}{2} \right| < \varepsilon n, \qquad \left| d^-(v) - \frac{n}{2} \right| < \varepsilon n .$$

Since

$$d^+(u) = |nd^+(u) \cap nd^+(v)| + |nd^+(u) \cap nd^-(v)| + \chi_T(u,v)$$

then by (22) we get

$$|nd^+(u) \cap nd^+(v)| = \frac{1}{2}(s(u,v) + d^+(u) - |nd^-(u) \cap nd^-(v)|$$
$$- |nd^+(u) \cap nd^-(v)| - \chi_T(u,v)) \qquad (23)$$
$$\leq \frac{1}{2}(s(u,v) + d^+(u) - d^-(v)) \leq \left(\frac{1}{4} + 3\varepsilon\right)n$$

for all but εn^2 pairs u, v. This then implies P_4 and the proof is complete. ∎

Finally, we come to the final link in our cycle of implications.

Fact 17. $P_3 \Rightarrow P_1(s)$.

Proof. Assume that we have

$$\sum_{u,v} \left| s(u,v) - \frac{n}{2} \right| = o(n^3). \tag{24}$$

We will show that for fixed s, if $T(s)$ is any tournament on s nodes, then $N_s := N_T^*(T(s))$ satisfies

$$N_s = (1 + o(1)) n^s 2^{-\binom{s}{2}}.$$

(We remark that this proof is virtually the same as one appearing in [9] for the case of graphs. We include it here for completeness.) Assume the node set of $T(s)$ is $\{v_1, \ldots, v_s\}$. For $1 \le r \le s$, define $T(r)$ to be the subtournament of $M(s)$ induced by the node set $V_r := \{v_1, \ldots, v_r\}$. We will prove by induction on r using a "second moment" method that

$$N_r := N_G^*(T(r)) = (1 + o(1)) n_{(r)} 2^{-\binom{r}{2}} \tag{25}$$

where, as usual,

$$n_{(r)} := n(n-1)\cdots(n-r+1).$$

For $r = 1$, (25) is immediate. Assume for some r, $1 \le r < s$, that (25) holds. Define $\alpha := (\alpha_1, \ldots, \alpha_r)$ where the α_i are distinct elements of $[n] := \{1, 2, \ldots, n\}$, which we take to be the node set N of T. Also define $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_r)$, $\varepsilon_i = \pm 1$, and

$$f_r(\alpha, \varepsilon) := \left|\{i \in [n] | i \ne \alpha_1, \ldots, \alpha_r \text{ and } \chi_T(i, \alpha_j) = \varepsilon_j;\ 1 \le j \le r\}\right|.$$

Note that N_{r+1} is a sum of exactly N_r quantities $f_r(\alpha, \varepsilon)$. Namely, for each embedding of $T(r)$ into T, say, $\lambda(v_j) = \alpha_j$, $1 \le j \le r$, $f_r(\alpha, \varepsilon)$ counts the number of ways of choosing $i \in [n]$ so that if we extend λ to V_{r+1} by setting $\lambda(v_{r+1}) = i$, and we take $\varepsilon_j = \chi_T(v_{r+1}, v_j)$, then λ becomes an embedding of $T(r+1)$ into T. Also note that there are just $n_{(r)} 2^r$ quantities $f_r(\alpha, \varepsilon)$, since there are $n_{(r)}$ choices for α and 2^r choices for ε. Our next step will be to compute the first and second moments of $f_r(\alpha, \varepsilon)$.

To begin with, we have

$$\begin{aligned} \bar{f}_r &:= \frac{1}{n_{(r)} 2^r} \sum_{\alpha, \varepsilon} f_r(\alpha, \varepsilon) = \frac{1}{n_{(r)} 2^r} \sum_{\alpha} \sum_{\varepsilon} f_r(\alpha, \varepsilon) \\ &= \frac{1}{n_{(r)} 2^r} \sum_{\alpha} (n - r) = \frac{n - r}{2^r} \end{aligned} \tag{26}$$

since every node $i \neq \alpha_1, \ldots, \alpha_r$ corresponds to a unique choice for ε. Thus,

$$\sum_{\alpha,\varepsilon} f_r(\alpha,\varepsilon) = (n - r)n_{(r)} = n_{(r+1)}. \tag{27}$$

