Quasi-Random Hypergraphs

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ABSTRACT

We introduce an equivalence class of varied properties for hypergraphs. Any hypergraph possessing any one of these properties must of necessity possess them all. Since almost all random hypergraphs share these properties, we term these properties quasi-random. With these results, it becomes quite easy to show that many natural explicit constructions result in hypergraphs which imitate random hypergraphs in a variety of ways.

Key words: hypergraph, random, quasi-random

1. INTRODUCTION

In [1], the authors introduced the concept of "quasi-random" graph properties. Roughly speaking, these are a particular set of properties shared by almost all random graphs which are *equivalent*, in the sense that any graph having any *one* of these properties must necessarily have *all* of them (see [2] for a thorough discussion of random graphs). Some examples of these properties (for graphs G with n vertices, n large) are the following:

- P_1: G has at least $(1+o(1))n^2/4$ edges and at most $(1+o(1))n^4/16$ 4-cycles;
- $P_2(s)$: For fixed s, each (ordered) graph $M(s)$ on s vertices occurs $(1+o(1))n^s/2^{\binom{s}{2}}$ times as an *induced* subgraph of G;
- P_3: For any subset S of vertices of G, the number $e(S)$ of edges spanned by S satisfies $e(S) = \frac{1}{4}|S|^2 + o(n^2)$;

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P_4: For almost all choices of vertices u and v, the number $s(u, v)$ of vertices adjacent to either both u and v, or neither u nor v, satisfies $s(u, v) = (1 + o(1))n/2$.

In particular, it turns out (rather unexpectedly) that $P_2(4) \Rightarrow P_1 \Rightarrow P_2(s)$ for any fixed s, as $n \to \infty$. (We defer precise definitions and interpretations of our terms until the next section).

As noted in [1], it is natural to try to extend these results to hypergraphs, that is, the analogues of graphs in which "edges" are *k-sets* of some ground set for some k, rather than just *pairs*, for the case of graphs. However, examples of Erdös/Sós [3] and Rödl [4] showed that as soon as k exceeds 2, new difficulties arise. In particular, if $P_t^{(k)}$ denotes the analogue of property P_t for k-graphs (i.e., hypergraphs with k-sets for edges), then Rödl [4] constructed 3-graphs satisfying $P_3^{(3)}$ but not $P_2^{(3)}(4)$ (in fact, failing to contain even a *single* induced copy of a certain 4-vertex 3-graph). Similarly, in Section 9, we give examples of k-graphs satisfying $P_4^{(k)}$ but not $P_2^{(k)}(k+1)$, and also examples satisfying $P_2^{(k)}(k+1)$ but not $P_2^{(k)}(k+2)$.

In this article, we show that nevertheless, it *is* possible to construct a meaningful theory of quasi-randomness for general hypergraphs. For example, we will show for k-graphs that $P_2^{(k)}(2k)$ implies $P_2^{(k)}(s)$ for *any* fixed s, as $n \to \infty$.

2. NOTATION AND PRELIMINARIES

A k-graph $G = (V, E)$ consists of a set $V = V(G)$, called the *vertices* of G, and a subset $E = E(G)$ of the set $\binom{V}{k}$ of k-element sets of V, called the *edges* of G. We use the notation $G(n)$ to denote the fact that V has n elements, i.e., $|V| = n$. For $X \subseteq V$, we let $G[X]$ denote the sub-k-graph of G *induced by* X, i.e, $G[X] = (X, E \cap \binom{X}{k})$. Let $\chi_G\colon \binom{V}{k} \to \{0, 1\}$ denote the *edge indicator* of G, i.e., for

$$e \in \binom{V}{k}, \ \chi_G(e) = \begin{cases} 1 & \text{if } e \in E\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For another k-graph $G' = (V', E')$, we let $N_G^*(G')$ denote the number of labelled occurrences of G' as an induced subgraph of G. In other words,

$$N_G^*(G') = |\{\lambda\colon V' \to V \mid G[\lambda(V)] \cong G'\}|$$

where $\cong$ denotes the natural notion of k-graph isomorphism. The quantity $N_G^*(G')$ is related to $\bar{N}_G^*(G')$, the number of unlabelled occurrences of G' in G, by

$$N_G^*(G') = \bar{N}_G^*(G')/|Aut(G)|$$

where $Aut(G)$ denotes the automorphism group of G. If $\mathcal{F}$ is a family of graphs then $N_G^*(\mathcal{F})$ denotes $\bigcup_{G' \in \mathcal{F}} N_G^*(G')$.

Further, we denote the number of copies of G' occurring as (not necessarily induced) sub-k-graphs of G by $N_G(G')$. Thus,

$$N_G(G') = \sum_H N_G^*(H)$$

where the sum is taken over all k-graphs

$$H = (V', E_H) \text{ where } E_H \supseteq E' .$$

There is a special k-graph on $2k$ vertices which will be important in what follows. This k-graph is called a *k-octahedron*, or just *octahedron*, for short, and is denoted by $\mathcal{O}_k = \mathcal{O}_k(x_1(0), x_1(1), x_2(0), x_2(1), \ldots, x_k(0), x_k(1)) = \mathcal{O}_k(\bar{x}(\bar{\epsilon}))$. The vertices of $\mathcal{O}_k$ are the $x_i(\epsilon_i)$, $1 \le i \le k$, $\epsilon_i \in \{0, 1\}$. The edges of $\mathcal{O}_k$ consist of all k-sets of the form $\{x_1(\epsilon_1), x_2(\epsilon_2), \ldots, x_k(\epsilon_k) \mid \epsilon_i \in \{0, 1\}, 1 \le i \le k\}$. Thus, $\mathcal{O}_k$ has 2^k edges.

Finally, we will call a k-graph an *even partial octahedron* (EPO) if it has the same vertex as $\mathcal{O}_k$, and has an edge set consisting of an *even number* of the edges of $\mathcal{O}_k$. We let $\mathcal{O}_k^e$ denote the set of all EPOs based on $\mathcal{O}_k$, although occasionally we will let $\mathcal{O}_k^e$ denote an individual EPO, as well.

3. STATEMENT OF THE MAIN RESULTS

We next list a set of properties which a k-graph $G = G(n)$ might satisfy. Each of the properties will contain occurrences of the asymptotic "little-oh" notation $o(1)$. However, the dependence of the different $o(1)$'s on the particular properties they refer to will ordinarily be suppressed. The use of these $o(1)$'s can be viewed in two essentially equivalent ways.

In the first way, suppose we have two properties P and P', each with occurrences of $o(1)$, so that $P = P(o(1))$, $P' = P'(o(1))$. The implication "$P \Rightarrow P'$" then means that for each $\epsilon > 0$ there is a $\delta > 0$ so that if $G(n)$ satisfies $P(\delta)$, then it must also satisfy $P'(\epsilon)$, provided $n > n_0(\epsilon)$.

In the second way, we can think of considering a family $\mathcal{F}$ of graphs $G(n)$ with $n \to \infty$. In this case, the interpretation of $o(1)$ is the usual one as $G = G(n)$ ranges over $\mathcal{F}$, with $n \to \infty$. As usual, "almost all" (abbreviated a.a) denotes a fraction $1 + o(1)$ of the elements of the set in question.

