Asymptotic Analysis of a Random Walk on a Hypercube with Many Dimensions

Persi Diaconis
Department of Mathematics, Harvard University, Cambridge, MA 02138

R. L. Graham, J. A. Morrison
AT & T Bell Laboratories, Murray Hill, NJ 07974

ABSTRACT

In nearest neighbor random walk on an n-dimensional cube a particle moves to one of its nearest neighbors (or stays fixed) with equal probability. The particle starts at $\mathbf{0}$. How long does it take to reach its stationary distribution? In fact, this occurs surprisingly rapidly. Previous analysis has shown that the total variation distance to stationarity is large if the number of steps N is $\ll \frac{1}{4}n \log n$ and close to 0 if $N \gg \frac{1}{4}n \log n$. This paper derives an explicit expression for the variation distance as $n \to \infty$ in the transition region $N \sim \frac{1}{4}n \log n$. This permits the first careful evaluation of a cutoff phenomenon observed in a wide variety of Markov chains. The argument involves Fourier analysis to express the probability as a contour integral and saddle point approximation. The asymptotic results are in good agreement with numerical results for n as small as 100.

Key Words: asymptotics, random walk, saddle point method

1. INTRODUCTION

Let $\mathbb{Z}_2^n$ denote the group of binary n-tuples under coordinatewise addition modulo 2. Nearest neighbor random walk on the n-cube $\mathbb{Z}_2^n$ is based on the probability distribution

$$Q(\mathbf{x}) = \begin{cases} 1/(n+1) & \text{if } |\mathbf{x}| \le 1\,, \\ 0 & \text{if } |\mathbf{x}| \ge 2 \end{cases} \tag{1.1}$$

Random Structures and Algorithms, Vol. 1, No. 1 (1990)

where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}_2^n$ and $|\mathbf{x}| := x_1 + \cdots + x_n$ (ordinary addition). The walk is allowed to stay where it is or move to nearest neighbor with equal probabilities (where the nearest neighbors $\mathbf{y}$ of a point $\mathbf{x}$ are those $\mathbf{y}$ with $|\mathbf{x} - \mathbf{y}| = 1$).

Repeated steps in the walk correspond to convolutions:

$$Q * Q(\mathbf{x}) = \sum_{y} Q(\mathbf{x} - \mathbf{y})Q(\mathbf{y}); \; Q^{*N} := Q * Q^{*(N-1)} .$$

This random walk approaches the uniform distribution $U(\mathbf{x}) = 1/2^n$. In this paper we give bounds on the speed of approach to uniformity. We show that roughly $\frac{1}{4}n \log n$ steps are necessary and sufficient to reach stationarity. Approach to stationarity will be measured by the total variation distance

$$\|P - Q\| = \sup_{A} |P(A) - Q(A)|$$

where for a probability P on a space $X \supseteq A$, $P(A)$ denotes $\Sigma_{a \in A} P(a)$.

Theorem 0. *For nearest neighbor random walk* (1.1) *let* $N = \frac{1}{4}(n+1)(\log n + c)$. *If* $c > 0$ *then*

$$\|Q^{*N} - U\|^2 \leq \tfrac{1}{2}(e^{e^{-c}} - 1) .$$

As $n \to \infty$, *for any* $\epsilon > 0$ *there is a* $C < 0$ *such that* $c < C$ *and* N *as above imply*

$$\|Q^{*N} - U\| \geq 1 - \epsilon .$$

Theorem 0 is proved in Diaconis and Shahshahani [5]. It is closely related to a body of classical work surveyed in Letac and Takacs [11]. Aldous [2a, b] develops an application of random walk to the analysis of algorithms. Mathews [14, 15] proves theorems like Theorem 0 by "pure probability" arguments.

Nearest neighbor random walk on the cube is essentially the same process as the Ehrenfest model of diffusion. Kac [9, 10] gives a motivated treatment of this. Siegert [18] is an early paper linking the Ehrenfest model with random walk. Diaconis [4] describes much of this background and other applications. The present paper refines Theorem 0 by deriving an asymptotic approximation to the variation distance in the transition region where $N \sim \frac{1}{4}n \log n$. The main result is:

Theorem 1. *For nearest neighbor random walk* (1.1), *let* $N = \frac{1}{4}n \log n + cn$. *Then for fixed* $c \in (-\infty, \infty)$, *as* $n \to \infty$,

$$\|Q^{*N} - U\| \sim \operatorname{Erf}(e^{-2c}/\sqrt{8}) .$$

Here, $\operatorname{Erf}(z) = 2/\sqrt{\pi} \int_0^z e^{-t^2}\, dt$ *denotes the error function* (*e.g.*, *see Ref.* 13, *p.* 349).

Theorem 1 allows us to explore what Aldous and Diaconis [3] have called the "cutoff phenomenon." To explain, consider $V(N) = \|Q^{*N} - U\|$ as a function of N. As N increases, of course, $V(N)$ decreases. However, $V(N)$ stays quite close to its maximum value 1 for N up to $\frac{1}{4}n \log n - cn$, where $c > 0$. Then, as N moves

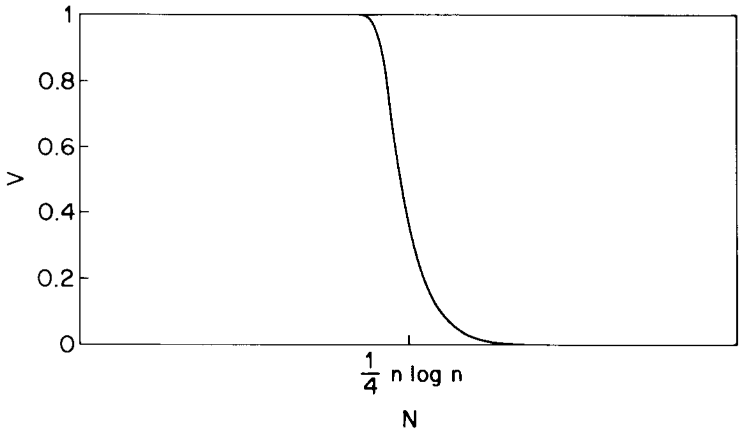

Fig. 1. The variation distance V as a function of N, for $n = 10^{12}$.

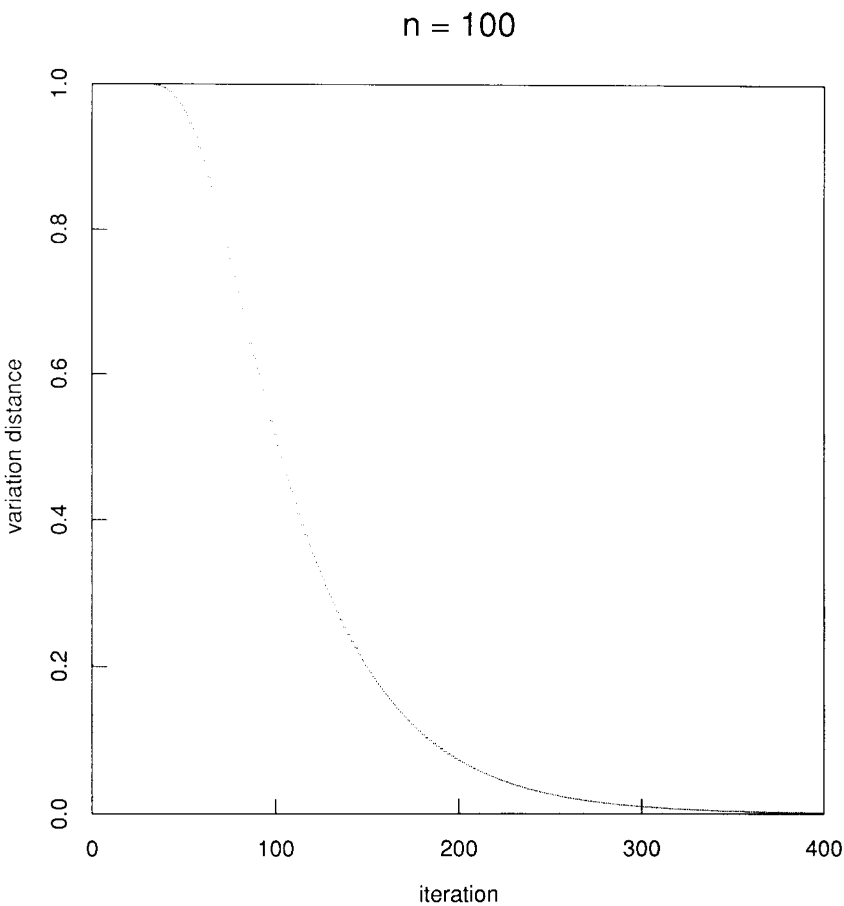

Fig. 2. The variation distance as a function of the iteration N, for $n = 100$.

from $\frac{1}{4}n \log n - cn$ to $\frac{1}{4}n \log n + cn$, $V(N)$ moves rapidly from close to 1 to close to 0. It then tends to 0 exponentially fast as N increases further.