Next, define

$$S_r := \sum_{\alpha,\varepsilon} f_r(\alpha,\varepsilon)\,(f_r(\alpha,\varepsilon) - 1).$$

We claim that

$$S_r = \sum_{i\neq j} s(i,j)_{(r)}. \tag{28}$$

To see this, we interpret S_r as counting the number of ways of choosing $\alpha = (\alpha_1, \ldots, \alpha_r)$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ and two other (ordered) nodes i and j in $[n]$ so that

$$\chi_T(i,\alpha_k) = \varepsilon_k = \chi_T(j,\alpha_k), \qquad 1 \leq k \leq r.$$

Summing over all possible ε reduces this to requiring just that

$$\chi_T(i,\alpha_k) = \chi_T(j,\alpha_k), \qquad 1 \leq k \leq r.$$

Now, think of choosing i and j first. The required additional r nodes $\alpha_1, \ldots, \alpha_r$ must come exactly from $\{v \in [n]: \chi_T(i,v) = \chi_T(j,v)\}$. Therefore, there are just $s(i,j)_{(r)}$ ways to choose them, which implies (28).

We next assert that (24) implies

$$\sum_{i\neq j} s(i,j)_{(r)} = (1 + o(1))n^{r+2}2^{-r}. \tag{29}$$

To see this, first define

$$\varepsilon_{ij} := s(i,j) - \frac{n}{2}.$$

By (24), $\Sigma_{i\neq j}|\varepsilon_{ij}| = o(n^3)$. Also, $|\varepsilon_{ij}| \leq n$. Therefore,

$$\sum_{i\neq j} |\varepsilon_{ij}|^a \leq n^{a-1}\sum_{i\neq j}|\varepsilon_{ij}| = o(n^{a+2}), \qquad a \text{ fixed}.$$

Hence,

$$\begin{aligned}
\sum_{i \neq j} s(i,j)_{(r)} &= \sum_{i \neq j} \left(\frac{n}{2} + \varepsilon_{ij} \right)_{(r)} \\
&= \sum_{k=0}^{r} \sum_{i \neq j} c_k \left(\frac{n}{2} \right)^k \varepsilon_{ij}^{r-k} \qquad \text{(for appropriate constants } c_k\text{)} \\
&= \left(\frac{n}{2} \right)^r n_{(2)} + \sum_{k=0}^{r-1} \sum_{i \neq j} c_k \left(\frac{n}{2} \right)^k \varepsilon_{ij}^{r-k} \\
&\leq \left(\frac{n}{2} \right)^r n_{(2)} + \sum_{k=0}^{r-1} \sum_{i \neq j} |c_k| \, |\varepsilon_{ij}|^{r-k} n^k \\
&\leq \left(\frac{n}{2} \right)^r n_{(2)} + c \sum_{k=0}^{r-1} n^k \sum_{i \neq j} |\varepsilon_{ij}|^{r-k} \\
&\leq \left(\frac{n}{2} \right)^r n_{(2)} + c \sum_{k=0}^{r-1} n^k \cdot o(n^{r-k+2}) \\
&= \left(\frac{n}{2} \right)^r n_{(2)} + o(n^{r+2}) \\
&= (1 + o(1)) n^{r-2} 2^{-} r
\end{aligned}$$

as claimed.

Note that by (28) and (29) we have

$$S_r = (1 + o(1)) n^{r+2} 2^{-r}. \tag{30}$$

Consequently,

$$\begin{aligned}
\sum_{\alpha, \varepsilon} (f_r(\alpha, \varepsilon) - \bar{f}_r)^2 &= \sum_{\alpha, \varepsilon} f_r^2(\alpha, \varepsilon) - \sum_{\alpha, \varepsilon} \bar{f}_r^2 \\
&= \sum_{\alpha, \varepsilon} (f_r^2(\alpha, \varepsilon) - f_r(\alpha, \varepsilon)) + \sum_{\alpha, \varepsilon} f_r(\alpha, \varepsilon) - n_{(r)} 2^r (n - r)^2 2^{-2r} \\
&= S_r + n_{(r+1)} - n_{(r)} (n - r)^2 2^{-r} = o(n^{r+2}).
\end{aligned}$$

