

Highly Irregular Graphs*

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ABSTRACT

A connected graph is highly irregular if each of its vertices is adjacent only to vertices with distinct degrees. In this paper we investigate several problems concerning the existence and enumeration of highly irregular graphs as well as their independence numbers, with particular focus on the corresponding problems for highly irregular trees.

1. INTRODUCTION

One of the best known classes of graphs is the class of regular graphs. These graphs have been studied extensively in a variety of contexts. Regular graphs of degree r and order n exist with only limited, but natural, restrictions. Indeed, for integers r and n with $0 \leq r \leq n - 1$, an r -regular graph of order n exists if and only if nr is even.

A graph that is not regular will be called *irregular*. It is well-known (see [1], for example) that all nontrivial graphs, regular or irregular, must contain at least two vertices of the same degree. In a regular graph, of course, every vertex is adjacent only to vertices having the same degree. On the other hand, it is possible for a vertex in an irregular graph to be adjacent only to vertices with distinct degrees. With these observations made, we now consider graphs that are opposite, in a certain sense, to regular graphs.

*This subject was inspired by the first author's regular use of the phrase "highly irregular."

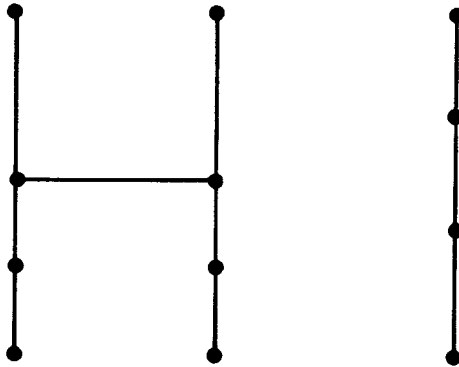


FIGURE 1. Two highly irregular graphs.

For a vertex v of a graph H we denote its neighborhood (i.e., the set of vertices adjacent to v) by $N(v)$. We define a connected graph H to be *highly irregular* if for every vertex v ,

$$u, w \in N(v), \quad u \neq w, \quad \text{implies that } \deg_H u \neq \deg_H w,$$

i.e., every vertex of H is adjacent only to vertices with distinct degrees. For example, the graphs H and I of Fig. 1 are *highly irregular*.

We begin by stating a sequence of easy observations.

Fact 1. If v is a vertex of maximum degree d in a highly irregular graph H , then v is adjacent to exactly one vertex of degree k for $1 \leq k \leq d$.

Fact 2. Let H be a highly irregular graph of order at least 4 having maximum degree d . Then H contains P_4 (the path of order 4) as an induced subgraph so that the vertices of degree 2 in P_4 have degree d in H and the vertices of degree 1 in P_4 have degree 1 in H .

Fact 3. A highly irregular graph H with maximum degree d has at least $2d$ vertices.

Fact 4. If H is a highly irregular graph with maximum degree d , then a highly irregular graph having maximum degree $d + 1$ can be obtained by taking two copies of H and joining vertices of degree d in the two copies. Furthermore, since K_1 is a highly irregular graph with maximum degree 0, we see that for every positive integer d there exists a highly irregular graph H_d (which may always be taken to be a tree) having maximum degree d .

Fact 5. There is no highly irregular graph of order 3 since K_3 and P_3 are the only connected graphs of order 3 and neither of these two graphs is highly irregular.

Fact 6. There is no highly irregular graph H of order 5 since, by Fact 3, H has maximum degree at most 2, and, by Fact 2, H must contain two vertices of degree 1 at a distance 3 from each other (which is impossible).

Fact 7. For $n = 1, 2, 4,$ and 6 there is a highly irregular graph of order n , namely, $K_1, K_2, P_4,$ and the graph of Fig. 2.

Fact 8. There is no highly irregular graph of order 7. For if such a graph exists, then it must have maximum degree 3 and, furthermore, there must be a vertex not adjacent to any vertex of degree 3 but adjacent to two vertices of degree 2, which is impossible.

Fact 9. For every positive integer $n \neq 3, 5,$ or 7 , there exists a highly irregular graph of order n .

Proof. It is sufficient to show there is a highly irregular graph of order n where $n \geq 8$.