Theorem 1 implies that a plot of V vs. N would look like the picture in Figure 1, which corresponds to $n = 10^{12}$. This is indeed the case for n large. If the plot is drawn on a scale that goes from 0 to $\frac{1}{2}n \log n$, the cutoff from 1 to 0 occurs on a scale of n.

Figures 2, 3, and 4 show the result of an exact computation of $V(N)$ for $n = 100$, 1000, and 10,000. These computations were carried out by A. M. Odlyzko using a clever iterative scheme. They show the same general shape but also show that for finite n, the cutoff is not as sharp as in Figure 1. To explain the disparity, observe that the factor differentiating the lead term, $\frac{1}{4}n \log n$, from the next term, cn, is $\frac{1}{4} \log n$. When $n = 10{,}000$, $\frac{1}{4} \log n = 2.3\ldots$. This is supposed to be large compared to $|c|$. For $n = 10^{12}$, $\frac{1}{4} \log n = 6.9\ldots$, which is sufficiently large. Figures 2, 3, and 4 suggest a smooth limiting value although a close look shows the effect of the parity of N. Comparison with the numerical results underlying Figures 2, 3, and 4 shows fairly close agreement with the limiting value guaranteed by Theorem 1 when $n = 100$, and very close agreement when $n = 1000$.

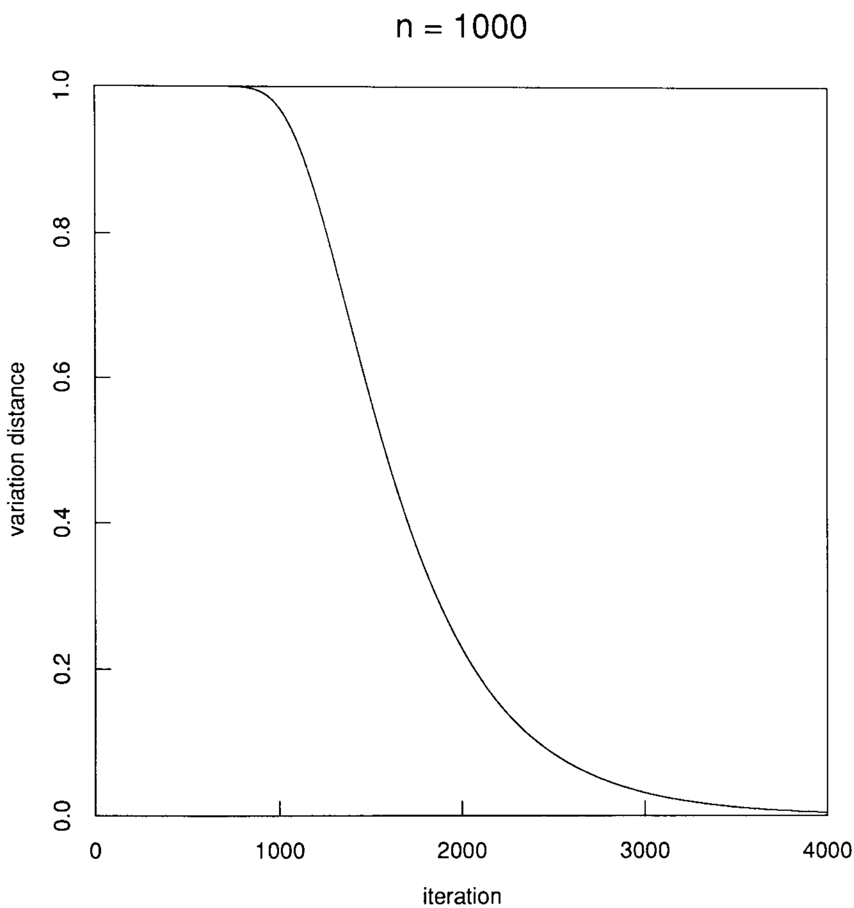

Fig. 3. The variation distance as a function of the iteration N, for $n = 1000$.

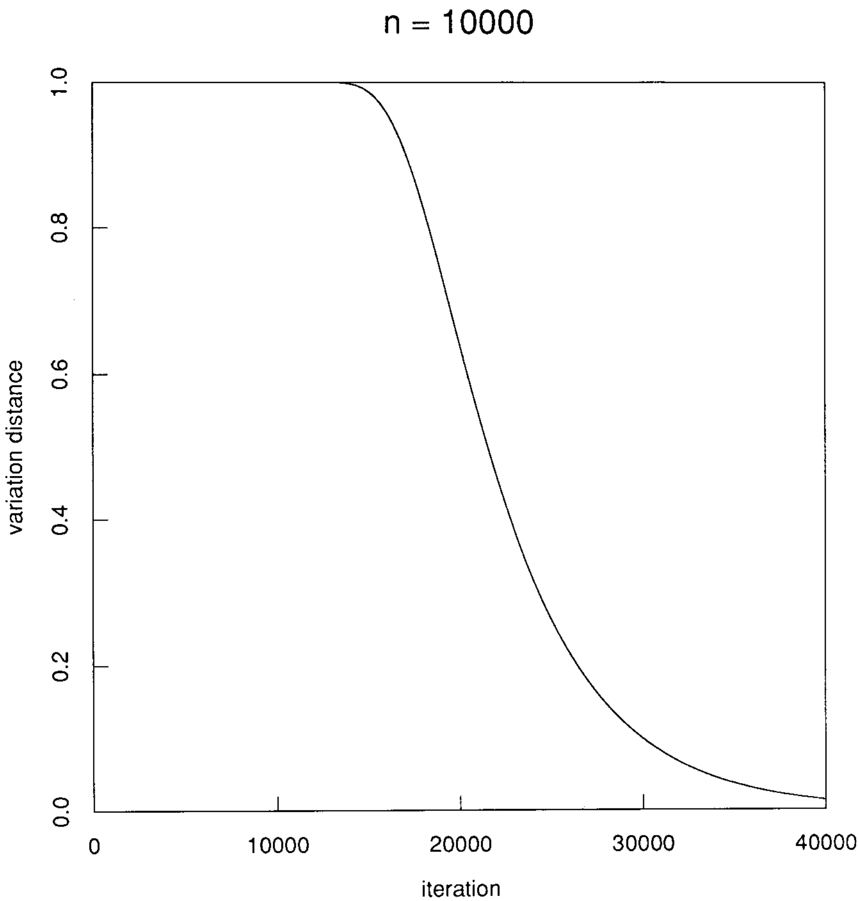

Fig. 4. The variation distance as a function of the iteration N, for $n = 10{,}000$.

Section 2 contains some elementary preliminary material. It also contains an analysis of the continuous time version of Theorem 1. The computations here are easy and "explain" the appearance of the error function through probabilistic considerations.

Theorem 1 is proved by expressing the exact probabilities as a contour integral. An asymptotic approximation to this integral is then derived by the method of steepest descent (the saddle point method). This is carried out in §3. In the Appendix the behavior of the paths of steepest descent from the real saddle points is investigated. The asymptotic approximation to the variation distance is derived in §4.

The random walk starts at the origin $\mathbf{0}$ and as time goes on, the probability spreads out over the cube. By symmetry the probability at any time at some point $\mathbf{x}$ depends only on $|\mathbf{x}|$. Furthermore, as is noted in Lemma 2, this probability for any fixed number of steps is a monotone decreasing function of $|\mathbf{x}|$. In numerical work we observed that the distance W at which points after N steps have probability "above average," i.e., greater than $1/2^n$, grows linearly with N for

$N < \alpha n$ where $\alpha = 0.32756\ldots$ satisfies $\alpha 2^{1/\alpha} = e$, and then slows down abruptly. For $N = \lambda n$ and $n \to \infty$, we have $W/n \to \rho$ for some ρ. In §2 it is shown by elementary arguments that $\rho = \lambda$ for $0 < \lambda < \alpha$. In §5 a parametric relationship is obtained between ρ and λ for $\lambda > \alpha$. Comparison with numerical results of Odlyzko [16] shows W/n is already very close to its limiting value for $n = 1000$.

2. ELEMENTARY FACTS

In this section we derive an exact formula in Lemma 1 for the underlying probabilities $P_N(\mathbf{x})$ using Fourier analysis on $\mathbb{Z}_2^n$. These probabilities are shown to be monotone in $|\mathbf{x}|$ for fixed N in Lemma 2. The exact crossover point W where $P_N(\mathbf{x}) \geq 1/2^n$ is discussed in Lemma 3. Finally, Proposition 1 carries out a proof of Theorem 1 in the continuous time case. This may serve as motivation for the argument used in the proof of Theorem 1.

