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Old and New Proofs of the Erdös-Ko-Rado Theorem

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ABSTRACT

The Erdös-Ko-Rado Theorem is a central result of combinatorics which opened the way for the rapid development of extremal set theory. Proofs of it are reviewed and a new generalization is given. For a survey of results related to the Erdös-Ko-Rado Theorem see [DF].

Key Words Erdös-Ko-Rado theorem, intersecting family, extremal set theory, combinatorics.
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1 Introduction

Let X be a finite set of n elements. Usually we suppose that $X=\{1, 2\cdots\cdots, n\}$. Let 2^X be the power set of X and $\binom{X}{k}$ the set of all k-subsets of X. A family $\mathscr{F}\subset 2^X$ is called intersecting if $F\cap F'\neq\phi$ holds for all $F, F'\in\mathscr{F}$.

Theorem 0 If $\mathscr{F}\subset 2^X$ is intersecting then

$$|\mathscr{F}|\leqslant 2^{n-1} \quad \text{holbs.} \tag{1}$$

Proof There are 2^{n-1} pairs $\{C, X-C\}$ of complementary subsets of X. Since $C\cap(X-C)=\phi$, $|\mathscr{F}\cap\{C, X-C\}|\leqslant 1$ holds for each of them. ■

Erdös, Ko and Rado [EKR] were the first to observe the validity of (1) and they proved that there are very many families $\mathscr{F}$, achieving equality in (1). More exactly, they proved that for every intersecting family $\mathscr{G}\subset 2^X$ there exists another intersecting family $\mathscr{F}$, $\mathscr{G}\subset\mathscr{F}\subset 2^X$, such that $|\mathscr{F}|=2^{n-1}$ holds.

The Erdös-Ko-Rado Theorem deals with the much more difficult case when $|F|=k$ is assumed for all $F\in\mathscr{F}$, i. e., $\mathscr{F}$ is a k-graph.

Theorem 1 (Erdös-Ko-Rado Theorem, special case). Suppose that $\mathscr{F}\subset$

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$\binom{X}{k}$ is intersecting, $n \geq 2k$. Then

$$|\mathscr{F}| \leqslant \binom{n-1}{k-1}. \tag{2}$$

The main purpose of the present paper is to review all known (to the authors) proofs and give some generalizations to other hypergraphs.

For an integer $t \geq 1$, a family $\mathscr{F}$ is called t-intersecting if $|F \cap F'| \geq t$ holds for all F, $F' \in \mathscr{F}$.

To close this section let us state the general case of the Erdös-Ko-Rado Theorem.

Theorem 2 (Erdös-Ko-Rado Theorem, general case). Suppose that $\mathscr{F} \subset \binom{X}{k}$ is t-intersecting and $n \geq n_0(k, t)$. Then

$$|\mathscr{F}| \leqslant \binom{n-t}{k-t}. \tag{3}$$

Remark By now it is known that the best possible value of $n_0(k, t)$ is $(k-t+1)(t+1)$(cf. [F1] and [W]).

2 Shifting

That is how the original proof went. Since then shifting has become one of the most powerful tools in extremal set theory.

Definition 2.1 *The (i, j)-shift*. For a family $\mathscr{F} \subset 2^X$ and $1 \leqslant i < j \leqslant n$, define $S_{ij}(\mathscr{F}) = \{S_{ij}(F) : F \in \mathscr{F}\}$ where

$$S_{ij}(F) = \begin{cases} F' = (F - \{j\}) \cup \{i\} & \text{if } j \in F,\ i \notin F \text{ and } F' \notin \mathscr{F}, \\ F & \text{otherwise.} \end{cases}$$

Proposition 2.2 (i) $|S_{ij}(F)| = |F|$; (ii) $|S_{ij}(\mathscr{F})| = |\mathscr{F}|$; (iii) If $\mathscr{F}$ is intersecting then so is $S_{ij}(\mathscr{F})$.

Proof (i) and (ii) are immediate from the definition. To prove (iii), suppose by contradiction that there exist sets F, G in the intersecting family $\mathscr{F}$ such that

$$S_{ij}(F) \cap S_{ij}(G) = \phi \text{ holds.} \tag{4}$$

Since $F \cap G \neq \phi$ by assumption, and the only element which can be deleted is j, it follows that $F \cap G = \{j\}$.