We next state a series of properties for k-graphs $G(n)$ which are shared by almost all random k-graphs $G_{1/2}(n)$ on n vertices. (For $G_{1/2}(n)$, each possible k-set is chosen to be an edge independently with probability $1/2$).

$Q_1(s)$: For all k-graphs $G'(s)$ on s vertices,

$$N^*_{G(n)}(G'(s)) = (1 + o(1))n^2/2^{\binom{s}{k}} . \tag{1}$$

Q_2: For all k-graphs $G'(2k)$,

$$N^*_{G(n)}(G'(2k)) = (1 + o(1))n^{2k}/2^{\binom{2k}{k}} . \tag{2}$$

Q_3:

$$N^*_{G(n)}(\mathcal{O}^e_k) \leq (1 + o(1))n^{2k}/2 . \tag{3}$$

In other words, the number of induced EPOs $\mathcal{O}^e_k$ occurring in $G(n)$ is at most $(1 + o(1))n^{2k}/2$, which is just the expected number occurring in the random k-graph $G_{1/2}(n)$.

For our final properties, we need another definition. Let $G = (V, E)$ be a k-graph and let $x, y \in V$. The *sameness* $(k-1)$*-graph* $G(x, y)$ is defined to be the $(k-1)$-graph $G' = (V', E')$ with $V' = V \backslash \{x, y\}$ and edge set

$$E' = \left\{ e' \in \binom{V'}{k-1} \,\middle|\, \chi_G(e' \cup \{x\}) = \chi_G(e' \cup \{y\}) \right\}$$

Q_4: For almost all choices of $x, y \in V$, the sameness $(k-1)$-graph $G(x, y)$ of $G(n) = (V, E)$ satisfies Q_2, with k replaced by $k - 1$.

Q_5: For $1 \leq r \leq 2k - 1$ and almost all $x, y \in V$,

$$N_{G(x, y)}(K_r^{(k-1)}) = (1 + o(1))n^r/2^{\binom{r}{k-1}} \tag{4}$$

where $G(x, y)$ is the sameness $(k-1)$-graph of $G(n) = (V, E)$, and $K_r^{(k-1)}$ denotes the complete $(k-1)$-graph on r vertices, i.e., having all the possible $\binom{r}{k-1}$ edges (and we use the convention that any set of $r < k - 1$ vertices forms a $K_r^{(k-1)}$).

Several implications among these properties are immediate or easily proved. For example,

$$Q_1(2k) = Q_2 \Rightarrow Q_3 \text{ and } Q_1(s + 1) \Rightarrow Q_1(s) .$$

Our main result asserts that for $s \geq 2k$, *all* of these properties are in fact *equivalent.*

Theorem 1. *For $s \geq 2k$,*

$$Q_1(s) \Rightarrow Q_2 \Rightarrow Q_3 \Rightarrow Q_4 \Rightarrow Q_5 \Rightarrow Q_1(s) .$$

Hypergraphs which satisfy any one (and therefore all) of these properties we call *quasi-random.*

4. BEGINNING THE PROOF

The proof of Theorem 1 will proceed by induction on k. The result for $k = 2$, our initial case, is essentially a consequence of Theorem 1 in [1]. Here, we give a simpler proof (which avoids the eigenvalue arguments in [1]). A key idea in this simplification is the following.

Suppose $G(n) = (V, E)$ satisfies Q_3 for $k = 2$, i.e.,

$$N_G^*(\mathcal{O}_2^e) \leq (1 + o(1))n^4/2\,. \tag{5}$$

For this case, an EPO $\mathcal{O}_2^e$ is just a sequence of four points y_0, y_1, y_2, y_3 in which an *even* number of the pairs y_0y_1, y_1y_2, y_2y_3, y_3y_0 are actually edges of $G(n)$.

Fact 1. *For any 2-graph $H = H(n)$,*

$$N_H^*(\mathcal{O}_2^e) \geq (1 + o(1))n^4/2\,.$$

Proof. For vertices $x, y \in (V, H)$, define

$$A_0(x, y) = \{v \in V(H) : \chi_H(x, v) = \chi_H(y, v)\}\,,$$

$$A_1(x, y) = \{v \in V(H) : \chi_H(x, v) \neq \chi_H(y, v)\}\,,$$

Then

$$N_H^*(\mathcal{O}_2^e) = \sum_{x,y} \{|A_0(x, y)|_{(2)} + |A_1(x, y)|_{(2)}\}$$

$$\geq \sum_{x,y} \left\{\binom{n}{2}_{(2)} + \binom{n}{2}_{(2)}\right\} \geq (1 + o(1))n^4/2$$

where $m_{(t)}$ denotes the falling factorial $m(m-1)\cdots(m-t+1)$. ∎

Therefore, if $G(n)$ satisfies Q_3 then by Fact 1,

$$N_G^*(\mathcal{O}_2^e) = (1 + o(1))n^4/2 \tag{6}$$

which implies that for almost all $x, y \in V$,

$$|A_0(x, y)| = (1 + o(1))n/2, |A_1(x, y)| = (1 + o(1))n/2\,. \tag{7}$$

However, the first equality is exactly what is needed for $G(n)$ to satisfy property P_4, which implies Q_4 as follows. For $x, y \in V$, the sameness 1-graph $G(x, y) = (V', E')$ has

$$V' = V \backslash \{x, y\} \text{ and } E' = \{z \in V' \mid \chi_{G(n)}(x, z) = \chi_{G(n)}(y, z)\}\,.$$

However, property P_4 (see [1]) implies

$$|E'| = (1 + o(1))n/2\,. \tag{8}$$

For $G(x, y)$ to satisfy Q_2 with 2 replaced by 1, we need to have for all 1-graphs $G'(2)$ on 2 vertices,

$$N^*_{G(x,y)}(G'(2)) = (1 + o(1))n^2/2 . \qquad (9)$$

However, there are exactly four possible (ordered) 1-graphs on $\{u, v\}$ (depending on whether or not u and/or v are "edges," i.e., elements of E'. In each case, (9) follows immediately from (8), and shows that $P_4 \Rightarrow Q_4$.

However, the proof that $P_4 \Rightarrow P_2(s)$ in [1] in fact shows that $Q_4 \Rightarrow Q_5 \Rightarrow Q_1(s)$, which in turn implies Q_2, and therefore Q_3 (we will see these arguments for general k later in the article). This completes our discussion of Theorem 1 for the case $k = 2$.

Assume now for a fixed value of $k \geq 3$ that Theorem 1 holds for all values less than k. We will find it convenient to work with the modified statement:

Q_4': For almost all choices of $x, y \in V$, the sameness $(k-1)$-graph $G(x, y)$ of $G = (V, E)$ is quasi-random.

(By the induction hypothesis, $Q_4' \Leftrightarrow Q_4$.) By what we have already noted, Theorem 1 will be proved if we can show:

$$Q_3 \Rightarrow Q_4' \Rightarrow Q_5 \Rightarrow Q_1(s) .$$

This we now do.