Lemma 1. *For P defined on $\mathbb{Z}_2^n$ by Eq.* (1.1), *let P_N denote the N-fold convolution. Then for $\mathbf{x} \in \mathbb{Z}_2^n$, $|\mathbf{x}| = x_1 + \cdots + x_n$,*

$$P_N(\mathbf{x}) = \frac{1}{2^n} \sum_{j=0}^{n} \left(1 - \frac{2j}{n+1}\right)^N \sum_{i=0}^{|\mathbf{x}|} (-1)^i \binom{|\mathbf{x}|}{i} \binom{n-|\mathbf{x}|}{j-i}. \tag{2.1}$$

Proof. For $\mathbf{y} \in \mathbb{Z}_2^n$, the Fourier transform of P at $\mathbf{y}$ is given by

$$\hat{P}(\mathbf{y}) = \sum_{\mathbf{x}} (-1)^{\mathbf{x}\cdot\mathbf{y}} P(\mathbf{x}).$$

For P defined by Eq. (1.1),

$$\hat{P}(\mathbf{y}) = \left(1 - \frac{2|\mathbf{y}|}{n+1}\right).$$

Since P is invariant under permutations, $P(\mathbf{x})$ and $\hat{P}(\mathbf{y})$ are functions of $|\mathbf{x}|$ and $|\mathbf{y}|$, respectively. The Fourier transform converts convolution into products. Thus, $\hat{P}_N = \hat{P}^N$. Now, the Fourier inversion theorem gives

$$\begin{aligned} P_N(\mathbf{x}) &= \frac{1}{2^n} \sum_{\mathbf{y}} (-1)^{\mathbf{x}\cdot\mathbf{y}} \hat{P}_N(\mathbf{y}) \\ &= \frac{1}{2^n} \sum_{j=0}^{n} \left(1 - \frac{2j}{n+1}\right)^N \sum_{|\mathbf{y}|=j} (-1)^{\mathbf{x}\cdot\mathbf{y}}. \end{aligned} \tag{2.2}$$

MacWilliams and Sloane [12] or Diaconis [4] give the inversion theorem in this form. These sources also show that the inner sum of Eq. (2.2), namely the transform of a "shell" of radius j, is just the Krawtchouk polynomial $K^n(|\mathbf{x}|, j)$. The formula for this appears in the inner sum in Eq. (2.1). ∎

In general for an integer k, $0 \leq k \leq n$, we define $P_N(k)$ to be $P_N(\mathbf{x})$ where $\mathbf{x} \in \mathbb{Z}_2^n$ has $|\mathbf{x}| = k$. This notation is well defined by our previous remarks and should cause no confusion.

Lemma 2. *For P defined on $\mathbb{Z}_2^n$ by* (1.1), *the Nth convolution $P_N(k)$ is monotone decreasing in k.*

Proof. More precisely, define

$$\Delta_N(k) := P_N(k) - P_N(k+1)\,, \quad 0 \le k < n\,.$$

We show

$$\Delta_N(k) \ge 0\,. \tag{2.3}$$

Assume without loss of generality that $\mathbf{x} = (0, x_2, \dots, x_n)$. For such $\mathbf{x}$, define $\mathbf{x}' := (1, x_2, \dots, x_n)$; then $|\mathbf{x} - \mathbf{x}'| = 1$.

Thus, by Eq. (1.1)

$$P_{N+1}(\mathbf{x}') = \frac{1}{n+1} \sum_{|\mathbf{z}-\mathbf{x}'| \le 1} P_N(\mathbf{z})$$

and consequently

$$\begin{aligned}
\Delta_{N+1}(\mathbf{x}) &= P_{N+1}(\mathbf{x}) - P_{N+1}(\mathbf{x}') \\
&= \frac{1}{n+1} \Big\{ \sum_{|\mathbf{y}-\mathbf{x}| \le 1} P_N(\mathbf{y}) - \sum_{|\mathbf{z}-\mathbf{x}'| \le 1} P_N(\mathbf{z}) \Big\} \\
&= \frac{1}{n+1} \Big\{ P_N(\mathbf{x}) + \sum_{|\mathbf{y}-\mathbf{x}| = 1} P_N(\mathbf{y}) - P_N(\mathbf{x}') - \sum_{|\mathbf{z}-\mathbf{x}'| = 1} P_N(\mathbf{z}) \Big\} \\
&= \frac{1}{n+1} \Big\{ \sum_{\substack{|\mathbf{y}-\mathbf{x}| = 1 \\ \mathbf{y} \ne \mathbf{x}'}} P_N(\mathbf{y}) - \sum_{\substack{|\mathbf{z}-\mathbf{x}'| = 1 \\ \mathbf{z} \ne \mathbf{x}}} P_N(\mathbf{z}) \Big\} \\
&= \frac{1}{n+1} \sum_{\substack{|\mathbf{y}-\mathbf{x}| = 1 \\ \mathbf{y} \ne \mathbf{x}'}} \Delta_N(\mathbf{y})\,.
\end{aligned}$$

Since $\Delta_0(\mathbf{y}) \ge 0$ for all $\mathbf{y}$ then by induction $\Delta_N(\mathbf{y}) \ge 0$ for all $\mathbf{y}$ and all $N \ge 0$. ∎

Lemma 3. (B. Poonen [17]). *Let W denote the largest value w for which $P_N(w) \ge 2^{-n}$. Then for $N = \lambda n$ with $0 < \lambda < \alpha$ where $\alpha = 0.32756\dots$ satisfies $\alpha 2^{1/\alpha} = e$, we have $W/n \to \lambda$ as $n \to \infty$.*

Proof. Any point $\mathbf{x} \in \mathbb{Z}_2^n$ with $|\mathbf{x}| = w > 0$ has w neighbors $\mathbf{y}$ with $|\mathbf{y}| = w - 1$. Thus,

$$P_w(w) = \frac{1}{n+1} \sum_{\substack{|\mathbf{y}-\mathbf{x}| = 1 \\ |\mathbf{y}| = w-1}} P_{w-1}(w-1)\,. \tag{2.4}$$

Since $P_0(0) = 1$ then we obtain the explicit expression

$$P_N(N) = \frac{N!}{(n+1)^N}, \quad 0 \le N \le n. \tag{2.5}$$

Set $N = \lambda n$ with $0 < \lambda < \alpha$. We show that in this case

$$\frac{N!}{(n+1)^N} \ge 2^{-n}, \quad n \to \infty, \tag{2.6}$$

which implies $W = N$.

Since $N! \ge (N/e)^N$ (e.g., see Ref. 7), it will suffice to show

$$(N/e)^N \ge (n+1)^N 2^{-n}, \quad n \to \infty$$

i.e.,

$$N(\log N - 1) \ge N \log(n+1) - n \log 2,$$
$$\lambda(\log \lambda n - 1) \ge \lambda \log(n+1) - \log 2,$$

or

$$\log \lambda + \frac{1}{\lambda} \log 2 - 1 \ge \log(n+1) - \log n, \quad n \to \infty. \tag{2.7}$$

However, $x2^{1/x}$ is monotone decreasing for $0 < x < \log 2$, and $\log 2 < 1$, so that

$$\lambda 2^{1/\lambda} > e \quad \text{for} \quad 0 < \lambda < \alpha < \log 2,$$

i.e.,

$$\log \lambda + \frac{1}{\lambda} \log 2 > 1.$$

Since the RHS of Eq. (2.7) goes to 0 as $n \to \infty$, then Eq. (2.6) follows, and we have $W = N = \lambda n$, which in particular implies that $W/n \to \lambda$ as $n \to \infty$. ∎

The final result in this section develops random walk on the n-cube where jumps take place in continuous time. This variant turns out to be easy to analyze and understand. The results help explain our findings for the discrete time case.

Consider the n-cube $\mathbb{Z}_2^n$. For $0 \le t < \infty$, let X_t be nearest neighbor random walk: a particle starts at $\mathbf{0}$ and jumps to one of its nearest neighbors at the times of occurrence of a Poisson process running at rate 1. Thus, there is an exponential wait between jumps with density e^{-x}. Of course, there are t jumps expected in $[0, t]$.