Suppose now that $n = 2d \geq 8$. Let H denote the bipartite graph of order n having partite sets $V = \{v_1, v_2, \dots, v_d\}$ and $U = \{u_1, u_2, \dots, u_d\}$ and edge set $E(H) = \cup_{i=1}^d E_i$, where $E_i = \{v_i u_j \mid d - i + 1 \leq j \leq d\}$. Hence, it follows that $deg_H v_i = deg_H u_i = i$ for $i = 1, 2, \dots, d$ and therefore that H is a highly irregular graph of order n . Note further that by subdividing the edge $v_2 u_{d-1}$ of H , we obtain a highly irregular graph of order $n + 1 = 2d + 1 \geq 9$. We may now conclude that there exists a highly irregular graph of every order $n \geq 8$. ■

Figure 3 illustrates the construction employed in the proof of Fact 9 for $n = 8$ and $n = 9$.

Fact 10. The size of a highly irregular graph of order n is at most $n(n + 2)/8$, with equality possible for n even.

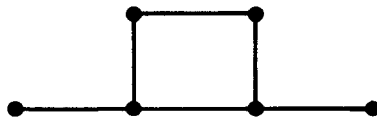


FIGURE 2. A highly irregular graph of order 6.

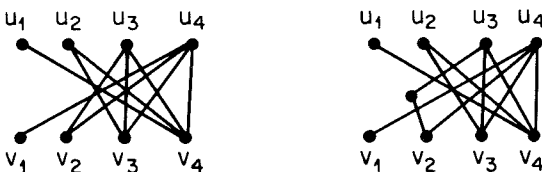


FIGURE 3. Highly irregular graphs of orders 8 and 9.

Proof. The construction of highly irregular graphs of even order n provided in Fact 8 shows that there exist highly irregular graphs with $n(n + 2)/8$ edges. Furthermore by Fact 3, every highly irregular graph of order n has maximum degree at most $n/2$. Since a vertex of every degree d in a highly irregular graph is adjacent to exactly one vertex of every degree k , $1 \leq k \leq d$, it follows that there are at least two vertices of every degree k . Hence a highly irregular graph of order $2m$ has size at most $(1/2) \cdot 2(1 + 2 + \cdots + m) = [m(m + 1)]/2$. Similarly, if the order is $2m + 1$, then the size is at most

$$\frac{m}{2} + \frac{m(m + 1)}{2} = \frac{m(m + 2)}{2}$$

and we are done. ■

2. HIGHLY IRREGULAR GRAPHS CONTAINING A GIVEN GRAPH AS AN INDUCED SUBGRAPH

In 1936 König [3] proved that if G is a graph with maximum degree d , then there exists a d -regular graph H containing G as an induced subgraph. In 1963 Erdős and Kelly [2] determined the minimum number of vertices required to add to a given graph G to produce such a graph H . We now present a result that may be considered an analogue to König's theorem for highly irregular graphs.

Theorem 1. Every graph of order $n \geq 2$ is an induced subgraph of a highly irregular graph of order $4n - 4$.

Proof. Let G be a graph of order $n \geq 2$. If $n = 2$, then $G \cong K_2$ or $G \cong \overline{K}_2$, and $H \cong P_4$ has the desired properties; thus we assume that $n \geq 3$. Let G' be another copy of G , where $V(G) = \{v_i \mid 1 \leq i \leq n\}$, $V(G') = \{v'_i \mid 1 \leq i \leq n\}$, and v'_i corresponds to v_i ($1 \leq i \leq n$). To the graphs G and G' we add $2n - 4$ new vertices u_1, u_2, \dots, u_{n-2} and $u'_1, u'_2, \dots, u'_{n-2}$. The vertices v_i and v'_i ($1 \leq i \leq n$) and the vertices u_j and u'_j ($1 \leq j \leq n - 2$) constitute the vertex set of the desired graph H . To $E(G) \cup E(G')$ we add several additional edges to complete the construction of H . First, for $1 \leq j \leq n$, we join v_j and v'_i for all i such that $v_j v_i \notin E(G)$, and for $1 \leq j \leq n - 1$ we also join v_j and v'_j . Then, for $1 \leq j \leq n - 2$, we join u_j to each v_i with $1 \leq i \leq j$ and join u'_j to each v'_i with $1 \leq i \leq j$.