5. $Q_3 \Rightarrow Q_4'$

Assume (3) holds for $G = G(n) = (V, E)$. We first need the following result.

Fact 2. *For any k-graph $F = F(n)$*

$$N^*_F(\mathcal{O}^e_k) \geq (1 + o(1))n^{2k}/2 . \qquad (10)$$

Proof of Fact 2. By Fact 1, we have (10) for $k = 2$. Suppose it holds for all values less than k. For $H = \mathcal{O}^e_k(x_1(0), x_1(1), \ldots, x_k(0), x_k(1))$, consider the sameness $(k-1)$-graph $H' = H(x_k(0), x_k(1))$. Thus, e' is an edge of H' if and only if

$$\chi_H(e' \cup \{x_k(0)\}) = \chi_H(e' \cup \}x_k(1)\}) ,$$

that is,

$$\chi_{H'}(e') \equiv \chi_H(e' \cup \{x_k(0)\}) + \chi_H(e' \cup \{x_k(1)\}) + 1 \pmod 2 . \qquad (11)$$

This implies by parity considerations that H' is in fact itself an EPO $\mathcal{O}^e_{k-1}$ (on the vertex set $(x_1(0), x_1(1), \ldots, x_{k-1}(0), x_{k-1}(1))$ in the sameness $(k-1)$-graph $F(x_k(0), x_k(1))$.

Therefore,

$$\begin{aligned} N_F^*(\mathcal{O}_k^e) &= \sum_{x_k(0),x_k(1)} N^*_{F(x_k(0),x_k(1))}(\mathcal{O}^e_{k-1}) \\ &\geq \sum_{x_k(0),x_k(1)} (1+o(1))n^{2k-2}/2 \quad \text{by induction} \qquad (12) \\ &\geq (1+o(1))n^{2k}/2 \end{aligned}$$

as claimed. ∎

Thus, by (3) we have in fact

$$N^*_{G(n)}(\mathcal{O}^e_k) = (1+o(1))n^{2k}/2 \qquad (13)$$

which by (12) implies that for almost all choices of $x = x_k(0)$, $y = x_k(1)$ in V,

$$N^*_{G(x,y)}(\mathcal{O}^e_{k-1}) = (1+o(1))n^{2k-2}/2\,. \qquad (14)$$

Hence, by the induction hypothesis of Theorem 1 applied to (3), (14) implies that almost all $G(x, y)$ are quasi-random. However, this is just Q_4'. ∎

6. $Q_4' \Rightarrow Q_5$

Assume Q_4' holds for $G = G(n) = (V, E)$. By the induction hypothesis (using $Q_1(2k-1)$ for $(k-1)$-graphs, and the fact that $Q_1(s+1) \Rightarrow Q_1(s)$), we have for almost all $x, y \in V$,

$$N_{G(x,y)}(K_r^{(k-1)}) = N^*_{G(x,\,y)}(K_r^{(k-1)}) = (1+o(1))n^r/2^{\binom{r}{k-1}}$$

for $1 \leq r \leq 2k-1$. Therefore, Q_5 holds. ∎

7. $Q_5 \Rightarrow Q_1(s)$

Let s be arbitrary but fixed. Assume that $G = G(n) = (V, E)$ satisfies Q_5. The plan is the following. We first show that $Q_5 \Rightarrow Q_2$. Since $Q_2 \Rightarrow Q_4'$ then almost all $G(x, y)$ are quasi-random. By the induction hypothesis, using Q_1 for $(k-1)$-graphs, we get

$$N_{G(x,\,y)}(K_t^{(k-1)}) = (1+o(1))n^t/2^{\binom{t}{k-1}} \qquad (15)$$

for any $t < s$. This then will allow us to prove $Q_1(s)$ for k-graphs.

The argument is modeled on the corresponding argument for 2-graphs in [1].

Let $G' = G'(2k)$ be a fixed (arbitrary) k-graph on the vertex set $V' = \{v_1, \ldots, v_{2k}\}$. For $1 \le r \le 2k$, define $G'(r)$ to be the sub-k-graph of G' induced by $V_r = \{v_1, \ldots, v_r\}$. Let $N_r := N_G^*(G'(r))$. We will prove by induction on r that

$$N_r = (1 + o(1))n_{(r)}/2^{\binom{r}{k}}. \tag{16}$$

For $r = 1$, (16) is immediate. Assume for some r, $1 \le r < 2k$, that (16) holds. Define $\alpha := (\alpha_1, \ldots, \alpha_r)$, where the α_i where the α_i are distinct elements of $[n] := \{1, 2, \ldots, N\}$, which we take to be V, the vertex set of $G = G(n)$. Also, define $\epsilon := (\epsilon(e_1), \epsilon(e_2), \ldots, \epsilon(e_z))$, where $\epsilon(e_i) \in \{0, 1\}$, $z = \binom{r}{k-1}$ and $e_1, e_2, \ldots, e_z$ denotes a fixed ordering of the edges of $K_r^{(k-1)}$. Finally, define

$$f_r(\alpha, \epsilon) := \left| \left\{ i \in [n] \mid i \ne \alpha_1, \ldots, \alpha_r \text{ and } \chi_G(\{i\} \cup e_j) = \epsilon(e_j), 1 \le j \le \binom{r}{k-1} \right\} \right|$$

Note that N_{r+1} is the sum of exactly N_r quantities $f_r(\alpha, \epsilon)$. Namely, for each embedding of $G'(r)$ into G, say $\lambda(v_u) = \alpha_u$, $1 \le u \le r$, $f_r(\alpha, \epsilon)$ counts the number of ways of choosing $i \in [n]$ so that if we extend λ to V_{r+1} by setting $\lambda(v_{r+1}) = i$, and take $\epsilon(e_j) = \chi_G(\{i\} \cup e_j)$, $1 \le j \le \binom{r}{k-1}$, then λ becomes an embedding of $G'(r+1)$ into G. Also note that there are just $n_{(r)} 2^{\binom{r}{k-1}}$ quantities $f_r(\alpha, \epsilon)$, since there are $n_{(r)}$ choices for α and $2^{\binom{r}{k-1}}$ choices for ϵ. Our next step will be to compute the first and second moments of $f_r(\alpha, \epsilon)$. To begin with, we have

$$\begin{aligned} \bar{f}_r &= \frac{1}{n_{(r)} 2^{\binom{r}{k-1}}} \sum_{\alpha, \epsilon} f_r(\alpha, \epsilon) \\ &= \frac{1}{n_{(r)} 2^{\binom{r}{k-1}}} \sum_{\alpha} \sum_{\epsilon} f_r(\alpha, \epsilon) \\ &= \frac{1}{n_{(r)} 2^{\binom{r}{k-1}}} \sum_{\alpha} (n - r) = \frac{n - r}{2^{\binom{r}{k-1}}} \end{aligned} \tag{17}$$

since every vertex $i \ne \alpha_1, \ldots, \alpha_r$, corresponds to a unique choice for ϵ. Thus,

$$\sum_{\alpha, \epsilon} f_r(\alpha, \epsilon) = n_{(r+1)}. \tag{18}$$

Next, define

$$S_r = \sum_{\alpha, \epsilon} f_r(\alpha, \epsilon)(f_r(\alpha, \epsilon) - 1).$$