If both F and G changed by the (i, j)-shift, then $i \in S_{ij}(F) \cap S_{ij}(G)$ would hold, contradicting(4). Thus we may assume that $S_{ij}(F) = F$, $S_{ij}(G) = (G - \{j\}) \cup \{i\}$.

Similarly, $i \in F$ would contradict (4). Thus the only reason not to replace F by $F' = (F - \{j\}) \cup \{i\}$ during the $i-j$-shift is because $F' \in \mathscr{F}$. However, $F' \cap G = F \cap S_{ij}(G) = S_{ij}(F) \cap S_{ij}(F) = \phi$, a contradiction. ■

We can now prove (2).

The first proof of the Erdös-Ko-Rado Theorem. Apply induction on n and prove it simultaneously for all $k \leqslant n/2$.

(a) $n=2k$. We argue as with the proof of (1). The $\binom{2k}{k}$ k-subsets of X can be partitioned into $\frac{1}{2}\binom{2k}{k}=\binom{2k-1}{k-1}$ pairs of complementary sets, not both of which can be in an intersecting family. This yields $|\mathscr{F}| \leqslant \binom{2k-1}{k-1}$, as desired.

(b) $n>2k$, Define $\mathscr{F}_0=\mathscr{F}$, $\mathscr{F}_i=S_{in}(\mathscr{F}_{i-1})$, $i=1, \cdots, n-1$. By Proposition 2.2 we have $|\mathscr{F}|=|\mathscr{F}_{n-1}|$, and $\mathscr{F}_{n-1} \subset \binom{X}{k}$ is intersecting.

Define $\mathscr{G}=\{F \in \mathscr{F}_{n-1}: n \notin F\}$, $\mathscr{H}=\{F-\{n\}: n \in F \in \mathscr{F}\}$.

Clearly $|\mathscr{F}|=|\mathscr{G}|+|\mathscr{H}|$, $\mathscr{G} \subset \mathscr{H}$, and thus by induction $|\mathscr{G}| \leqslant \binom{n-2}{k-1}$ holds. Consequently, $|\mathscr{H}| \leqslant \binom{n-2}{k-2}$ would be sufficient to show $|\mathscr{F}| \leqslant \binom{n-2}{k-1}+\binom{n-2}{k-2}=\binom{n-1}{k-1}$.

The desired upper bound for the cardinality of $\mathscr{H} \subset \binom{\{1, 2, \cdots, n-1\}}{k-1}$ will follow from the induction hypothesis once we prove the following.

Proposition 2.3 $\mathscr{H}$ is intersecting.

Proof Suppose the contrary, i.e., there exist disjoint sets H, $H' \in \mathscr{H}$. Since $|H \cup H'|=2(k-1)<n-1$, there exists some i, $1 \leqslant i<n$ satisfying $i \notin H \cup H'$. By definition $F=H \cup \{n\}$ is in $\mathscr{F}_{n-1}$. Since $n \in F$ then $F \in \mathscr{F}$, and consequently, $F \in \mathscr{F}_i$ holds for all $1 \leqslant i \leqslant n-1$. This means that $S_{in}(F)=F$, i.e., F did not get replaced during the (i, n)-shift. This can happen only if $(F-\{n\}) \cup \{i\}=(H \cup \{i\}) \in \mathscr{F}_{i-1}$ and consequently $(H \cup \{i\} \in \mathscr{F}_{n-1}$ hold.

However, $(H \cup \{i\}) \cap (H' \cup \{n\})=\phi$, a contradiction. ■

3 Shadows

Given a k-graph $\mathscr{F}$ and an integer l, $1 \leqslant l \leqslant k$, the *l-shadow* $\sigma_l(\mathscr{F})$ is defined as follows.

$$\sigma_l(\mathscr{F})=\{G: |G|=l, \quad \text{and for some } F \in \mathscr{F},\ G \subset F\}.$$

Given an integer m and a k-graph $\mathscr{F}$ of cardinality m, what can one say about $|\sigma_l(\mathscr{F})|$? Clearly, $|\sigma_l(\mathscr{F})| \leqslant \binom{k}{l}|\mathscr{F}|$ holds, with equality if and only if $|F \cap F'| < l$ holds for all distinct $F, F' \in \mathscr{F}$.