The resulting graph H contains G as an induced subgraph. Moreover, for $1 \leq i \leq n$,

$$\deg_H v_i = \deg_H v'_i = 2n - 1 - i,$$

whereas for $1 \leq i \leq n - 2$,

$$\deg_H u_i = \deg_H u'_i = i,$$

so that H has exactly two vertices of degree i for each i , $1 \leq i \leq 2n - 2$. By the construction of H , no vertex is adjacent to vertices of the same degree, and H is connected, i.e., H is highly irregular. ■

The proof of the preceding theorem shows that for every graph G of order $n \geq 2$, there exists a highly irregular graph H of order $4n - 4$ and maximum degree $2n - 2$ containing G as an induced subgraph. Furthermore, it follows from the preceding construction that given such an H , we can uniquely recover the original graph G . This implies that there are at least as many highly irregular graphs of order $4n - 4$ as there are graphs of order n . If, for $n \geq 4$, in the graph H constructed in the proof of the preceding theorem, we (1) subdivide the edge v_1u_2 , (2) subdivide the edges v_1u_2 and $v'_1u'_2$ or (3) subdivide the edges v_1u_2 , $v'_1u'_2$, and $v_{n-1}v'_{n-1}$, we obtain a highly irregular graph containing G as an induced subgraph and having order $4n - 3$, $4n - 2$, or $4n - 1$, respectively. Consequently, there are at least as many highly irregular graphs of order k , $4n - 4 \leq k \leq 4n - 1$, as there are graphs of order n .

By a suitable choice of the graph G in Theorem 1, the proof of this theorem shows that there exists a highly irregular graph of order $4n - 4$ having clique number k , for every k with $2 \leq k \leq n$. We now show that for $n \geq 3$, no larger clique number is possible.

Corollary 1a. For $n \geq 3$, the smallest order of a highly irregular graph with clique number n is $4n - 4$.

Proof. We have already remarked that there exists a highly irregular graph of order $4n - 4$ having clique number n . Let H be a highly irregular graph containing a set S of n (≥ 3) mutually adjacent vertices. Since each vertex of S has degree at least $n - 1$ in H and no two vertices of S have the same degree in H , it follows that the maximum degree is at least $2n - 2$, which implies, by Fact 3, that H has order at least $4n - 4$. ■

Corollary 1a may be restated as follows.

Corollary 1b. For $n \geq 3$, the smallest order of a highly irregular graph containing K_n as a (n induced) subgraph is $4n - 4$.

Corollary 1b shows that the bound presented in Theorem 1 cannot then be improved in general. We next present a class of graphs where the bound given in Theorem 1 is not sharp.

Proposition 1. For $r \geq 1$, the minimum order of a highly irregular graph containing the regular complete bipartite graph $K(r, r)$ of order $n = 2r$ as an induced subgraph is $2n - 2$.

Proof. For $n = 2$, the result is immediate since $K(1, 1) \cong K_2$ is itself highly irregular; thus we assume that $n \geq 4$. Denote the partite sets of $K(r, r)$

by $\{v_1, v_2, \dots, v_r\}$ and $\{v'_1, v'_2, \dots, v'_r\}$. Since $K(r, r)$ is r -regular and v'_1 , say, is adjacent to v_i ($1 \leq i \leq r$), the maximum degree of any highly irregular graph H containing $K(r, r)$ as an induced subgraph is at least $2r - 1$, implying, by Fact 3, that the order of H is at least $4r - 2 = 2n - 2$. We now construct such a graph H of order $2n - 2$. To $K(r, r)$ we add $n - 2$ new vertices $u_1, u_2, \dots, u_{r-1}, u'_1, u'_2, \dots, u'_{r-1}$. To complete the construction of H , for $1 \leq j \leq r - 1$, we join u_j to each v_i with $1 \leq i \leq j$ and u'_j to each v'_i with $1 \leq i \leq j$. Then $K(r, r)$ is an induced subgraph of H , and H is highly irregular. ■

For even a regular graph G of order n , the problem of determining the minimum order of a highly irregular graph containing G as an induced subgraph appears to be very difficult. Even $G \cong \bar{K}_n$ poses great difficulty. We shall describe this situation in more detail in the last two sections.