We claim that:

$$S_r = \sum_{x, y} N_{G(x,y)}(K_r^{(k-1)}) \tag{19}$$

where, as usual, $G(x, y)$ is the sameness $(k-1)$-graph with respect to the vertices x and y. To see this, interpret S_r as counting the number of ways of choosing $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\epsilon = (\epsilon(e_1), \ldots, \epsilon(e_{\binom{r}{k-1}}))$ and two other (ordered) vertices x and y in $[n]$ so that

$$\chi_G(\{x\} \cup e_j) = \epsilon(e_j) = \chi_G(\{y\} \cup e_j)\,, \quad 1 \le j \le \binom{r}{k-1}\,.$$

Summing over all possible ϵ reduces this to requiring just that

$$\chi_G(\{x\} \cup e_j) = \chi_G(\{y\} \cup e_j)\,, \quad 1 \le j \le \binom{r}{k-1}\,. \tag{20}$$

Now, think of choosing x and y first. Then by (20), the required additional r vertices $\alpha_1, \ldots, \alpha_r$ must form a $K_r^{(k-1)}$ in $G(x, y)$. This proves (19).

Now, by the hypothesis (that Q_5 holds), we have

$$S_r = (1 + o(1))n^{r+2}/2^{\binom{r}{k-1}}\,. \tag{21}$$

We now compute the variance:

$$\begin{aligned}
&\sum_{\alpha,\epsilon} (f_r(\alpha, \epsilon) - \bar{f}_r)^2 \\
&\quad = \sum_{\alpha,\epsilon} f_r^2(\alpha, \epsilon) - \sum_{\alpha,\epsilon} \bar{f}_r^2 \\
&\quad = \sum_{\alpha,\epsilon} f_r(\alpha, \epsilon)(f_r(\alpha, \epsilon) - 1) + \sum_{\alpha,\epsilon} f_r(\alpha, \epsilon) - (n-r)^2 n_{(r)}/2^{\binom{r}{k-1}} \qquad (22) \\
&\quad = S_r + n_{(r+1)} - (n-r)n_{(r+1)}/2^{\binom{r}{k-1}} = o(n^{r+2})
\end{aligned}$$

by (21). Finally, since from our earlier observations that

$$N_{r+1} = \sum_{\substack{N_r \text{ choices} \\ \text{of } (\alpha,\epsilon)}} f_r(\alpha, \epsilon)$$

then

$$\begin{aligned}
|N_{r+1} - N_r\bar{f}_r|^2 &= \Bigg| \sum_{N_r \text{ terms}} (f_r(\alpha, \epsilon) - \bar{f}_r) \Bigg|^2 \\
&\le N_r \sum_{N_r \text{ terms}} (f_r(\alpha, \epsilon) - \bar{f}_r)^2 \quad \text{by Cauchy-Schwarz} \\
&\le N_r \sum_{\substack{\text{all} \\ \alpha,\epsilon}} (f_r(\alpha, \epsilon) - \bar{f}_r)^2 \qquad (23) \\
&= o(N_r \cdot n^{r+2}) \quad \text{by (22)} \\
&= o(n^{2r+2})
\end{aligned}$$

by induction, i.e., (16). Consequently,

$$|N_{r+1} - N_r\bar{f}_r| = o(n^{r+1}) \tag{24}$$

and so,

$$\begin{aligned} N_{r+1} &= N_r \bar{f}_r + o(n^{r+1}) \\ &= (1+o(1)) n_{(r)}/2^{\binom{r}{k}} \cdot (n-r)/2^{\binom{r}{k-1}} + o(n^{r+1}) \\ &= (1+o(1)) n_{(r+1)}/2^{\binom{r+1}{k}} \end{aligned}$$

as desired.

We can continue this argument as long as Q_5 applies, i.e., until $r = 2k-1$, at which point we have

$$N_{2k} = N_G^*(G'(2k)) = (1+o(1)) n_{(2k)}/2^{\binom{2k}{k}} . \tag{25}$$

Since $G'(2k)$ was arbitrary then (25) implies (2), and the proof that $Q_5 \Rightarrow Q_2$ is complete.

Now, because we have shown $Q_2 \Rightarrow Q_3 \Rightarrow Q_4'$ then Q_5 implies almost all $G(x, y)$ are quasi-random. This in turn implies (15) so that the preceding argument can in fact now be continued to obtain $Q_1(s)$, as desired. This completes the proof that $Q_5 \Rightarrow Q_1(s)$. ∎

We have now completed the proof of Theorem 1, since we have shown for any $s \geq 2k$,

$$\begin{array}{c} Q_4 \\ \Updownarrow \\ Q_1(s) \Rightarrow Q_2 \Rightarrow Q_3 \Rightarrow Q_4' \Rightarrow Q_5 \Rightarrow Q_1(s). \end{array}$$

∎

8. SOME CONSEQUENCES

Quasi-random k-graphs share a variety of other properties with random k-graphs, which are typically weaker than quasi-randomness. In this section we will discuss some of these.

Corollary 1. *Let $G = G(n) = (V, E)$ be a quasi-random k-graph and let $X \subset V$ with $|X| = \alpha n$, $\alpha > 0$. Then the restriction $G[X]$ is quasi-random.*

Proof. By Q_4', the sameness graphs $G(x, y)$ are quasi-random $(k-1)$-graphs for almost all choices of $x, y \in V$. However, for $x, y \in X$,

$$G(x, y)[X] = G[X](x, y) . \tag{26}$$

By Theorem 1 of [1], the corollary holds for $k = 2$. Assume for some value of $k \geq 3$ it holds for all values less than k. Thus, by induction $G(x, y)[X]$ is quasi-random for almost all choices of $x, y \in X$. Therefore, by (26) and Q_4', $G[X]$ is quasi-random. ∎

Note that by Corollary 1, the edges in a quasi-random k-graph are "well distributed," i.e., any αn vertices must span $(\frac{1}{2} + o(1)]\binom{\alpha n}{k}$ edges. On the other hand, this condition is *not* enough to guarantee quasi-randomness for $k \geq 3$, as the examples of Erdös/Sós [3] and [4] show. In Section 9 we will give a different construction.

Corollary 2. *A k-graph $G = G(n)$ is quasi-random if and only if*

$$\sum_{z} \sum_{\mathcal{O}^e_{k-1}=(V', E')} \left| \{z \mid \sum_{e' \in E'} \chi_G(\{z\} \cup e') \equiv 0 (mod\, 2)\} \right| = (1 + o(1))n^{2k-1}/2 . \tag{27}$$

The proof of this corollary follows rather directly by considering property Q_3. Note that (27) is equivalent to saying that almost all $\mathcal{O}^e_{k-1} = (V', E')$ satisfy the following property:

Q_6:

$$\left| \{z \in V \mid \sum_{e' \in E'} \chi_G(\{z \cup e'\}) \equiv 0 (\text{mod } 2)\} \right| = (1 + o(1))n/2 . \tag{28}$$

For a k-graph $G = (V, E)$, define for $x \in V$, the *projection* $G(x)$ as follows. $G(x)$ is a $(k-1)$-graph which has $V' = V \backslash \{x\}$ as its vertex set, and $E' = \{e' \in \binom{V'}{k-1} \mid \chi_G(e' \cup \{x\}) = 1\}$ as its edge set.