The real problem is to get best possible lower bounds. The answer is given by the Kruskal-Katona Theorem, one of the most widely used results concerning finite sets. We shall only state and prove a numerical consequence of it which is due to Lovász.

Kruskal-Katona Theorem ([Kr], [Ka2], [L1]). Let $\mathscr{F}$ be a k-graph, and suppose $|\mathscr{F}| \geqslant \binom{x}{k}$ with $x \geqslant k$, real. Then

$$|\sigma_l(\mathscr{F})| \geqslant \binom{x}{l} \text{ holds for all } 0 \leqslant l \leqslant k. \tag{5}$$

First note that it is sufficient to prove (5) for the case $l = k-1$ (and then apply this case $k-l$ times noting the monotonicity of $\binom{x}{s}$ as a function of x).

The proof which we are going to present is from [F2] and is based upon the fact that the (i, j)-shift does not increase the shadow.

Proposition 3.1 Let $\mathscr{F} \subset \binom{X}{k}$ be a k-graph, and suppose $1 \leqslant i < j \leqslant n$. Then

$$\sigma_{k-1}(S_{ij}(\mathscr{F})) \subset S_{ij}(\sigma_{k-1}(\mathscr{F})). \tag{6}$$

Equation (6) can be proved by a relatively simple case by case analysis, which we leave to the reader.

Define inductively $\mathscr{F}_1 = \mathscr{F}$ and $\mathscr{F}_i = S_{1i}(\mathscr{F}_{i-1})$, $2 \leqslant i \leqslant n$. In view of (6) we have $|\sigma_{k-1}(\mathscr{F}_n)| \leqslant |\sigma_{k-1}(\mathscr{F})|$. Therefore it is sufficient to deal with $\mathscr{F}_n$.

Recall the definitions: $\mathscr{F}_n(1) = \{F - \{1\} : F \in \mathscr{F}_n\}$, $\mathscr{F}_n(\bar{1}) = \{F \in \mathscr{F}_n : 1 \notin F\}$.

Claim 3.2. (i) $\sigma_{k-1}(\mathscr{F}_n) \geqslant |\mathscr{F}_n(1)| + |\sigma_{k-2}(\mathscr{F}_n(1))|$; (ii) $\sigma_{k-1}(\mathscr{F}_n(\bar{1})) \subset \mathscr{F}_n(1)$.

Proof of the claim By definition, $\mathscr{F}_n(1) \subset \sigma_{k-1}(\mathscr{F}_n)$ and $\{1 \cup G : G \in \sigma_{k-2}(\mathscr{F}_n(1))\} \subset \sigma_{k-1}(\mathscr{F}_n)$ hold. Moreover, these two families are disjoint, proving (i).

To prove (ii) choose (G, H) with $H \in \mathscr{F}_n(\bar{1})$, $G \subset H$, $|G| = k-1$. Let i be the unique element of $H - G$.

The only way H was not replaced by $H' = G \cup \{i\}$ when S_{1i} was applied is that $H' \in \mathscr{F}_{i-1}$ and, consequently, $H' \in \mathscr{F}_n$. This proves $G \in \mathscr{F}_n(1)$, as desired. ■

Now the proof of (6) is easy. Apply induction on k and for given k, on $|\mathscr{F}|$; of course, the case $|\mathscr{F}| = 1$ is trivial. We distinguish two cases.

(a) $|\mathscr{F}_n(1)| \geqslant \binom{x-1}{k-1}$. By induction $|\sigma_{k-2}(\mathscr{F}_n(1))| \geqslant \binom{x-1}{k-2}$ and thus by (i), $|\sigma_{k-1}(\mathscr{F}_n)| \geq \binom{x-1}{k-1} + \binom{x-1}{k-2} = \binom{x}{k-1}$ follow.