By standard properties of the Poisson distribution, the coordinates $(X_t^1, \ldots, X_t^n)$ are independent binary processes with

$$Pr\{X_t^i = 1\} = \tfrac{1}{2}(1 - e^{-2t/n}), \quad 1 \le i \le n. \tag{2.8}$$

It follows that

$$Pr\{X_t = \mathbf{x}\} = \frac{1}{2^n}(1 - e^{-2t/n})^{|\mathbf{x}|}(1 + e^{-2t/n})^{n-|\mathbf{x}|}. \tag{2.9}$$

From this, for fixed t, $Pr\{X_t = \mathbf{x}\}$ is decreasing in $|\mathbf{x}|$ so $\mathbf{0}$ is always the most likely point, followed by points at distance 1, etc. The monotonicity of the binomial distribution in p implies that for any fixed j, $Pr\{|X_t| \le j\}$ is decreasing in t.

The stationary distribution is uniform on the cube. As will emerge, it takes $\frac{1}{4}n \log n + cn$ steps to get close to uniform. The main result of this section gives the asymptotics of the total variation distance as a function of c.

Proposition 1. *Let $t = \frac{1}{4}n \log n + cn$. For P^t, the law of nearest neighbor random walk on $\mathbb{Z}_2^n$ we have as $n \to \infty$,*

$$\|P^t - U\| = \operatorname{Erf}(e^{-2c}/\sqrt{8}) + o(1), \tag{2.10}$$

where $\operatorname{Erf}(z) := (2/\sqrt{\pi}) \int_0^z e^{-t^2}\, dt$ *denotes the error function,* $U(\mathbf{x}) = 1/2^n$ *is the uniform distribution and* $\|P^t - U\| = \sup_A |P^t(A) - U(A)|$.

Proof. The total variation distance to uniform can be represented as

$$\sum (Pr\{X_t = \mathbf{x}\} - 1/2^n) \text{ summed over } \mathbf{x} \text{ with } Pr\{X_t = \mathbf{x}\} \ge 1/2^n. \tag{2.11}$$

From Eq. (2.9), $Pr\{X_t = \mathbf{x}\} \ge 1/2^n$ if and only if

$$|\mathbf{x}| \le \frac{n \log(1 + e^{-2t/n})}{\log(1 + e^{-2t/n}) - \log(1 - e^{-2t/n})}. \tag{2.12}$$

Set $t = \frac{1}{4}n \log n + cn$. Straightforward calculation now gives $Pr\{X_t = \mathbf{x}\} \ge 1/2^n$ if and only if

$$|\mathbf{x}| \le \frac{n}{2} - \frac{e^{-2c}}{4}\sqrt{n} + O(1). \tag{2.13}$$

Denote the largest value of $|\mathbf{x}|$ consistent with Eq. (2.12) by $j^* = j^*(n, c)$. Let $Y(p)$ be a binomial random variable with parameters n and p. From Eq. (2.11) we have

$$\|P^t - U\| = Pr\left\{Y\left(\frac{1}{2}\left(1 - \frac{e^{-2c}}{\sqrt{n}}\right)\right) \le j^*\right\} - Pr\left(Y\left(\frac{1}{2}\right) \le j^*\right). \tag{2.14}$$

From the central limit theorem for coin tossing with the standard normal cumulative $\Phi(z) = (1/\sqrt{2\pi}) \int_{-\infty}^z e^{-t^2/2}\, dt$, we obtain

$$\begin{aligned} \|P^t - U\| &= \Phi(\tfrac{1}{2}e^{-2c}) - \Phi(-\tfrac{1}{2}e^{-2c}) + o(1) \\ &= 2\Phi(\tfrac{1}{2}e^{-2c}) - 1 + o(1). \end{aligned}$$

Converting from Φ to Erf completes the proof. ∎

Remarks

A. This argument explains where the normal distribution comes from. Under the uniform distribution, the number of ones is binomial $(n, 1/2)$; under P^t, the number of ones is binomial $(n, \frac{1}{2}(1 - e^{-2t/n}))$. For n large, each of these is approximately normal, and the variation distance is well approximated by the variation distance between these two normal limits.

B. If c is large and positive,

$$\operatorname{Erf}(e^{-2c}/\sqrt{8}) \sim \frac{e^{-2c}}{\sqrt{2\pi}} .$$

This gives exponential convergence to 0. If c is large and negative,

$$\operatorname{Erf}(e^{-2c}/\sqrt{8}) \sim 1 - \frac{4e^{2c}}{\sqrt{2\pi}} e^{-e^{-4c}/8} .$$

This tends to 1 extremely rapidly. In general, the limiting form exhibits a sharp cutoff as shown in Figure 5.

C. The continuous time argument is related to the discrete time argument in §3 and §4. If P^t denotes the law of the continuous process at time t, and P^k denotes the law of the discrete time process at time k, then

$$P^t = \sum_{k=0}^{\infty} P^k e^{-t} t^k / k! .$$

The Poisson mixing measure $e^{-t}t^k/k!$ is peaked at $k = t$, and most of its mass is within $c\sqrt{t}$ of t. Our argument determined the behavior of P^t for large t. It is possible to use Tauberian arguments for Borel summability to draw conclusions about P^k for large k. The argument of the next section is an even closer parallel [compare Eqs. (3.3) and (2.9)].

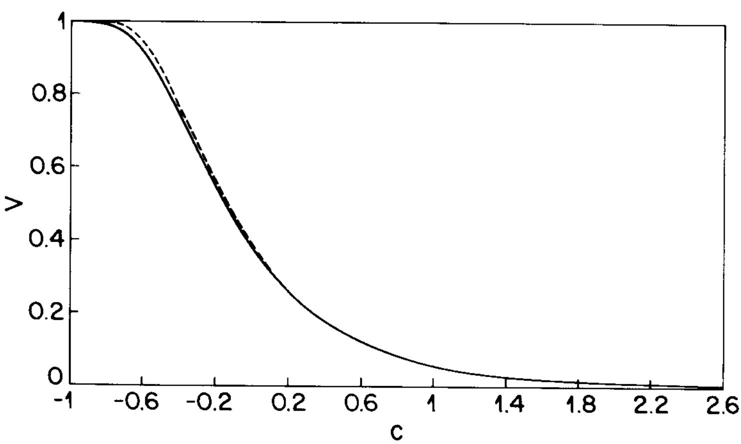

Fig. 5. The variation distance V as a function of $c = N/n - \frac{1}{4}\log n$, for $n = 100$ (broken curve) and $n = 1000$ (solid curve).

D. Here is a feature of the random walk that is quite different in discrete and continuous time. To bound variation distance, it is important to determine the largest value W of $|\mathbf{x}|$ such that $Pr\{X_N = \mathbf{x}\} \geq 1/2^n$. As we have seen for the discrete time process, $W = N$ up to about $N = \alpha n$, at which time the behavior changes.

In the continuous case, $Pr\{X_t = \mathbf{x}\} \geq 1/2^n$ if and only if Eq. (2.12) is satisfied. For t of the order $\frac{1}{4}n \log n + cn$, the continuous and discrete cases give the same answer. However, for $t \ll n$ they differ. For example, if $t = 1$, the RHS of Eq. (2.12) is asymptotic to $n \log 2/\log n$. Now, if $t = 1$ only one step is expected. If X_N^* is the discrete time process, $Pr\{X_1^* = \mathbf{x}\} \geq 1/2^n$ if and only if $|\mathbf{x}| \leq 1$. This shows the need for a careful argument in the discrete case.

3. AN ASYMPTOTIC APPROXIMATION

We start from the explicit expression for the probability $P_N(w)$ of being at a point $\mathbf{x}$ of weight $w = |\mathbf{x}|$ on the n-dimensional cube after N steps,

$$P_N(w) = \frac{1}{2^n} \sum_{j=0}^{n} \left[1 - \frac{2j}{(n+1)}\right]^N \sum_{i=0}^{w} (-1)^i \binom{w}{i} \binom{n-w}{j-i}, \quad 0 \leq w \leq n . \tag{3.1}$$

We let $j = s + i$, and note that the second binomial coefficient in Eq. (3.1) is zero unless $0 \leq s \leq n - w$. Hence we obtain

$$P_N(w) = \frac{1}{2^n} \sum_{i=0}^{w} \sum_{s=0}^{n-w} \left[1 - \frac{2(i+s)}{(n+1)}\right]^N (-1)^i \binom{w}{i} \binom{n-w}{s} . \tag{3.2}$$

It follows from Eq. (3.2) that

$$\sum_{N=0}^{\infty} \frac{(n+1)^N}{N!} P_N(w) z^N = \frac{1}{2^n} e^{(n+1)z} (1 - e^{-2z})^w (1 + e^{-2z})^{n-w} . \tag{3.3}$$

Consequently, from Cauchy's formula, we have

$$P_N(w) = \frac{N!}{2(n+1)^N \pi i} \int_C e^z \sinh^w z \cosh^{n-w} z \frac{dz}{z^{N+1}} , \tag{3.4}$$

where C is a contour enclosing the origin.