Corollary 3. *If G is quasi-random then so are almost all projections $G(x)$.*

Proof. Suppose G is a quasi-random. Since *any* projection $G(z)$ has

$$N^*_{G(z)}(\mathcal{O}^e_{k-1}) \geq (1 + o(1))n^{2k-2}/2 . \tag{29}$$

then (28) implies almost all z satisfy

$$N^*_{G(z)}(\mathcal{O}^e_{k-1}) = (1 + o(1))n^{2k-2}/2 .$$

However, this implies $G(z)$ is quasi-random (by property Q_3) for almost all z, as claimed. ∎

On the other hand, it is possible to give examples of k-graphs G which are *not* quasi-random but for which all projections $G(z)$ *are* quasi-random (see the next section).

Corollary 4. *Let $G = G(n) = (V, E)$ be a quasi-random k-graph, and let t be arbitrary but fixed. Then for almost all choices of distinct $B_i \in \binom{V}{k-1}$, $1 \leq i \leq t$,*

$$\left| \left\{ z \in V \mid \prod_{i=1}^{t} \chi_G(\{z\} \cup B_i) = 1 \right\} \right| = (1 + o(1))n/2^t . \tag{30}$$

This says roughly that the B_i almost always behave independently when we try to extend them simultaneously to become edges of G by adding a common vertex z to each of them.

Proof. Define $B = \{b_1, b_2, \ldots, b_r\} = \cup_{j=1}^{t} B_j$. For $X = \{x_1, x_2, \ldots, x_r\} \subset V$, let $m(X)$ denote the number of $z \in V$ so that each k-set $\{z\} \cup X_j$ is an edge of G, where $X_j = \{x_{i_1}, \ldots, x_{i_{k-1}}\}$ if and only if $B_j = \{b_{i_1}, \ldots, b_{i_{k-1}}\}$, $1 \le j \le t$. Let $H = (V', E')$ denote the k-graph formed by setting $V' = B \cup \{z_1, z_2\}$ where z_1, z_2 are distinct vertices in $V \backslash B$, and E' consists of all k-sets $\{z_i\} \cup B_j$, $1 \le i \le 2$, $1 \le j \le t$, and let H^- denote $H[B \cup \{z_i\}]$.

By this construction, we have

$$N_G(H) = \sum_{|X|=r} m(X)_{(2)} \tag{31}$$

where $N_G(H)$ denotes the number of (possibly noninduced) copies of H in G.

Since G is quasi-random then property $Q_1(r+2)$ for $N_G^*(H^+)$, as H^+ ranges over all k-graphs (V', E^+) with $E^+ \supset E'$, implies

$$N_G(H) = (1 + o(1)) n^{r+2} / 2^{2t} . \tag{32}$$

Now, applying the Cauchy–Schwarz inequality to the RHS of (31), we obtain

$$\sum_{|X|=r} m(X)_{(2)} \ge (1 + o(1)) \frac{1}{n^r} \Big(\sum_{|X|=r} m(X) \Big)^2 . \tag{33}$$

Therefore, by (31), (32), and (33),

$$\sum_{|X|=r} m(X) \le (1 + o(1)) n^{r+1} / 2^t . \tag{34}$$

However,

$$\sum_{|X|=r} m(X) = N_G(H^-) = \sum_z N_{G(z)}(B^*) \tag{35}$$

where $B^* = (B, E^*)$ denotes the $(k-1)$-graph on B with edges B_j, $1 \le j \le t$. By Corollary 3, we have

$$\sum_z N_{G(z)}(B^*) = (1 + o(1)) n^{r+1} / 2^t . \tag{36}$$

Putting these all together gives

$$\sum_{|X|=r} m(X)_{(2)} = (1 + o(1)) n^{r+2} / 2^{2t} .$$

Thus, by Cauchy–Schwarz, almost all the terms $m(X)$ must essentially equal their average, i.e.,

$$m(X) = (1 + o(1)) n / 2^t \tag{37}$$

holds for almost all choices of $|X| = r$. However, this is just the content of (30). ∎

In the next section we will show (30) is not sufficient to guarantee quasi-randomness.

9. EXAMPLES OF NON-QUASI-RANDOM *k*-GRAPHS

We next construct several classes of k-graphs which, while not quasi-random, behave in many ways like quasi-random k-graphs.

To begin, let $G = G_{1/2}(n) = (V, E)$ denote a *random* $(k-1)$-graph on an n-set V. Form the k-graph $H^* = H^*(G) = (V, E^*)$ by defining

$$E^* = \left\{ e^* \in \binom{V}{k} \middle| e^* \text{ contains an \textit{even number} of edges of } G \right\}. \qquad (38)$$

Symbolically, for $e^* \in \binom{V}{k}$

$$\chi_{H^*}(e^*) \equiv 1 + \sum_{e \in \binom{e^*}{k-1}} \chi_G(e) \pmod 2. \qquad (39)$$

Fact 3. *For almost all choices of $G = G_{1/2}(n)$, $H^*(z)$ is quasi-random for all $z \in V$.*

Proof. The result is immediate for $k = 2$. Assume $k \geq 3$ and fix $z \in V$. Note that for $e^* = \{z\} \cup e$, $\chi_{H^*(z)}(e) = \chi_{H^*}(e^*)$. We will estimate the probability that

$$N_{H^*(z)}(\mathcal{O}^e_{k-1}) = (1 + o(1))n^{2k-2}/2. \qquad (40)$$

We abbreviate the dependence

$$\mathcal{O}^e_{k-1} = \mathcal{O}^e_{k-1}(x_1(0), x_1(1), \ldots, x_{k-1}(0), x_{k-1}(1))$$

by $\mathcal{O}^e_{k-1}(\bar{x}(\bar{\epsilon}))$ where $\bar{\epsilon} = (\epsilon_1, \ldots, \epsilon_{k-1})$, $\epsilon_i \in \{0, 1\}$.