(b) $|\mathscr{F}_n(1)| < \binom{x-1}{k-1}$. In view of (ii), $|\mathscr{F}_n(1)| \geq k$ and thus $x-1>k$ follows. On the other hand,

$$|\mathscr{F}_n(\bar{1})| = |\mathscr{F}| - |\mathscr{F}_n(1)| \geq \binom{x}{k} - \binom{x-1}{k-1} = \binom{x-1}{k}.$$

Applying the induction hypothesis and using (ii), $|\mathscr{F}_n(1)| \geq |\sigma_{k-1}(\mathscr{F}_n(\bar{1}))| \geq \binom{x-1}{k-1}$ follows, a contradiction. ■

Erdös-Ko-Rado from Kruskal-Katona (cf. [Da], [Ka1]). Suppose that $\mathscr{F} \subset \binom{X}{k}$, $n \geq 2k$, and $|\mathscr{F}| > \binom{n-1}{k-1} = \binom{n-1}{n-k}$. Define the complementary family $\mathscr{G} = \{X-F : F \in \mathscr{F}\} \subset \binom{X}{n-k}$. By(5) $|\sigma_k(\mathscr{G})| \geq \binom{n-1}{k}$, and thus $|\mathscr{F}| + |\sigma_k(\mathscr{G})| > \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$ holds.

Consequently there is some $F \in \mathscr{F} \cap \sigma_k(\mathscr{G})$. Since $F \in \sigma_k(\mathscr{G})$, $F \subset X - F'$ for some $F' \in \mathscr{F}$. This implies $F \cap F' = \phi$, i. e., $\mathscr{F}$ is not intersecting. ■

4 Cyclic Permutations

In this section we reproduce the short proof of (2) given by Katona [Ka3].

For any of the $(n-1)!$ cyclic permutations $\pi: a_1, \cdots, a_n$ of 1, 2, ⋯, n, consider the k-graph $\mathscr{F}(\pi)$ consisting of the n blocks of length k (for each i there is one such block starting at a_i namely, a_i, a_{i+1}, $\cdots a_{i+k-1}$ (index addition is performed modulo n).

Claim 4.1 Suppose that $\mathscr{G} \subset \mathscr{F}(\pi)$ is an intersecting subfamily. Then $|\mathscr{G}| \leq k$.

Proof Without loss of generality, we may assume that $A = \{a_1 \cdots, a_k\} \in \mathscr{G}$. Being intersecting implies that for each $G \in \mathscr{G}$ either its first or last element is in A. A priori this gives $2(k-1)$ more candidates outside A for membership in $\mathscr{G}$. We can group them into $k-1$ pairs by associating the one ending in a_i with the one starting in a_{i+1}, $1 \leq i \leq k-1$. Since the two k-sets are disjoint in each pair, $|\mathscr{G}| \leq 1 + k - 1 = k$ follows. ■

Let us now estimate the number M of pairs (F, π), where $F \in \mathscr{G}$, π is a cyclic permutation and $F \in \mathscr{F}(\pi)$.

For each k-set F there are $k!(n-k)!$ cyclic permutations π with $F \in \mathscr{F}(\pi)$, Thus $M = |\mathscr{F}|k!(n-k)!$.

On the other hand, claim 4. 1 implies $M \leqslant k(n-1)!$. Consequently, $|\mathscr{F}| \leqslant \frac{k\cdot(n-1)!}{k!(n-k)!} = \binom{n-1}{k-1}$, follows as desired.

5 Delsarte's Linear Programming Bound: Lov sz' Proof

Let $A_1, \cdots, A_{\binom{n}{k}}$ be an arbitrary fixed ordering of all the k-subsets of X. For an intersecting family $\mathscr{F} \subset \binom{X}{k}$, let $v(F) = (v_1, \cdots, v_{\binom{n}{k}})$ be its characteristic vector, i. e., $v_i = 1$ or 0 according to whether $A_i \in \mathscr{F}$ or $A_i \notin \mathscr{F}$ holds.

Let B be an $\binom{n}{k}$ by $\binom{n}{k}$ real symmetric matrix whose general entry b_{ij} satisfies $b_{ij} = 0$ whenever $A_i \cap A_j \neq \phi$.

Let I and J denote the $\binom{n}{k}$ by $\binom{n}{k}$ identity, and all 1's matrices, respectively.