We consider

$$n \gg 1 , \quad w = \rho n , \quad 0 \leq \rho \leq 1 \quad \text{and} \quad N = \lambda n , \quad \lambda > \rho , \tag{3.5}$$

and let

$$G(z) = \rho \log \sinh z + (1 - \rho) \log \cosh z - \lambda \log z . \tag{3.6}$$

Then from Eq. (3.4),

$$P_N(w) = \frac{N!}{2(n+1)^N \pi i} \int_C e^z e^{nG(z)} \frac{dz}{z} . \tag{3.7}$$

We will obtain an asymptotic approximation to the integral in Eq. (3.7) by the method of steepest descents [6]. The saddle points are given by $G'(z) = 0$. But,

$$G'(z) = \rho \coth z + (1-\rho) \tanh z - \lambda / z , \tag{3.8}$$

and

$$G''(z) = (1-\rho) \operatorname{sech}^2 z - \rho \operatorname{cosech}^2 z + \lambda / z^2 . \tag{3.9}$$

Since $\lambda > \rho$ and $\sinh x > x$ for $x > 0$, it follows that $G''(x) > 0$ for $x > 0$. But, $G'(x) \to -\infty$ as $x \to 0+$, since $\lambda > \rho$, and $G'(x) \to 1$ as $x \to +\infty$. Hence there is a unique positive zero of $G'(x)$, which we denote by β. Consequently, there are two real saddle points $z = \pm\beta$, where

$$\rho \coth \beta + (1-\rho) \tanh \beta - \lambda / \beta = 0 , \quad \beta > 0 . \tag{3.10}$$

Next,

$$iG'(iy) = \rho \cot y - (1-\rho) \tan y - \lambda / y \equiv g(y) . \tag{3.11}$$

Since $\lambda > \rho$, $g(y) \to \mp\infty$ as $y \to 0\pm$. For $0 < \rho \le 1$, $g(y) \to \pm\infty$ as $y \to m\pi\pm$, for $m = \pm 1, \pm 2, \ldots$. Also, for $0 \le \rho < 1$, $g(y) \to \pm\infty$ as $y \to (m + 1/2)\pi\pm$, for $m = 0, \pm 1, \ldots$. Hence there are infinitely many saddle points on the imaginary axis. We note that alternate saddle points are not present if $\rho = 0$ or $\rho = 1$. Now,

$$\operatorname{Re} G(iy) = \rho \log |\sin y| + (1-\rho) \log |\cos y| - \lambda \log |y| . \tag{3.12}$$

Since $\lambda > \rho$, $\operatorname{Re} G(iy) \to +\infty$ as $y \to 0\pm$. For $0 < \rho \le 1$, $\operatorname{Re} G(iy) \to -\infty$ as $y \to m\pi\pm$, for $m = \pm 1, \pm 2, \ldots$. Also, for $0 \le \rho < 1$, $\operatorname{Re} G(iy) \to -\infty$ as $y \to (m + 1/2)\pi\pm$, for $m = 0, \pm 1, \ldots$. Hence there are infinitely many "sinks" on the imaginary axis, and alternate sinks are not present if $\rho = 0$ or $\rho = 1$.

The steepest paths are curves along which $\operatorname{Im} G(z)$ is constant. From Eq. (3.6), the curve $\operatorname{Im} G(x + iy) = 0$ is given by

$$\rho \tan^{-1} (\coth x \tan y) + (1-\rho) \tan^{-1} (\tanh x \tan y) - \lambda \tan^{-1} (y/x) = 0 . \tag{3.13}$$

For $x > 0$ a solution is $y = 0$. Another solution, in polar coordinates

$$x = r \cos \theta , \quad y = r \sin \theta , \quad -\pi/2 < \theta < \pi/2 , \tag{3.14}$$

is given by

$$\rho \tan^{-1} [\coth (r \cos \theta) \tan (r \sin \theta)] + (1-\rho) \tan^{-1} [\tanh (r \cos \theta) \tan (r \sin \theta)] = \lambda\theta . \tag{3.15}$$

We note that the solution $r(\theta)$ is a symmetric function of θ. In view of (3.10), this curve passes through the saddle point at $r=\beta$, $\theta=0$, and corresponds to the paths of steepest descent from this point, since $G''(\beta)>0$.

The behavior of the curve given in Eq. (3.15) as $\theta \to \pi/2-$ is investigated in the Appendix. We write

$$\lambda = k+\mu\ , \quad 0<\mu\leq 1\ , \quad k \text{ a nonnegative integer}\ . \tag{3.16}$$

It is shown that the curve in general goes to a sink on the imaginary axis, corresponding to Re $G(re^{i\theta}) \to -\infty$, but goes to a saddle point on the imaginary axis when k is even and $\mu=\rho$ $(0<\rho\leq 1)$, or when k is odd and $\mu = 1-\rho$ $(0\leq\rho<1)$. Since the steepest paths are given by Im $G(z)=$ const., it follows from Eq. (3.6) that the paths of steepest descent from the saddle point at $z=-\beta$ are given by the image in the imaginary axis of the curve corresponding to Eqs. (3.14) and (3.15). Consequently, we take the contour of integration C in Eq. (3.7) as the union of these two curves.

Since $n \gg 1$, the main contributions to the integral in Eq. (3.7) arise from the neighborhoods of the saddle points at $z=\pm\beta$. These contributions may be evaluated asymptotically by Laplace's method [6], and, to leading order, are found to be

$$i\left[\frac{2\pi}{G''(\beta)n}\right]^{1/2}\frac{e^{\beta}}{\beta}\,e^{nG(\beta)}\ ,\quad i\left[\frac{2\pi}{G''(-\beta)n}\right]^{1/2}\frac{e^{-\beta}}{\beta}\,e^{nG(-\beta)}\ , \tag{3.17}$$

respectively. But, from (3.9), $G''(-\beta)=G''(\beta)$ and, from Eqs. (3.5) and (3.6),

$$e^{nG(-\beta)} = e^{in(\rho-\lambda)\pi}e^{nG(\beta)} = (-1)^{N+w}e^{nG(\beta)}\ . \tag{3.18}$$

It follows from Eqs. (3.5)–(3.7) that, for $n\gg 1$,

$$P_N(w) \sim \frac{N!\sinh^w\beta\cosh^{n-w}\beta}{(n+1)^N\beta^{N+1}[2\pi G''(\beta)n]^{1/2}}\,[e^{\beta}+(-1)^{N+w}e^{-\beta}]\ , \tag{3.19}$$

where β is given by Eq. (3.10), and $G''(\beta)$ by Eq. (3.9). Although further terms in the asymptotic expansion may be derived, the approximation in Eq. (3.19) suffices for our purposes.

4. THE VARIATION DISTANCE

According to Lemma 2, $P_N(w)$ is a nonincreasing function of w, and the crossover point is defined to be

$$W=\max\{w \,|\, P_N(w)\geq 2^{-n}\}\ . \tag{4.1}$$

The variation distance, from Eq. (2.11), is

$$V = \sum_{w=0}^{W} \binom{n}{w} [P_N(w) - 2^{-n}] . \tag{4.2}$$

We will derive the limiting value of V as $n \to \infty$ when

$$\lambda = \tfrac{1}{4} \log (bn) , \quad 0 < b_0 \leq b = O(1) . \tag{4.3}$$

We first use Eq. (3.19) to obtain an asymptotic approximation to $P_N(w)$, subject to Eq. (4.3). Since $\lambda \gg 1$, it follows from Eq. (3.10) that $\beta \gg 1$ and

$$\lambda / \beta \sim 1 + 2(2\rho - 1)e^{-2\beta} + 2e^{-4\beta} + \cdots . \tag{4.4}$$

The inversion of Eq. (4.4) leads to

$$\beta \sim \lambda[1 + 2(1 - 2\rho)e^{-2\lambda} - 2\kappa e^{-4\lambda} + \cdots] , \quad \kappa = 1 + 2(1 - 2\rho)^2(2\lambda - 1) . \tag{4.5}$$