Let

$$EPO(\bar{x}(\bar{\epsilon})) = \begin{cases} 1 & \text{if all edges of } \mathcal{O}^e_{k-1}(\bar{x}(\bar{\epsilon})) \text{ are edges of } H^*(z), \\ 0 & \text{otherwise}, \end{cases}$$

and let

$$X := \sum_{\bar{x}(\bar{\epsilon})} EPO(\bar{x}(\bar{\epsilon})).$$

Thus, X counts the number of EPOs $\mathcal{O}^e_{k-1}$ occurring in $H^*(z)$. We next estimate the mean and variance of X.

$$E(X) = \sum_{\bar{x}(\bar{\epsilon})} E(EPO(\bar{x}(\bar{\epsilon}))) = \sum_{\bar{z}(\bar{\epsilon})} \frac{1}{2} = \frac{1}{2}\, n_{(2k-2)}. \qquad (41)$$

$$
\begin{aligned}
Var(X) &= E(X^2) - E(X)^2 \\
&= E\Big(\sum_{\substack{\bar{x}(\bar{\epsilon}) \\ \bar{x}'(\bar{\epsilon}')}} EPO(\bar{x}(\bar{\epsilon}))EPO(\bar{x}'(\bar{\epsilon}'))\Big) - E(X)^2 \\
&= \sum_{\substack{\bar{x}(\bar{\epsilon}) \\ \bar{x}'(\bar{\epsilon}')}} E(EPO(\bar{x}(\bar{\epsilon}))EPO(\bar{x}'(\bar{\epsilon}'))) - E(X)^2 \qquad (42) \\
&= \sum_{|\bar{x}(\bar{\epsilon}) \cap \bar{x}'(\bar{\epsilon}')| < k-1} E(EPO(\bar{x}(\bar{\epsilon})EPO(\bar{x}'(\bar{\epsilon}'))) \\
&\quad + \sum_{|\bar{x}(\bar{\epsilon}) \cap \bar{x}'(\bar{\epsilon}')| \geq k-1} E(EPO(\bar{x}(\bar{\epsilon})EPO(\bar{x}'(\bar{\epsilon}'))) - E(X)^2 .
\end{aligned}
$$

In the first sum, the two factors $EPO(\bar{x}(\bar{\epsilon}))$ and $EPO(\bar{x}'(\bar{\epsilon}'))$ are *independent*, since edges in H (and $H(z)$) depend only on edges (=$(k-1)$-sets) in G; by (39) each factor is equal to 1/2. Continuing (42) we have

$$
\begin{aligned}
Var(X) &\leq \sum_{\substack{\bar{x}(\bar{\epsilon}) \\ \bar{x}'(\bar{\epsilon}')}} 1/4 + \sum_{|\bar{x}(\bar{\epsilon}) \cap \bar{x}'(\bar{\epsilon}')| \geq k-1} E(X)^2 \\
&\leq E(X)^2 + c_k n^{3(k-1)} - E(X)^2 , \quad c_k \text{ constant} \qquad (43) \\
&= O(n^{3(k-1)}) .
\end{aligned}
$$

Thus, by Chebyshev's inequality (see [5]),

$$Pr\{|X - E(X)| \geq c\} \leq Var(X)/c^2 , \quad c > 0 . \qquad (44)$$

Setting $c = n^{2k-2}/\log n$ we have

$$Pr\Big\{\Big|X - \frac{1}{2} n_{(2k-2)}\Big| \geq n^{2k-2}/\log n\Big\} = 0\Big(\frac{\log^2 n}{n^{k-1}}\Big) \qquad (45)$$

where, of course, $X = X(z)$. Summing over all $z \in V$, we get

$$\sum_{z \in V} Pr\Big\{\Big|X(z) - \frac{1}{2} n_{(2k-2)}\Big| \geq n^{2k-2}/\log n\Big\} = O\Big(\frac{\log^2 n}{n^{k-2}}\Big) . \qquad (46)$$

Thus, for $k \geq 3$, almost all choices of G have

$$X(z) = (1 + o(1))n^{2k-2}/2$$

for *all* $z \in V$. However, this implies by property Q_3 of Theorem 1 that all $H(z)$ are quasi-random. This proves Fact 3. ∎

Corollary 5. *Let $H^* = (V, E^*)$ be as above. Then for almost all $G = G_{1/2}(n)$, if $\alpha > 0$ is fixed then any αn vertices of V span $(\frac{1}{2} + o(1))\binom{\alpha n}{k}$ edges of H^*.*

Proof. By Fact 3, almost all choices of G result in all $H^*(z)$ being quasi-random. By the remark following Corollary 1, each such $H^*(z)$ has its edges "well distributed" in the above sense. This implies that the edges of H^* are themselves "well distributed," i.e., any αn vertices span $(\frac{1}{2} + o(1))\binom{\alpha n}{2}$ edges of H^*. ∎

The same techniques used in the proof of Fact 3 can also be used to prove the following (compare with Corollary 4).

Fact 4. *Almost all $H^* = H^*(G)$ have the property that for fixed t and almost all choices of distinct $B_i \in \binom{V}{k-1}$, $1 \le i \le t$,*

$$\left|\left\{z \in V \,\Big|\, \prod_{i=1}^{t} \chi_{H^*}(\{z\} \cup B_i) = 1\right\}\right| = (1 + o(1))n/2^t .$$

We now show that in spite of the preceding properties of $H^*(G_{1/2}(n))$, it is far from being quasi-random. Let K_{k+1}^- denote the k-graph on a set W of $k+1$ vertices which has all but one of the $k+1$ k-subsets of W as its edges.

Fact 5.

$$N_{H^*}^*(K_{k+1}^-) = 0 .$$

Proof. Construct a matrix M with rows indexed by k-sets $A_i \subset W$ and columns indexed by $(k-1)$-sets $B_j \subset W$. Define

$$M(A_i, B_j) = \begin{cases} 1 & \text{if } A_i \subset B_j , \\ 0 & \text{otherwise} . \end{cases}$$

Note that each column B_j has exactly two 1's in it, since $|W \backslash B_j| = 2$, and consequently there are just 2 choices with which to augment B_j to get an A_i.

Suppose now that in fact $W \subset V$ induces a copy of K_{k+1}^- in H^*, with the k-set A_1 being the edge missing in K_{k+1}^-. Since edges of H^* depend only on the edges of G they contain, we will replace M by M', by restricting M just to those columns e_j which are *edges* of G. The sum of all rows of M' is the vector $(2, 2, \ldots, 2)$ of all 2's. Also, since all A_i, $i > 1$, are edges of K_{k+1}^-, then each of these rows must contain an *even* number of 1's. Similarly, since A_1 is not an edge of K_{i+1}^- then its row must contain an odd number of 1's. However, this is clearly impossible (since the total sum is $(2, 2, \ldots, 2)$), so we conclude that K_{k+1}^- does not occur as an induced sub-k-graph of H^*. This proves Fact 5. ∎

Of course, the non-occurrence of any particular induced $(k+1)$-vertex k-graph in H^* is a blatant violation of quasi-randomness (property $Q_1(k+1)$). As we have seen in Theorem 1, property $Q_1(2k)$ is enough to force a k-graph to be quasi-random. In our next example, we exhibit a non-quasi-random k-graph $G^*(n)$ which satisfies property $Q_1(k+1)$. Whether this is possible for $Q_1(k+2)$ is currently unknown. (This has now been settled. See note at the end of the article.)

In forming $G^*(n)$ there are two possibilities, depending on the parity of k. For k *odd*, we do the following. The vertex set V of $G^*(n)$ will consist of two disjoint sets A and $\bar{A}$, each of size $n/2$. The *edges* of $G^*(n)$ consist of those k-sets e such that $|e \cap A|$ is *even*.

For k *even*, the construction is slightly more complex. In this case, V consists of four disjoint sets A, A', B, B', each of size $n/4$. To form the edge set of $G^*(n)$ we choose those k-sets intersecting both $A \cup A'$ and $B \cup B'$ (independently) with probability $1/2$, together with those k-sets e satisfying either:

$$|e \cap A| \text{ is even}, \quad e \cap B = e \cap B' = \emptyset;$$

or

$$|e \cap B| \text{ is odd}, \quad e \cap A = e \cap A' = \emptyset;$$

In either case, $G^*(n)$ is not quasi-random since $G^*(n)$ contains a clique, i.e., complete k-graph, of size $n/4$.