Claim 5.1 If $B + I - cJ$ is positive semi-definite for some positive c, then $|\mathscr{F}| \leqslant 1/c$.

Proof Consider $y = v(B + I - cJ)v^T$. By assumption, $vBv^T = 0$. Also $vIv^T = |\mathscr{F}|$ and $v\mathscr{F}v^T = |\mathscr{F}|^2$ are immediate from the definition. Thus $y = |\mathscr{F}| - c|\mathscr{F}|^2$.

On the other hand, $y \geqslant 0$ follows from positive semi-definiteness, i.e. $c|\mathscr{F}|^2 \leqslant |\mathscr{F}|$, or equivalently, $|\mathscr{F}| \leqslant 1/c$. ■

Following Lovàsz [L2] we define $B = (b_{ij})$ by
$$b_{ij} = \begin{cases} \binom{n-k-1}{k-1}^{-1} & \text{if } A_i \cap A_j = \phi \\ 0 & \text{otherwise.} \end{cases}$$

To prove (2) we need to compute the eigenvalues of B. Obviously, the all 1's vector is a common eigenvector of B, I and J with respective eigenvalues $\frac{n-k}{k}$, 1 and $\binom{n}{k}$. Thus it is annihilated by $B = B + I - \binom{n-1}{k-1}^{-1} J$.

To prove positive semi-definiteness we have to show that all the remaining eigenvalues of B are at least -1. It is easy to give eigenvectors having eigenvalue -1, namely for each pair (x, y), $x, y \in X$ define $v(x, y) = (v_1, \cdots, v_{\binom{n}{k}})$ by

$$v_i = \begin{cases} 1 & \text{if } A_i \cap \{x, y\} = \{x\}, \\ -1 & \text{if } A_i \cap \{x, y\} = \{y\}. \\ 0 & \text{otherwise.} \end{cases}$$

Direct computation shows that $v(x, y)B = -v(x, y)$ holds. The $v(x, y)$ span a vector space of dimension $n-1$. To find the remaining $\binom{n}{k} - (n-1) - 1$ eigenvectors we have to do some more work. For $2\leqslant i\leqslant k$ and any two disjoint i-element subsets $C=\{x_1,\cdots,x_i\}$, $D=\{y_1,\cdots,y_i\}$ of X, define the vector $u(C, D)=(u_1,\cdots,u_{\binom{n}{k}})$ by

$$u_j=\begin{cases}(-1)^{|D\cap A_j|} & \text{if } |A_j\cap\{x_l, y_l\}|=1 \text{ holds for } 1\leqslant l\leqslant i,\\ 0 & \text{otherwise.}\end{cases}$$

Claim 5.2

$$u(C, D)B=(-1)^i\frac{\binom{n-k-i}{k-i}}{\binom{n-k-1}{k-1}}u(C, D). \tag{7}$$

Proof Set $\delta=\binom{n-k-1}{k-1}^{-1}$ and compute the r^{th} entry v_r of $u(C, D)B$. This is simply the dot product of u_j and the column of B belonging to the set A_r.

Suppose first that $|A_r\cap\{x_l, y_l\}|=1$ for $1\leqslant l\leqslant i$. The only way to get a non-zero (actually δ or $-\delta$) is for $A\in\binom{X}{k}$ to satisfy $A\cap(C\cup D)=(C\cup D)-A_r$, i.e., if A and A_r are complementary inside $C\cup D$. Consequently, $v_r=(-1)^i\delta\binom{n-k-i}{k-i}u_r$ follows.

If $|A_r\cap\{x_l, y_l\}|=2$ for some l, then there is no way to get a non-zero term, yielding $v_r=0$.