With the help of Eqs. (4.3) and (4.5) it is found that, for $N = \lambda n$ and $n \gg 1$,

$$\left(\frac{\beta}{\lambda}\right)^N \sim \exp\left\{2(1 - 2\rho)\lambda\left(\frac{n}{b}\right)^{1/2} - \frac{2\lambda}{b} [\kappa + (1 - 2\rho)^2]\right\} , \tag{4.6}$$

$$(1 - e^{-2\beta})^{\rho n}(1 + e^{-2\beta})^{(1-\rho)n}$$

$$\sim \exp\left\{(1 - 2\rho)\left(\frac{n}{b}\right)^{1/2} - \frac{1}{2b} [1 + 8(1 - 2\rho)^2\lambda]\right\} , \tag{4.7}$$

$$n\beta \sim N + 2(1 - 2\rho)\lambda\left(\frac{n}{b}\right)^{1/2} - \frac{2\lambda\kappa}{b} , \tag{4.8}$$

$$\left(1 + \frac{1}{n}\right)^N \sim e^\lambda , \tag{4.9}$$

and, from Eq. (3.9),

$$G''(\beta) \sim 1/\lambda . \tag{4.10}$$

But, from Stirling's formula [13, p. 12],

$$N! \sim (2\pi N)^{1/2} N^N e^{-N} . \tag{4.11}$$

Consequently, from Eqs. (3.5), (3.19), and (4.6)–(4.11), we obtain the asymptotic approximation

$$P_N(w) \sim 2^{-n} \exp\left\{(1 - 2\rho)\left(\frac{n}{b}\right)^{1/2} - \frac{1}{2b} [1 + 4(1 - 2\rho)^2\lambda]\right\} . \tag{4.12}$$

We now consider $0 < \rho < 1$, with $w = \rho n \gg 1$ and $n - w = (1 - \rho)n \gg 1$. Then, again from Stirling's formula,

$$\binom{n}{w} \sim [2\pi\rho(1-\rho)n]^{-1/2} \exp\{-n[\rho \log \rho + (1-\rho) \log (1-\rho)]\} . \tag{4.13}$$

We define

$$f(\rho) = \log 2 + \rho \log \rho + (1-\rho) \log (1-\rho) . \tag{4.14}$$

Then, from Eqs. (4.12)–(4.14), we obtain

$$\binom{n}{w} P_N(w) \sim [2\pi\rho(1-\rho)n]^{-1/2} \exp\left\{(1-2\rho)\left(\frac{n}{b}\right)^{1/2} - \frac{1}{2b}[1+4(1-2\rho)^2\lambda] - nf(\rho)\right\} . \tag{4.15}$$

Also,

$$f'(\rho) = \log\left(\frac{\rho}{1-\rho}\right), \quad f''(\rho) = \frac{1}{\rho} + \frac{1}{(1-\rho)} . \tag{4.16}$$

Hence $f(\frac{1}{2}) = 0$, $f'(\frac{1}{2}) = 0$ and $f''(\rho) > 0$ for $0 < \rho < 1$. It follows from Eq. (4.15) that $\binom{n}{w} P_N(w)$ is exponentially small unless

$$1 - 2\rho = \frac{\chi}{\sqrt{n}} , \quad \text{where} \quad \chi = O(1) . \tag{4.17}$$

Also, from Eq. (4.12),

$$P_N(w) \sim 2^{-n} \exp\left(\frac{\chi}{\sqrt{b}} - \frac{1}{2b}\right) . \tag{4.18}$$

Hence, from Eqs. (3.5), (4.1), (4.17), and (4.18), it follows that

$$\frac{W}{n} = \frac{1}{2}\left(1 - \frac{\gamma}{\sqrt{n}}\right), \quad \text{where} \quad \gamma \sim \frac{1}{2\sqrt{b}} . \tag{4.19}$$

From Eqs. (4.14) and (4.17) we find that

$$nf(\rho) = \sum_{l=0}^{\infty} \frac{\chi^{2l+2}}{(2l+1)(2l+2)n^l} = F(\chi, n) . \tag{4.20}$$

Then, from Eq. (4.15), we have

$$\binom{n}{w} P_N(w) \sim [\pi(n-\chi^2)/2]^{-1/2} \exp\left[\frac{\chi}{\sqrt{b}} - \frac{1}{2b}\left(1 + \frac{4\lambda}{n}\chi^2\right) - F(\chi, n)\right] . \tag{4.21}$$

Also, from Eqs. (4.17) and (4.19),

$$\chi = \frac{2}{\sqrt{n}}(W - w) + \gamma . \tag{4.22}$$

From the Euler–Maclaurin summation formula [1], we obtain the approximation

$$\sum_{w=0}^{W} \binom{n}{w} P_N(w) \sim (2\pi)^{-1/2} e^{-1/2b} \times \int_{\gamma}^{\sqrt{n}} \exp\left[\frac{\chi}{\sqrt{b}} - \frac{2\lambda\chi^2}{nb} - F(\chi, n)\right] \frac{d\chi}{(1-\chi^2/n)^{1/2}} . \tag{4.23}$$

Following Friedman [8], we let

$$\frac{\zeta}{2}(\zeta - 2\sigma) = F(\chi, n) - \frac{\chi}{\sqrt{b}} = nf\left[\frac{1}{2}\left(1 - \frac{\chi}{\sqrt{n}}\right)\right] - \frac{\chi}{\sqrt{b}} , \tag{4.24}$$

from Eqs. (4.17) and (4.20), where $\zeta = \sigma$ when $\chi = \psi$, and

$$f'\left[\frac{1}{2}\left(1 - \frac{\psi}{\sqrt{n}}\right)\right] = -\frac{2}{\sqrt{bn}} . \tag{4.25}$$

Hence,

$$\frac{\sigma^2}{2} = \frac{\psi}{\sqrt{b}} - nf\left[\frac{1}{2}\left(1 - \frac{\psi}{\sqrt{n}}\right)\right], \tag{4.26}$$

$$\zeta - \sigma = \operatorname{sgn}(\chi - \psi)\left[\frac{2}{\sqrt{b}}(\psi - \chi) + 2n\left\{f\left[\frac{1}{2}\left(1 - \frac{\chi}{\sqrt{n}}\right)\right] - f\left[\frac{1}{2}\left(1 - \frac{\psi}{\sqrt{n}}\right)\right]\right\}\right]^{1/2}, \tag{4.27}$$

and the transformation is analytic in the neighborhood of $\chi = \psi$.

From Eqs. (4.16) and (4.25) we find that

$$\psi = \sqrt{n} \tanh\left(\frac{1}{\sqrt{bn}}\right) \sim \frac{1}{\sqrt{b}} . \tag{4.28}$$

Hence, from Eqs. (4.17), (4.20), and (4.26), it follows that

$$\sigma \sim \frac{1}{\sqrt{b}} , \tag{4.29}$$

and, from Eq. (4.24), that

$$\zeta \sim \chi , \quad \frac{d\zeta}{d\chi} \sim 1 . \tag{4.30}$$

Finally, from Eqs. (4.19), (4.23), (4.24), (4.29), and (4.30), we obtain

$$\sum_{w=0}^{W} \binom{n}{w} P_N(w) \sim (2\pi)^{-1/2} e^{-1/2b} \int_{\gamma}^{\infty} e^{-\zeta(\zeta - 2\sigma)/2}\, d\zeta \sim (2\pi)^{-1/2} \int_{\gamma}^{\infty} e^{-(\zeta-\sigma)^2/2}\, d\zeta \sim \pi^{-1/2} \int_{-(8b)^{-1/2}}^{\infty} e^{-\xi^2}\, d\xi . \tag{4.31}$$

Next, from Eqs. (4.13), (4.14), (4.17), and (4.20), we find that

$$2^{-n}\binom{n}{w} \sim [\pi(n-\chi^2)/2]^{-1/2} \exp[-F(\chi, n)] . \tag{4.32}$$

Hence, from Eq. (4.22) and the Euler–Maclaurin summation formula, we obtain the approximation

$$\sum_{w=0}^{W} 2^{-n}\binom{n}{w} \sim (2\pi)^{-1/2} \int_{\gamma}^{\sqrt{n}} \exp[-F(\chi, n)] \frac{d\chi}{(1-\chi^2/n)^{1/2}} . \tag{4.33}$$

We now let

$$\tfrac{1}{2}\eta^2 = F(\chi, n) , \tag{4.34}$$

so that, from Eq. (4.20),

$$\eta \sim \chi , \quad \frac{d\eta}{d\chi} \sim 1 . \tag{4.35}$$

Consequently, from Eqs. (4.19) and (4.33)–(4.35), we obtain

$$\sum_{w=0}^{W} 2^{-n}\binom{n}{w} \sim (2\pi)^{-1/2} \int_{\gamma}^{\infty} e^{-\eta^2/2}\, d\eta \sim \pi^{-1/2} \int_{(8b)^{-1/2}}^{\infty} e^{-\xi^2}\, d\xi . \tag{4.36}$$

Finally, from Eqs. (4.2), (4.31), and (4.36), we have

$$V \sim \pi^{-1/2} \int_{-(8b)^{-1/2}}^{(8b)^{-1/2}} e^{-\xi^2}\, d\xi = \frac{2}{\sqrt{\pi}} \int_{0}^{(8b)^{-1/2}} e^{-\xi^2}\, d\xi = \mathrm{Erf}[(8\mathrm{b})^{-1/2}] , \tag{4.37}$$

in terms of the error function [13, p. 349]. With $b = e^{4c}$ we obtain Theorem 1.