3. HOW MANY HIGHLY IRREGULAR GRAPHS ARE THERE?

Let us denote by $HI(n)$ the number of (nonisomorphic) highly irregular graphs with n vertices. On one hand, one naturally suspects (and we will prove shortly) that highly irregular graphs are quite rare in the sense that if $G(n)$ denotes the total number of graphs with n vertices, then

$$HI(n)/G(n) \rightarrow 0$$

as $n \rightarrow \infty$ (in fact, exponentially rapidly). On the other hand, an unexpected corollary of Theorem 1 shows that there are quite a few highly irregular graphs. In particular,

$$HI(n) \geq (1 + o(1))2^{n^{2/32}}$$

and so,

$$HI(n) \geq (1 + o(1))G(n)^{1/16}.$$

We formalize this in our next result.

Theorem 2.

$$\frac{1}{16} + o(1) < \frac{\log HI(n)}{\log G(n)} < 2 - \frac{3}{4} \log_2 3 + o(1) = 0.8112 \dots$$

Proof. To prove the right-hand side, observe that, by Fact 10, a highly irregular graph with n vertices can have at most $\frac{n^2}{8} + O(n)$ edges. Then

$$HI(n) \leq \sum_{1 \leq j \leq n^{2/8+O(n)}} \binom{\binom{n}{2}}{j} \sim 2^{(n^2/2)H(1/4)}$$

where $H(x)$ denotes the binary entropy function $-(x \log_2 x + (1 - x) \cdot \log_2(1 - x))$ [so that $H(1/4) = 2 - (3/4) \log 3$]. Since $G(n) = 2^{(n^2/2)(1+o(1))}$, then the claim follows. By more careful consideration of the vertices of small degree, it is possible to improve this exponent slightly.

To prove the left-hand side, note that by the remark following Theorem 1,

$$HI(4n - 4) \geq G(n).$$

Since

$$G(t) = (1 + o(1))2^{\binom{t}{2}}$$

then

$$HI(n) \geq (1 + o(1))2^{n^2/32}$$

and the theorem is proved. ■

It is very likely that $\log HI(n) \sim cn^2$ for some constant c .

4. THE INDEPENDENCE NUMBERS OF HIGHLY IRREGULAR GRAPHS

The definition of highly irregular graphs suggests that these graphs tend to contain large independent sets of vertices. We investigate the independence number of highly irregular graphs in this section. We begin by showing the existence of a family $\{H_m\}$ of highly irregular graphs in which almost all vertices are independent, i.e.,

$$\lim_{m \rightarrow \infty} \frac{\beta(H_m)}{\nu(H_m)} = 1,$$

where $\beta(H_m)$ denotes the independence number of H_m and $\nu(H_m)$ denotes its order. It is convenient to first describe a class $\{G_m\}$ of graphs.

It is known (see [1]) that for each positive integer m , there exists exactly one connected graph F_m of order m with the property that for every pair u, v of distinct vertices, $\deg u \neq \deg v$, with exactly one exception. In particular, we may let $V(F_m) = \{u_1, u_2, \dots, u_m\}$, where

$$\deg u_i = \begin{cases} m - i & \text{if } 1 \leq i \leq \lceil m/2 \rceil \\ m - i + 1 & \text{if } \lceil m/2 \rceil + 1 \leq i \leq m. \end{cases}$$

Thus F_m has two vertices of degree $\lfloor m/2 \rfloor$ and one vertex of degree i for each $i \neq \lfloor m/2 \rfloor$ and $1 \leq i \leq m - 1$. Moreover, F_m has clique number $\lfloor m/2 \rfloor + 1$ as every two vertices u_i and u_j are adjacent, if $1 \leq i < j \leq \lfloor m/2 \rfloor + 1$. Let F'_m be another copy of F_m , where $V(F'_m) = \{v_1, v_2, \dots, v_m\}$ and v_i corresponds to u_i ($1 \leq i \leq m$).