Finally, since from our earlier observation that

$$N_{r+1} = \sum_{\substack{N_r \text{ choices} \\ \text{of } (\alpha, \varepsilon)}} f_r(\alpha, \varepsilon)$$

then

$$|N_{r+1} - N_r\bar{f}_r|^2 = \left|\sum_{N_r \text{ terms}} (f_r(\alpha,\varepsilon) - \bar{f}_r)\right|^2$$

$$\leq N_r \sum_{N_r \text{ terms}} (f_r(\alpha,\varepsilon) - \bar{f}_r)^2 \text{ by Cauchy–Schwarz}$$

$$\leq N_r \sum_{\alpha,\varepsilon} (f_r(\alpha,\varepsilon) - \bar{f}_r)^2$$

$$= o(N_r \cdot n^{r+2}) = o(n^{2r+2})$$

by induction. Consequently

$$|N_{r+1} - N_r\bar{f}_r| = o(n^{r+1})$$

and so,

$$N_{r+1} = N_r\bar{f}_r + o(n^{r+1})$$

$$= (1 + o(1))n_{(r)}2^{-\binom{r}{2}} \cdot (n - r)2^{-r} + o(n^{r+1})$$

$$= (1 + o(1))n_{(r+1)}2^{-\binom{r+1}{2}}.$$

This completes the induction step, and Fact 17 is proved. ∎

Finally, since the four (unlabeled) implications $P_1(s) \Rightarrow P_2$, $P_5 \Rightarrow P_6$, $P_8 \Rightarrow P_7$, and $P_{10} \Rightarrow P_{11}$, are all immediate (each implied property is a special case of the implying property), then all the links in Figure 3 have been established, and so Theorem 1 is proved. ∎

6. CONCLUDING REMARKS

We close with a collection of remarks and open problems. To begin with, it would be of interest to expand our family of quasi-random properties for tournaments. For example is the following property quasi-random:

$P_{12}(2k)$:

$$|\{(v_0, v_1, \ldots, v_{2k-1}) \in N^{2k} \mid \Pi_i \chi_T(v_i, v_{i+1}) = 1\}| = (1 + o(1))\frac{n^{2k}}{2}?$$

Of course, $P_{12}(4)$ is just P_2. We remark that if a *graph* G has the expected number of edges and $2k$-cycles, then it is quasi-random (see [9]). Also in this connection one could ask for other directed graphs $D = (N_D, A_D)$

(besides E4Cs) so that

$$N_T^*(D) = (1 + o(1))n^{|N_D|}2^{-\binom{|A_D|}{2}}$$

implies T is quasi-random.

An interesting difference occurs between graphs and tournaments with respect to properties P_7 and P_8. It is not difficult to show that for any fixed $\alpha \in (0,1)$, the following special case of P_8 is quasi-random:

$P_8(\alpha)$: For all $X \subset N$ with $|X| = \lfloor \alpha n \rfloor$,

$$\sum_{v \in X} |d^+(v, \overline{X}) - d^-(v, \overline{X})| = o(n^2).$$

Of course, P_7 is just $P_8(1/2)$. It turns out that the analogous theorem does not quite hold for graphs. Namely, consider the corresponding property for graphs $G(n) = (V, E)$.

$P(\alpha)$: For all $X \subset V$ with $|X| = \lfloor \alpha n \rfloor$,

$$e(X, \overline{X}) = (1 + o(1))\alpha(1 - \alpha)n^2$$

where $e(X, \overline{X})$ denotes the number of edges between X and $\overline{X} = V \backslash X$.

It is shown in [8] that $P(\alpha)$ is a quasi-random property for graphs for all $\alpha \in (0,1)$ *except* for $\alpha = 1/2$! A counterexample for this case can be constructed by placing a random (or quasi-random) bipartite graph between a complete graph of size $n/2$ and an independent set of size $n/2$.