If $A_r\cap\{x_l, y_l\}=\phi$ holds, then we can associate to every position A giving a non-zero term and satisfying $A\cap\{x_l, y_l\}=\{x_l\}$ the position $(A-\{x_l\})\cup\{y_l\}$ giving a non-zero term exactly of the opposite sign. This gives again $v_r=u_r=0$. ■

Next we are going to exhibit $\binom{n}{i}-\binom{n}{i-1}$ linearly independent vectors $u(C, D)$. This will show that the eigenspace belonging to the eigenvalue $(-1)^i\binom{n-k-i}{k-i}/\binom{n-k-1}{k-1}$ has dimension at least $\binom{n}{i}-\binom{n}{i-1}$ for $i=0,1,\cdots,k$. Since these numbers sum up to $\binom{n}{k}$ equality holds everywhere. Consequently, we have found all the eigenvalues and the positive semi-definiteness follows from $\binom{n-k-i}{k-i}\leqslant\binom{n-k-1}{k-1}$, valid for $1\leqslant i\leqslant k$.

If $C=\{x_1, \cdots, x_i\}$ and $D=\{y_1, \cdots, y_i\}$ with $x_j<y_j$ then we write $C<D$. It is not hard to see that there are exactly $\binom{n}{i}-\binom{n}{i-1}$ sets $D\in\binom{X}{i}$ for which some $C\in\binom{X}{i}$ satisfying $C<D$ exists. Fix some $C=C(D)$ with this property for each such D.

Then the vectors $u(C(D), D)$ are linearly independent, and we are done. ■

6 Stronger results for large n

Let $L=\{l_1, \cdots, l_s\}$ with $0\leqslant l_1<\cdots<l_s<k$. A family $\mathscr{F}\subset\binom{X}{k}$ is called an *L-system* if $|F\cap F'|\in L$ holds for all distinct $F, F'\in\mathscr{F}$.

In this terminology $\mathscr{F}$ is t-intersecting iff it is an L-system with $L=\{t, t+1, \cdots, k-1\}$.

In this section we are going to prove the following general result.

Theorem 6.1 [DEF]. Suppose that $\mathscr{F}\subset\binom{X}{k}$ is an L-system. Then

$$|\mathscr{F}|\leqslant\prod_{l\in L}\frac{n-l}{k-l}\quad\text{holds for } n\geqslant k\binom{3k}{k}.$$

Note that for $L=\{t, \cdots, k-1\}$, the upper bound becomes $\binom{n-t}{k-t}$, so that this result generalizes the Erdös-Ko Rado Theorem.

Proof Apply induction on $|L|=s$. In the case $s=0$, the upper bound $|\mathscr{F}|\leqslant 1$ is trivially true. Supposing the theorem is true for $s-1$, we attack the case $|L|=s$.

(a) $l_1=0$. For each $x\in X$ the family $\widetilde{\mathscr{F}}(x)=\{F\in\mathscr{F}: x\in F\}$ is an L'-system with $L'=\{l_2, \cdots, l_s\}$. Thus by induction

$$|\widetilde{\mathscr{F}}(x)|\leqslant\prod_{2\leqslant i\leqslant s}\frac{n-l_i}{k-l_i}.$$

Since $\sum_{x\in X}|\widetilde{\mathscr{F}}(x)|=k|\mathscr{F}|$ holds, we obtain

$$|\mathscr{F}|\leqslant\frac{n}{k}\prod_{2\leqslant i\leqslant s}\frac{n-l_i}{k-l_i}=\prod_{l\in L}\frac{n-l}{k-l}$$

as desired.

(b) $l_1>0$. If $|F\cap F'|\neq l_1$ holds for all $F, F'\in\mathscr{F}$ then $\mathscr{F}$ is actually an $\{l_2, \cdots, l_s\}$-system and the much stronger bound $|\mathscr{F}|\leqslant\prod_{2\leqslant i\leqslant s}\frac{n-l_i}{k-l_i}$ follows by induction.

Suppose next that $|F_1\cap F_2|=l_1$ for some $F_1, F_2\in\mathscr{F}$. Set $G=F_1\cap F_2$. If $G\subset F$ for all $F\in\mathscr{F}$, then replacing $\mathscr{F}$ by $\{F-G: F\in\mathscr{F}\}$, k by $k-l_1$ and L by $\{0, l_2-l_1, \cdots, l_s-l_1\}$ brings us back to case (a).

Suppose finally that $G\not\subset F_3$ holds for some $F_3\in\mathscr{F}$.

Claim 6.2 $|F\cap(F_1\cup F_2\cup F_3)|>l_1$ holds for all $F\in\mathscr{F}$.