Fact 6. *$G^*(n)$ satisfies property $Q_1(k+1)$.*

That is, for any k-graph $H = H(k+1)$ on $k+1$ vertices,

$$N^*_{G^*(n)}(H) = (1 + o(1))n^{k+1}/2^{k+1}. \tag{47}$$

Proof. We only treat the case of k odd. The case of k even is similar (though slightly more complicated). Let H have vertex set $W = \{w_1, \ldots, w_{k+1}\}$ and edges $e(i_1), \ldots, e(i_r)$ where $e(i_j)$ denotes the edge $W \backslash \{w_{i_j}\}$. Define $W' = \{w_{i_j} | 1 \le j \le r\}$ and $W'' := W \backslash W'$. Suppose $\lambda : W \to V$ induces a copy of H in $G^*(n)$. Parity considerations show that there are only two possibilities:

$$\text{If } r \text{ is } odd \text{ then} \quad \lambda : W' \to A \text{ and } \lambda : W'' \to \bar{A};$$

$$\text{If } r \text{ is } even \text{ then} \quad \lambda : W' \to \bar{A} \text{ and } \lambda : W'' \to A.$$

Thus, in either case there are $(1 + o(1))(n/2)^{k+1}$ choices for λ. This is just what (47) claims, and therefore Fact 6 is proved. ∎

10. EXAMPLES OF QUASI-RANDOM k-GRAPHS

In this section we give examples of several explicit classes of quasi-random k-graphs. Many other similar classes can be constructed but we limit ourselves to a few of the simplest here. Of course, almost all random k-graphs are quasi-random.

The "even intersection" k-graph $I_k(n) = (V, E)$ is defined as follows. For V we take $2^{[n]}$, the class of all subsets of $[n]$. A k-set $\{X_1, \ldots, X_k\} \in E$ if and only if

$$|X_1 \cap \cdots \cap X_k| \equiv 0 (\text{mod } 2)$$

(where $X_i \subset V$).

Fact 7. *$I_k(n)$ is quasi-random.*

Proof. We will use the characterization of Corollary 4, and more specifically, equation (28). Let $X_i(\epsilon_j)$, $\epsilon_j \in \{0, 1\}$, $1 \le j \le 2$, $1 \le i \le k-1$, be distinct subsets of $[n]$. We consider an octahedron $\mathcal{O}_{k-1} = \mathcal{O}_{k-1}(X_1(0), X_1(1), \ldots, X_{k-1}(0), X_{k-1}(1))$ in $I_k(n)$. Recall that edges of $\mathcal{O}_{k-1}$ are sets of the form $\{X_1(\epsilon_1), \ldots, X_{k-1}(\epsilon_{k-1})\}$, $\epsilon_j \in \{0, 1\}$. For a subset $Z \subset [n]$, the k-set $\{Z, X_1(\epsilon_1), \ldots, X_{k-1}(\epsilon_{k-1})\}$ is an edge of $I_k(n)$ provided $|Z \cap X_i(\epsilon_i) \cap \cdots \cap X_{k-1}(\epsilon_{k-1})| \equiv 0 (\mathrm{mod}\ 2)$. The total number of edges of $I_k(n)$ formed by Z together with edges of $\mathcal{O}_{k-1}$ is just

$$\sum_{\epsilon_1, \ldots, \epsilon_{k-1}} |Z \cap X_1(\epsilon_1) \cap \cdots \cap X_{k-1}(\epsilon_{k-1})| \bmod 2 , \tag{48}$$

where $a \bmod 2$ is 0 if a is even, and 1 if a is odd. What we want to count is the number of Z for which the expression in (48) is *even*, since in this case an even number of terms in the sum must then be even.

First, note that any element $x \in X_i(0) \cap X_i(1)$ contributes to an *even* number of terms in (48). Thus, we do not change the parity of the expression if we replace

$$\begin{cases} X_i(0) \text{ by } X'_i(0) := X_i(0) \backslash X_i(1) , \\ X_i(1) \text{ by } X'_i(1) := X_i(1) \backslash X_i(0) . \end{cases}$$

Note that $X'_i(0) \cap X'_i(1) = \emptyset$, $1 \le i \le k-1$.

So, we have reduced our problem to counting the number of Z for which

$$\sum_{\epsilon_1, \ldots, \epsilon_{k-1}} |Z \cap X'_1(\epsilon_i) \cap \cdots \cap X'_{k-1}(\epsilon_{k-1})| \tag{49}$$

is even. Fact 7 will be proved if we can show that for almost all choices of the $X_i(\epsilon_j)$, the number of such Z is $(1 + o(1)) \frac{|V|}{2} = (1 + o(1))2^{n-1}$.

Let

$$s(\epsilon_1, \ldots, \epsilon_{k-1}) := |X'_1(\epsilon_1) \cap \cdots \cap X'_{k-1}(\epsilon_{k-1})| . \tag{50}$$

Since all the 2^{k-1} expressions on the RHS of (50) are disjoint, then the number of such Z is just

$$\sum_{i(\bar\epsilon)}{}' \prod_{\bar\epsilon} \binom{s(\bar\epsilon)}{i(\bar\epsilon)} 2^{n-s} \tag{51}$$

where the sum Σ' is taken over all $i(\bar\epsilon)$ such that $\Sigma_{\bar\epsilon}\, i(\bar\epsilon) \equiv 0 (\mathrm{mod}\ 2)$, $\bar\epsilon$ denotes $(\epsilon_1, \ldots, \epsilon_{k-1})$, and $s := \Sigma_{\bar\epsilon}\, s(\bar\epsilon)$. The interpretation of (51) is simply that the sum counts the number of ways of choosing a Z which has $i(\bar\epsilon)$ elements in $X'_1(\epsilon_1) \cap \cdots \cap X'_{k-1}(\epsilon_{k-1})$. (Of course, the expression in (48) is not affected if Z is changed by any subset of $[n] \backslash \cup_{i,j} X_i(\epsilon_j)$; this accounts for the factor 2^{n-s}.

However, observe that

$$\sum_{i(\bar{\epsilon})} (-1)^{\sum_{\bar{\epsilon}} i(\bar{\epsilon})} \prod_{\bar{\epsilon}} \binom{s(\bar{\epsilon})}{i(\bar{\epsilon})} 2^{n-s} = 0 \tag{52}$$

since this is just the value of the expression

$$\prod_{s(\bar{\epsilon})} (x-1)^{s(\bar{\epsilon})}$$

(expanded using the binomial theorem) when $x = 1$. Thus, the expression in (51), summed over all $i(\bar{\epsilon})$ with $\Sigma_{\bar{\epsilon}}\, i(\bar{\epsilon})$ *even* is just 1/2 of the total sum

$$\sum_{i(\bar{\epsilon})} \prod_{\bar{\epsilon}} \binom{s(\bar{\epsilon})}{i(\bar{\epsilon})} 2^{n-s} = 2^n \tag{53}$$

(again, by the binomial theorem), as required. Since almost all random choices of the $X_j(\epsilon_j)$ will result in distinct sets, then the preceding argument shows that (28) holds, and consequently, $I_k(n)$ is quasi-random. ∎

We point out that it was shown by Bollobás and Thomason [6] that for $k = 2$, $N^*_{I_2(2^r)}(G(r)) > 0$ for any r-vertex graph $G(r)$, which points to the potential quasi-randomness of $I_2(k)$.