We have not said much about the explicit construction of quasi-random tournaments. Of course, any quasi-random graph yields a quasi-random tournament (by converting edges to increasing arcs), as does any large subtournament of a quasi-random tournament. We should point out here that in fact we can reverse this process and use it as a powerful way of constructing quasi-random *graphs*. Namely, from a given quasi-random graph G on $[n]$, form the tournament $T = T(G)$ on $[n]$ by replacing each edge $\{i, j\}$, $i < j$, of G by the arc (i, j), apply an arbitrary ordering π to T, and then generate the new increasing arc graph $G_\pi = T_\pi^+$ of T. We plan to discuss the various properties of the members G_π in the orbit of G in a later paper.

Perhaps the most well-known example of a quasi-random tournament (e.g., see [16] and [17]) is the so-called Paley tournament $Q_p = (\mathbb{Z}_p, A)$. For a prime $p \equiv 3 \pmod 4$, the nodes of Q_p consist of the integers modulo p. A pair (i, j) is an arc iff $i - j$ is a quadratic residue modulo p. To see that Q_p is quasi-random, we verify property P_3. A node $z \in S(u, v)$ iff $(z - x)/(z - y)$ is a quadratic residue modulo p. However, for any of the $\frac{1}{2}(p - 3)$

quadratic residues $r \neq 0, 1$, there is always a unique z such that

$$\frac{z - x}{z - y} = 1 + \frac{y - x}{z - y} \equiv r \pmod{p}.$$

Thus, $s(x, y) = \frac{1}{2}(p - 3)$ and P_3 follows. (This argument, due to R.M. Wilson, appears in [9].)

The tournament Q_p was used in [15] (also see [3]) as a concrete example of tournament $T = (N, A)$ in which

$S(k)$: For any $x_1, \ldots, x_k \in N$, some $z \in N$ satisfies $(z, x_i) \in A$, $1 \leq i \leq k$ (i.e., any k players all lost to some other player).

However, the character sum estimates of Weil [22] and Burgess [4] used in the proof there required that $p > k^2 2^{2k-2}$ in order for Q_p to satisfy (S_k). Can p in fact be taken to be significantly smaller? It is known that

(i) a tournament on $ck^2 2^k$ nodes exists that satisfies $S(k)$ (see [10]);
(ii) any tournament that satisfies $S(k)$ must have at least $(k + 2)2^{k-1} - 1$ nodes (see [19]).

More precisely, it would certainly be of interest to obtain more quantitative forms for various quasi-random tournament properties, much in the spirit of the recent work for graphs by Thomason [20], [21], and Spencer–Tetali [18]. Of course, lower bounds for various quasi-random properties offer substantial challenges. It is known [12], for example, that in any tournament $T(n) = (N, A)$, there is always a subset $X \subset N$ such that

$$|d^+(X, \bar{X}) - d^-(X, \bar{X})| > cn^{3/2}.$$

In the other direction, how can the *non*-quasi-randomness of a tournament be expressed quantitatively? More precisely, if $T(n)$ fails to satisfy some quasi-random property, to what extent does it fail the others? For example, for graphs the following has been shown [7]:

Suppose a graph $G(n) = (V, E)$ *fails* to contain some graph $H(s)$ as an induced subgraph. Then for some subset $S \subset V$ with $|S| = \lfloor n/2 \rfloor$, we have

$$\left| e(S) - \frac{n^2}{16} \right| > 2^{-(2t^2+27)} n^2,$$

(where $e(S)$ denotes the number of edges spanned by S).

Recent results of this type are available [1], [11] for more drastic deviations from quasi-randomness. For example, it has been shown [1] that if $\varepsilon < 10^{-21}$ and $G(n)$ is a graph with at most εn^2 distinct induced subgraphs, then $G(n)$ contains either a clique or independent set of size at least $(1 - 4\varepsilon)n$. What are the analogous results for tournaments?

Finally, this whole line of investigation could be carried out for directed graphs in general (tournaments being an interesting but rather special case). Of course, the natural domain for these questions is the set of (binary) matrices, and some work has begun in [14].

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