The expression in Eq. (4.37) gives the limiting value of the variation distance V as $n \to \infty$, when $N/n = \lambda = \frac{1}{4}\log(bn)$, $0 < b_0 \le b = O(1)$. Odlyzko [16] has obtained numerical values of V, for various values of n, by calculating the probabilities $P_N(w)$, $w = 0, \ldots, n$, iteratively with respect to N. The value of V is depicted in Figure 5 as a function of

$$c = \frac{N}{n} - \frac{1}{4}\log n = \frac{1}{4}\log b . \tag{4.38}$$

The broken curve corresponds to $n = 100$ and the solid curve to $n = 1000$. The limiting value corresponding to Eq. (4.37), that is

$$V_\infty = \mathrm{Erf}[(8b)^{-1/2}] = \mathrm{Erf}(8^{-1/2}e^{-2c}) , \tag{4.39}$$

and the numerical value corresponding to $n = 10{,}000$ are graphically indistinguishable from each other for the range of c in Figure 5, and are essentially indistinguishable from the solid curve.

5. THE CROSSOVER POINT

We now turn our attention to the crossover point W defined by Eq. (4.1), and consider $N = \lambda n \gg 1$. According to Lemma 3, $W/n \rightarrow \lambda$ as $n \rightarrow \infty$ for $0 < \lambda < \alpha$, where $\alpha = 0.32756\ldots$ satisfies $\alpha 2^{1/\alpha} = e$. In order to determine the limiting crossover point for $\lambda > \alpha$, we will use the asymptotic approximation Eq. (3.19) to $P_N(w)$, subject to Eq. (3.5), where β is given by Eq. (3.10), and $G''(\beta)$ by Eq. (3.9). Thus, using Eqs. (4.9) and (4.11),

$$P_N(w) \sim \frac{e^{-\lambda}}{\beta}\left[\frac{\lambda}{G''(\beta)}\right]^{1/2}[e^{\beta} + (-1)^{N+w}e^{-\beta}]\left(\frac{\lambda}{\beta e}\right)^{\lambda n} \sinh^{\rho n}\beta \cosh^{(1-\rho)n}\beta\,. \tag{5.1}$$

In the limit $n \rightarrow \infty$, the crossover point corresponds to $W/n \rightarrow \rho$ where $0 \leq \rho \leq 1$ satisfies

$$\left(\frac{\lambda}{\beta e}\right)^{\lambda} \sinh^{\rho}\beta \cosh^{1-\rho}\beta = \frac{1}{2}\,, \tag{5.2}$$

i.e.,

$$\lambda \log(\lambda/\beta) - \lambda + \rho \log(\tanh\beta) + \log(2\cosh\beta) = 0\,. \tag{5.3}$$

The relationship between ρ and λ is given by Eqs. (3.10) and (5.3). We may rewrite Eq. (3.10) in the form

$$\rho = \sinh\beta\left(\frac{\lambda}{\beta}\cosh\beta - \sinh\beta\right). \tag{5.4}$$

We let

$$A = 1 + \frac{1}{\beta}\sinh\beta\cosh\beta\log(\coth\beta) + \log\beta\,, \tag{5.5}$$

and

$$B = \log(2\cosh\beta) + \sinh^2\beta\log(\coth\beta)\,. \tag{5.6}$$

Then, from Eqs. (5.3)–(5.6), we obtain

$$F(\lambda) \equiv \lambda(A - \log\lambda) - B = 0\,. \tag{5.7}$$

If we solve Eq. (5.7) for λ as a function of β, then Eq. (5.4) gives ρ as a function of β, and the relationship between ρ and λ is given parametrically. But, from Eq. (5.6), $B > 0$ since $\beta > 0$, $\coth\beta > 1$ and $\cosh\beta > 1$. Hence, from Eq. (5.7), $F(0) = -B < 0$, and $F(\lambda) \rightarrow -\infty$ as $\lambda \rightarrow +\infty$. Also, $F'(\lambda) = A - 1 - \log\lambda$, so that $F'(\lambda) = 0 \Rightarrow \lambda = e^{A-1} > 0$. Hence $F(\lambda)$ has either no or two positive zeros. Since the crossover point exists, $F(\lambda)$ must have two positive zeros, one smaller and one larger than e^{A-1}. From Eq. (5.4), since $\beta > 0$, $0 \leq \rho \leq 1$ implies that

$$\beta \tanh \beta \le \lambda \le \beta \coth \beta \,. \tag{5.8}$$

But, from Eq. (5.5),

$$A - 1 > \log(\coth \beta) + \log \beta \,, \quad e^{A-1} > \beta \coth \beta \,. \tag{5.9}$$

Hence the limiting crossover point corresponds to the zero of $F(\lambda)$ which satisfies Eq. (5.8).

Before discussing the numerical results, we investigate the limiting crossover point for $\beta \gg 1$, and for $0 \ll \beta < 1$. First, for $\beta \gg 1$, it is found from Eqs. (5.5)–(5.7) that

$$\frac{\lambda}{\beta}\left[1 + \frac{1}{2\beta}\left(1 - \frac{2}{3}e^{-4\beta} + \cdots\right) - \log\left(\frac{\lambda}{\beta}\right)\right] \sim 1 + \frac{1}{2\beta}\left(1 + \frac{1}{3}e^{-4\beta} + \cdots\right). \tag{5.10}$$

It follows from Eq. (5.8) that

$$\lambda/\beta \sim 1 + e^{-4\beta} + \cdots \,, \quad \beta \gg 1 \,. \tag{5.11}$$

Then, from Eq. (5.4), we obtain

$$\rho \sim \tfrac{1}{2} - \tfrac{1}{4}e^{-2\beta} + \cdots \,, \quad \beta \gg 1 \,. \tag{5.12}$$

Hence,

$$\rho \sim \tfrac{1}{2} - \tfrac{1}{4}e^{-2\lambda} + \cdots \,, \quad \lambda \gg 1 \,. \tag{5.13}$$

We note that this is consistent with Eq. (4.19) for $\lambda = \frac{1}{4}\log(bn)$.

Next, for $0 < \beta \ll 1$, it is found from Eqs. (5.5)–(5.7) that

$$\lambda\left[1 + \frac{\beta^2}{3}(1 - 2\log\beta) + \cdots - \log\lambda\right] \sim \log 2 + \frac{\beta^2}{2}(1 - 2\log\beta) + \cdots \,. \tag{5.14}$$

It follows from Eq. (5.8) that $\lambda \to \alpha$ as $\beta \to 0+$. More precisely,

$$\lambda \sim \alpha + \frac{(1 - 2\alpha/3)}{\log(1/\alpha)}\,\beta^2\left[\frac{1}{2} + \log\left(\frac{1}{\beta}\right)\right] + \cdots \,, \quad 0 < \beta \ll 1 \,. \tag{5.15}$$

Then, from Eq. (5.4), we obtain

$$\rho \sim \lambda - (1 - 2\alpha/3)\beta^2 + \cdots \,, \quad 0 < \beta \ll 1 \,. \tag{5.16}$$

From Eqs. (5.15) and (5.16) it is found, for $-\log\beta \gg 1$, that

$$\rho \sim \lambda - \frac{2(\lambda - \alpha)\log\alpha}{\log(\lambda - \alpha)} + \cdots \,, \quad \text{for} \quad -\log(\lambda - \alpha) \gg 1 \,. \tag{5.17}$$

Hence $d\rho/d\lambda \to 1$ as $\lambda \to \alpha +$, but $d^2\rho/d\lambda^2 \to -\infty$.

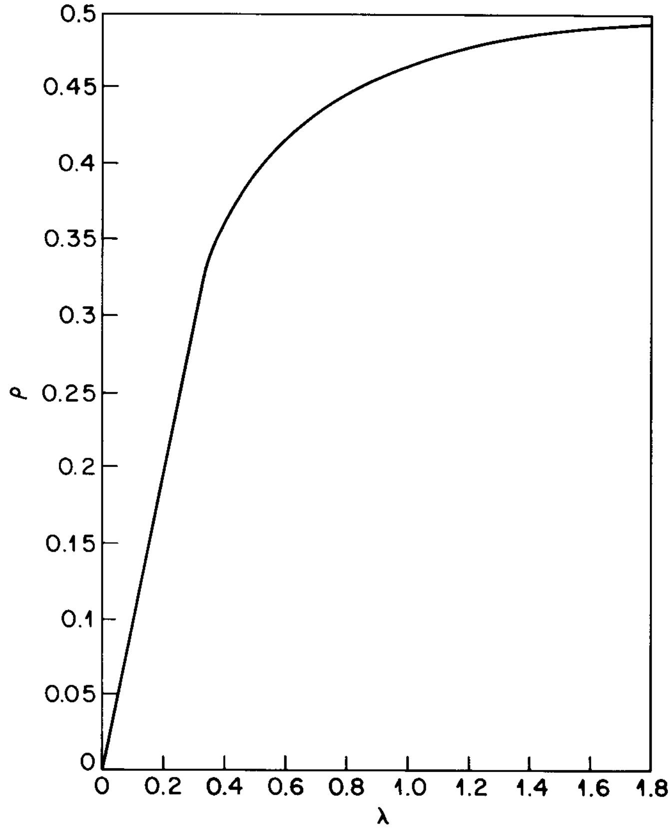

Fig. 6. The limiting crossover point ρ as a function of $\lambda = N/n$, corresponding to the limit of W/n as $n \to \infty$.