Proof Since $\mathscr{F}$ is an L-system, $|F\cap F_i|\geqslant l_1$ for $i=1, 2, 3$. If $|F\cap(F_1\cup$

$F_2)| = l_1$, then $F\cap F_1 = F\cap F_2 = G$ follows. In view of $|F\cap F_3| \geqslant l_1$, this forces $F\cap (F_3 - G) \neq \phi$, yielding the claim. ■

For each $H \in \binom{F_1\cup F_2\cup F_3}{l_1+1}$, define $\widetilde{\mathscr{F}}(H) = \{F \in \mathscr{F} : H\subset F\}$. Then $\widetilde{\mathscr{F}}(H)$ is an $\{l_2, \cdots, l_s\}$-system. Thus the induction hypothesis and the claim imply:

$$|\mathscr{F}| \leqslant \sum_{\substack{H\subset(F_1\cup F_2\cup F_3),\\ |H| = l_1+1}} \prod_{2\leqslant i\leqslant s} \frac{n-l_i}{k-l_i} < \binom{3k}{k} \prod_{2\leqslant i\leqslant s} \frac{n-l_i}{k-l_i},$$

and the statement of the theorem follows provided $\frac{n-l_1}{k-l_1} > \binom{3k}{k}$. ■

Remark From the proof it is clear that for $l_1 > 0$ epuality can hold only if all $F \in \mathscr{F}$ contain a fixed l_1-element set.

7 Edge-fillings by pairs

Let $\mathscr{F}$ be an intersecting k-graph. we call $\mathscr{F}$ *non-trivial* if $\bigcap_{F\in\mathscr{F}} F = \phi$ holds.

Let us call a family $\mathscr{G}$ an *edge-filling* of $\mathscr{F}$ if for every $F\in\mathscr{F}$ there exists some $G\in\mathscr{G}$ satisfying $G\subset F$.

Theorem 7.1 Every non-trivial intersecting family $\mathscr{F}\subset\binom{X}{k}$ possesses an edge-filling $\mathscr{G}\subset\binom{X}{2}$ satisfying $|\mathscr{G}|\leqslant k^2-k+1$.

Proof If $\mathscr{F}$ is 2-intersecting, then $\binom{F}{2}$ is an edge-filling of $\mathscr{F}$ for every $F \in \mathscr{F}$. Thus we may assume that there exist F_1, $F_2\in\mathscr{F}$ with $F_1\cap F_2 = \{x\}$ for some $x \in X$.

Since $\mathscr{F}$ is non-trivial, $x \notin F_3\in\mathscr{F}$ holds for some $F_3\in\mathscr{F}$.

Now the theorem follows from the following claim.

Claim 7.2 $\mathscr{G} = \{\{y, z\} : y\in F_1 - \{x\},\ z\in F_2 - \{x\}\}\cup\{\{x, u\} : u\in F_3\}$ is an edge-filling of $\mathscr{F}$.

Proof Take an arbitrary $F \in\mathscr{F}$ and distinguish two cases.

(a) $x\notin F$. Since $\mathscr{F}$ is intersecting, $F\cap(F_i - \{x\})\neq\phi$ holds for $i = 1, 2$. Consequently F contains one of the 2-subsets in the first part of $\mathscr{G}$.

(b) $x\in F$. Again, since $\mathscr{F}$ is intersecting, $F\cap F_3\neq\phi$, so that F must contain one of the 2-subsets in the second part of $\mathscr{G}$. ■ F has to contain one of the 2-subsets in the second part of $\mathscr{G}$. ■

Remark More careful analysis shows that one can find an edge-filling $\mathscr{G}$ with $|\mathscr{G}| < k^2 - k + 1$ unless $\mathscr{F}$ consists of the $k^2 - k + 1$ lines of a projective plane of order $k-1$. We hope to return to this and more general problems in a subsequent paper.

8 The Erdös-Ko-Rado Theorem for General Hypergraphs

Let $\mathscr{H}$ be a k-graph. For a vertex x of $\mathscr{H}$ its degree is the number of edges of $\mathscr{H}$ which contain x. Clearly, these edges form an intersecting hypergraph. Let $\Delta(\mathscr{H})$ denote the maximum degree of $\mathscr{H}$.