For our second example we will define for primes p, a (generalized) Paley k-graph $P_k(p)$ as follows. The vertex set of $P_k(p)$ is the set $\mathbb{Z}_p$ of integers modulo p. A k-set $\{i_1, \ldots, i_k\}$ is an edge of $P_k(p)$ if and only if $i_1 + \cdots + i_k$ is a *quadratic residue* modulo p.

Fact 8. *$P_k(p)$ is quasi-random.*

Proof. The proof will be a straightforward application of the following character sum estimate of Burgess [7] (see also Weil [8]). Let χ denote the nonprincipal character modulo p given by

$$\chi(a) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1 & \text{otherwise.} \end{cases}$$

Then for distinct $a_1, \ldots, a_s$ in $\mathbb{Z}_p$,

$$\left| \sum_{x \in \mathbb{Z}_p} \chi(x + a_1) \cdots \chi(x + a_s) \right| \le (s-1)\sqrt{p}\,. \tag{54}$$

(Of course, (54) holds when for nondistinct a_i, provided the product is not identically 1.) We will use (54) to verify property Q_3. To do this, we need to estimate the number of $\mathcal{O}_k^e = \mathcal{O}_k^e(x_1(0), x_i(1), \ldots, x_k(0), x_k(1))$ in $P_k(p)$. It follows from the definition of $P_k(p)$ that this number is just

$$\begin{aligned}N^*_{P_k(p)}(\mathcal{O}^e_k) &= \sum_{i=1}^{k} \sum_{x_i(0),x_i(1)} \frac{1}{2}\left(1 + \prod_{\epsilon_1,\ldots,\epsilon_k} \chi(x_1(\epsilon_1)+\cdots+x_k(\epsilon_k))\right)\\ &= \frac{1}{2}\sum_{i=1}^{k} \sum_{x_i(0),x_i(1)} 1\\ &\quad + \frac{1}{2}\sum_{x_i(0)}\sum_{i=2}^{k}\sum_{x_i(0),x_i(1)}\prod_{\epsilon_2,\ldots,\epsilon_k}\chi(x_1(0)+x_2(\epsilon_2)+\cdots+x_k(\epsilon_k))\\ &\quad + \frac{1}{2}\sum_{x_i(1)}\sum_{i=2}^{k}\sum_{x_i(0),x_i(1)}\prod_{\epsilon_2,\ldots,\epsilon_k}\chi(x_1(1)+x_2(\epsilon_2)+\cdots+x_k(\epsilon_k))\\ &\le \frac{1}{2}\,p^{2k} + O(p^{2k-1/2})\end{aligned}$$

by (54). Thus, (3) holds for almost all choices of the $x_i(\epsilon_j)$, and Fact 8 is proved. ∎

We remark that Fact 8 for $k=2$ follows from the results in Graham/Spencer [9] (see also [6]). The techniques used in proving Fact 7 can also be used to show that other related k-graphs are quasi-random, e.g., the k-graph having as its vertices the n-sets of $[2n]$, and edges $\{X_1,\ldots,X_k\}$, $X_i \in \binom{[2n]}{n}$, where $|X_1 \cap \cdots \cap X_k| \equiv 0 \pmod 2$.

11. QUESTIONS

We conclude with several questions about quasi-random k-graphs we have not resolved.

To begin with, regarding property $Q_1(s)$, how large must s be before $Q_1(s)$ implies quasi-randomness? By Theorem 1, $s=2k$ is sufficient. By the example in Section 9 (Fact 6), $s=k+1$ is not sufficient. In particular, does property $Q_1(k+2)$ imply quasi-randomness? Rephrasing this question for $k=3$: Is it true that if a 3-graph $G(n)$ contains all 5-vertex 3-graphs as induced 3-graphs asymptotically equally often, then the same is true for all 6-vertex 3-graphs?

For each fixed t, is there a 2-graph $G(n)$ so that $N_{G(n)}(K_r) = (1+o(1))n^r/2^{\binom{r}{2}}$, $2 \le r \le t$, but so that $G(n)$ is not quasi-random? Or even stronger, so that $N_{G(n)}(K_{t+1}) \ne (1+o(1))n^{t+1}/2^{\binom{t+1}{2}}$? (Here, K_r denotes the (ordinary) complete graph on r vertices.) We certainly believe that such $G(n)$ exist but we do not at present have a proof of this, even for the case $t=5$!

This problem is related to property Q_5 in the following way. If a 3-graph $H=H(n)$ has almost all its sameness 2-graphs $H(x,y)$ satisfying $N_{H(x,y)}(K_r) = (1+o(1))n^r/2^{\binom{r}{2}}$, for $1 \le r \le 5$, then H is quasi-random. In particular, this implies almost all $H(x,y)$ are quasi-random, and so, for example, $N_{H(x,y)}(K_6) = (1+o(1))n^6/2^{\binom{6}{2}}$ (or $(1+o(1))n^t/2^{\binom{t}{2}}$ for K_t with fixed t) for almost all $H(x,y)$. However, this certainly does not imply that this should hold for any *particular* $H(x,y)$. Not surprisingly, we do not know the answer to these questions for general k-graphs either.

In the spirit of [10] and [3], it would be of interest to have explicit estimates for how non-quasi-random k-graphs deviate from the various properties Q_i.

Perhaps even more fundamental is the problem of finding additional properties which characterize quasi-randomness. For example, the following property FR has been suggested by Frankl and Rödl [11] as a possible candidate (in the case of 3-graphs).

Suppose $G = G(n) = (V, E)$ is a 3-graph so that for every 2-graph $H = (V, E')$,

$$FR: \qquad \left|\left\{e \in E \middle| \binom{e}{2} \subset E'\right\}\right| = \frac{1}{2} N_H(K_3) + o(n^3). \qquad (56)$$

In other words, for any H which has cn^3 triangles, about half of them correspond to edges in G. (Of course, the extension of this property to k-graphs is clear.) At present we cannot prove either $FR \Rightarrow$ quasi-random (which we believe) or quasi-random $\Rightarrow FR$ (which we do not).

It also seems to the authors that it would be profitable to explore quasi-randomness extended to simulating random k-graphs $G_p(n)$ for $p \neq 1/2$, or more generally, for $p = p(n)$, especially along the lines carried out so fruitfully by Thomason [12, 13]. It is the author's belief that the surface of this interesting topic has thus far only been scratched.

Note added in proof: It is now known that $Q_1(5)$ is *not* sufficient to imply quasi-randomness of a 3-graph. Also, it has now been shown that property FR (and its natural extensions to k-graphs) is *equivalent* to quasi-randomness. Details will appear in Ref. 14. □

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