The limiting crossover point is depicted in Figure 6, that is ρ is plotted as a function of λ, where $N = \lambda n$ and $W/n \to \rho$ as $n \to \infty$. As pointed out earlier, $\rho = \lambda$ for $0 < \lambda < \alpha = 0.32756\ldots$. Numerical values of W, as a function of N, were obtained by Odlyzko [16], for different values of n. For $n = 1000$ the values of W/n are very close to the corresponding values of ρ depicted in Figure 6, and for $n = 10{,}000$ they are extremely close.

Acknowledgments

The authors thank Fan Chung for many insightful comments, Jim Fill and Andrew Odlyzko for help that borders on collaboration, and J. B. Seery for writing the program to evaluate the limiting crossover point.

APPENDIX

We investigate here the behavior of the curve given by Eq. (3.15) as $\theta \to \pi/2-$. We first note from Eq. (3.15) that if $r > 0$ and $0 \le \theta < \pi/2$, and m is a nonnegative integer, then $r \sin \theta = m\pi \Rightarrow \lambda\theta = m\pi$, and $r \sin \theta = (m + 1/2)\pi \Rightarrow \lambda\theta = (m + 1/2)\pi$. Hence, from (3.16),

$$r \sin \theta = \frac{s\pi}{2} \Rightarrow \lambda\theta = \frac{s\pi}{2}, \quad s = 0, \ldots, k. \tag{A.1}$$

We now let $\theta = \pi/2 - \phi$, and use Eq. (A.1) with $s = k$. Then, from Eqs. (3.15) and (3.16), we obtain

$$\rho \tan^{-1} [\coth (r \sin \phi) \tan (r \cos \phi)] + (1 - \rho) \tan^{-1} [\tanh(r \sin \phi) \tan (r \cos \phi)]$$
$$= \lambda\left(\frac{\pi}{2} - \phi\right) - \frac{k\pi}{2} = \frac{\mu\pi}{2} - \lambda\phi, \quad \text{if } k \text{ is even}, \tag{A.2}$$

and

$$\rho \cot^{-1} [\coth (r \sin \phi) \tan (r \cos \phi)] + (1 - \rho) \cot^{-1} [\tanh (r \sin \phi) \tan (r \cos \phi)]$$
$$= \frac{k\pi}{2} - \lambda\left(\frac{\pi}{2} - \phi\right) = \lambda\phi - \frac{\mu\pi}{2}, \quad \text{if } k \text{ is odd}. \tag{A.3}$$

Also, from Eq. (3.15), we have

$$\tanh (r \sin \phi) \tan (r \cos \phi) = \tan\left[\lambda\left(\frac{\pi}{2} - \phi\right)\right], \quad \text{if } \rho = 0, \tag{A.4}$$

and

$$\coth (r \sin \phi) \tan (r \cos \phi) = \tan\left[\lambda\left(\frac{\pi}{2} - \phi\right)\right], \quad \text{if } \rho = 1. \tag{A.5}$$

There are three possibilities as $\phi \to 0+$, i.e., as $\theta \to \pi/2-$; 1) $\tan r \to 0$, 2) $\tan r \to \pm\infty$, 3) $\tan r$ remains finite and does not tend to zero. Since $\lambda > \rho$, it follows from Eqs. (A.2) and (A.3) that r does not tend to zero, as is otherwise evident, since Eq. (3.15) gives the paths of steepest descent from the saddle point at $z = \beta$, and $\text{Re } G(re^{i\theta}) \to +\infty$ as $r \to 0$. Hence $\tan r \to 0 \Rightarrow r \to m\pi$, where m is a positive integer, and Eq. (A.4) implies that $\rho \neq 0$, so that in this case, from Eq. (3.12), $\text{Re } G(re^{i\theta}) \to -\infty$ as $\theta \to \pi/2-$, corresponding to a sink. Next, $\tan r \to \pm\infty \Rightarrow r \to (m + 1/2)\pi$, where m is a nonnegative integer, and Eq. (A.5) implies that $\rho \neq 1$, so that in this case, also, $\text{Re } G(re^{i\theta}) \to -\infty$ as $\theta \to \pi/2-$. It remains to consider the third case. If k is even then, from Eq. (A.2), as $\phi \to 0+$ we obtain

$$\rho\left(\frac{\pi}{2} - \frac{r\phi}{\tan r} + \cdots\right) + (1 - \rho)(r\phi \tan r + \cdots) \sim \frac{\mu\pi}{2} - \lambda\phi. \tag{A.6}$$

Hence $\mu = \rho$, which implies that $0 < \rho \leq 1$, and

$$\lambda = \rho r \cot r - (1 - \rho) r \tan r, \tag{A.7}$$

which, from Eq. (3.11), corresponds to a saddle point on the imaginary axis. If k is odd then, from Eq. (A.3), as $\phi \to 0+$ we obtain

$$\rho\left(\frac{r\phi}{\tan r}+\cdots\right)-(1-\rho)\left(\frac{\pi}{2}+r\phi\tan r+\cdots\right)\sim\lambda\phi-\frac{\mu\pi}{2}. \tag{A.8}$$

Hence $\mu = 1-\rho$, which implies that $0 \le \rho < 1$, and Eq. (A.7) holds.

To summarize, in the first two cases the curve given by Eq. (3.15) goes to a sink on the imaginary axis, but in the third case it goes to a saddle point on the imaginary axis when k is even and $\mu = \rho$ $(0 < \rho \le 1)$, or when k is odd and $\mu = 1-\rho$ $(0 \le \rho < 1)$. □

REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, DC, 1964, p. 886.

[2a] D. Aldous, Random walk on finite groups and rapidly mixing Markov chains, in *Seminaire de Probabilities XVII. Lecture Notes in Mathematics 986*, Springer, New York, 1983, pp. 2113–2297.

[2b] D. Aldous, Minimization algorithms and random walk on the d-cube, *Ann. Prob.*, **11**, 403–413 (1983).

[3] D. Aldous and P. Diaconis, Shuffling cards and stopping times, *Amer. Math. Monthly*, **93**, 333–348 (1986).

[4] P. Diaconis, *Group Representations in Probability and Statistics*, Institute of Mathematical Statistics, Hayward, CA, 1988.

[5] P. Diaconis and M. Shahshahani, Time to reach stationarity in the Bernoulli–Laplace diffusion model, *SIAM J. Math. Anal.*, **18**, 208–218 (1987).

[6] A. Erdélyi, *Asymptotic Expansions*, Dover, New York, 1956.

[7] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 1, 2nd. ed., Wiley, New York, 1950, p. 52.

[8] B. Friedman, Stationary phase with neighboring critical points, *J. Soc. Indust. Appl. Math.*, **7**, 280–289 (1959).

[9] M. Kac, Random walk and the theory of Brownian motion, *Amer. Math. Monthly*, **54**, 369–391 (1947).

[10] M. Kac, *Probability and Related Topics in Physical Sciences*, American Mathematical Society, Providence, RI, 1959.

[11] G. Letac and L. Takacs, Random walks on the m-dimensional cube, *J. Reine Angew. Math.* **310**, 187–195 (1979).

[12] J. MacWilliams and N. J. A. Sloane, *The Theory of Error Correcting Codes*, North-Holland, Amsterdam, The Netherlands, 1977.

[13] W. Magnus, F. Oberhettinger, and R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer-Verlag, New York, 1966.

[14] P. Mathews, Mixing rates for random walk on the cube, *SIAM J. Alg. Disc. Meth.*, **8**, 746–752 (1987).

[15] P. Mathews, (1988), Some sample path properties of a random walk on the cube, Tech. Rep. 88-17, Dept. of Mathematics, University of Maryland, Baltimore County Campus. *Theoretical Prob.* (to appear).

[16] A. M. Odlyzko (personal communication).

[17] B. Poonen (personal communication).

[18] A. Siegert, On the approach to statistical equilibrium, *Phys. Rev.*, **76**, 1708–1714 (1949).

Received January 10, 1989