We say that $\mathscr{H}$ has the Erdös-Ko-Rado property if $\Delta(\mathscr{H})\geqslant|\mathscr{F}|$ holds for all intersecting subfamilies $\mathscr{F}\subset\mathscr{H}$.

For a set I, recall the definition $\mathscr{H}(I)=\{H-I : I\subset H\in\mathscr{H}\}$. Let $\Delta_i(\mathscr{H})$ denote the maximum of $|\mathscr{H}(I)|$ over all i-sets I. Clearly, $\Delta_1(\mathscr{H})=\Delta(\mathscr{H})$ and $\Delta_k(\mathscr{H})=1$

Theorem 8.1 Suppose that $(k^2-k+1)\Delta_2(\mathscr{H})\leqslant\Delta_1(\mathscr{H})$ holds. Then $\mathscr{H}$ has the Erdös-Ko-Rado property.

Proof Let $\mathscr{F}\subset\mathscr{H}$ be intersecting. If $\bigcap_{F\in\mathscr{F}} F\neq\phi$, then $|\mathscr{F}|\leqslant\Delta_1(\mathscr{H})$ is immediate. Thus we may suppose that $\mathscr{F}$ is non-trivial. By Theorem 7.1 there exists an edge-filling $\mathscr{G}$ of $\mathscr{F}$, consisting of 2-element sets and satisfying $|\mathscr{G}|\leqslant k^2-k+1$. This implies

$$|\mathscr{F}|\leqslant\sum_{G\in\mathscr{G}}|\mathscr{F}(G)|\leqslant\sum_{G\in\mathscr{G}}|\mathscr{H}(G)|\leqslant(k^2-k+1)\Delta_2(\mathscr{H})\leqslant\Delta_1(\mathscr{H}).\ \blacksquare$$

Remark Since $\Delta_1(\mathscr{H})=\binom{n-1}{k-1}$ and $\Delta_2(\mathscr{H})=\binom{n-2}{k-2}$ hold for $\mathscr{H}=\binom{X}{k}$, Theorem 8.1 implies (2) for $n>(k-1)(k^2-k+1)$. As we shall see later, it implies (3) as well.

Now we give a construction showing that the theorem is in a sense best possible. Let $k-1$ be an integer such that there is a projective plane of order $k-1$.

If n is a sufficiently large multiple of k^2-k+1 then one can find $k-1$ orthogonal partitions of X into (k^2-k+1)-element sets. That is, there exist $B_{ij}\in\binom{X}{k^2-k+1}$, $1\leqslant i<k, 1\leqslant j\leqslant n/(k^2-k+1)$: $=r$, such that $B_{i1}\cup\cdots\cup B_{ir}=X$, $1\leqslant i<k$, and $|B_{ij}\cap B_{st}|\leqslant 1$ for $i\neq s$ and all j, t.

Now form a k-graph $\mathscr{H}$ by replacing each B_{ij} by the set of lines of a projective plane of order $k-1$ on B_{ij}. Then $|\mathscr{H}|=(k-1)n$, $\Delta(\mathscr{H})=(k-1)k=k^2-k$ and $\Delta_2(\mathscr{H})=1$. However, the size of the largest intersecting subfamily of $\mathscr{H}$ is k^2-k+1, namely, each projective plane of order $k-1$ gives such an example.

Let us conclude this section with a well known open problem. Call $\mathscr{H}\subset 2^X$ a *complex*, if $G\subset H\in\mathscr{H}$ implies $G\in\mathscr{H}$.

Conjecture 8.2 (Chvátál [C] Suppose that $\mathscr{H}$ is a complex, and $\mathscr{F}\subset\mathscr{H}$ is intersecting. Then $|\mathscr{F}|\leqslant\Delta(\mathscr{H})$ holds.

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厄多斯-柯-拉多定理的新老证明

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摘 要

厄多斯—柯—拉多定理是组合论的一个主要结果，它开辟了极值集论迅速发展的道路。本文回顾了它的多种证明，并给出了一个新的推广。有关该定理的综合报告见[DF]。

关键词 厄多斯－柯－拉多定理，交族，极值集论，组合论。