For $m \geq 1$, define G_m to consist of F_m and F'_m together with the edges $u_i v_i$, where $1 \leq i \leq \lfloor m/2 \rfloor$. The graph G_m so constructed has order $2m$, maximum degree m , and is highly irregular. The graphs G_4 and G_5 are shown in Fig. 4.

We are now prepared to present the aforementioned result.

Theorem 3. There exists a family $\{H_m\}$ of highly irregular graphs in which almost all vertices are independent.

Proof. For a fixed positive integer m , let $M = m!$, and consider the graph G_M defined above, where then $V(G_m) = \{u_1, u_2, \dots, u_M, v_1, v_2, \dots, v_M\}$. For $1 \leq k \leq m$, we introduce a set

$$A_k = \{a_k(1), a_k(2), \dots, a_k(M/k)\}$$

of M/k vertices. We join $a_k(1)$ to u_1, u_2, \dots, u_k , the vertex $a_k(2)$ to $u_{k+1}, u_{k+2}, \dots, u_{2k}$, vertex $a_k(3)$ to $u_{2k+1}, u_{2k+2}, \dots, u_{3k}$, and so on. In general, then, for each k ($1 \leq k \leq m$), the vertices of A_k are joined to disjoint sets of k vertices u_i of G_m and, moreover, each u_i is joined to some $a_k(j) \in A_k$. We then proceed in a symmetric manner by introducing sets B_k , $1 \leq k \leq m$, of M/k vertices that are joined as above to the vertices v_i of G_m . The resulting graph is highly irregular and is denoted by H_m .

Since $|A_k| = |B_k| = M/k$,

$$\begin{aligned} \nu(H_m) &= \nu(G_m) + 2 \sum_{k=1}^m (M/k) \\ &= 2m! \left[1 + \sum_{k=1}^m (1/k) \right] \\ &= 2m! [1 + o(1)] \log m. \end{aligned}$$

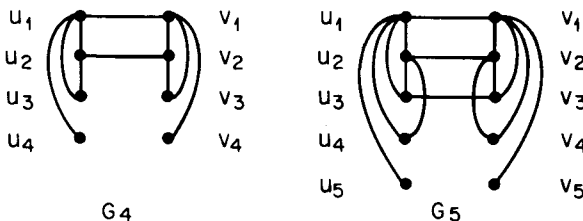


FIGURE 4. Constructing highly irregular graphs.

The set

$$\left(\bigcup_{k=1}^m A_k\right) \cup \left(\bigcup_{k=1}^m B_k\right)$$

is independent in H_m and has cardinality

$$2 \sum_{k=1}^m (M/k).$$

Thus,

$$\beta(H_m) = 2m! [1 + o(1)] \log m$$

and, consequently,

$$\lim_{m \rightarrow \infty} \frac{\beta(H_m)}{\nu(H_m)} = 1. \quad \blacksquare$$

By Theorem 3 we know that there are highly irregular graphs G for which $\beta(G)/\nu(G)$ is close to 1. We show that the maximum degree $\Delta(G)$ of such a graph G is large.

Theorem 4. If a maximum independent set I of vertices in a highly irregular graph G satisfies

$$\frac{|I|}{\nu(G)} \geq 1 - \frac{1}{m},$$

then $\Delta(G) \geq e^{m-2}$.

Proof. Partition I as $\bigcup_{k=1}^l I_k$, where I_k is the set of vertices of I having degree k . Next, partition $J = V(G) \setminus I$ as $\bigcup_{k=1}^s J_k$, where $x \in J_k$ if and only if x is joined to exactly k vertices in I . Thus, if $x \in J_k$, then $\deg x \geq k$.

If x and y are distinct vertices in I_k , then, since G is highly irregular, x and y must have disjoint neighborhoods (in J). Let $n = \nu(G)$. By hypothesis, then, $|I| \geq n - n/m$. Now since

$$x \in I_k \Rightarrow |N(x)| = k,$$

it follows that

$$k|I_k| \leq |J| = n - |I| \leq n - (n - n/m) = n/m.$$

Therefore,

$$(n - n/m) \leq n - |J| = |I| = \sum_{k=1}^t |I_k| \leq \frac{n}{m} \sum_{k=1}^t \frac{1}{k} \leq \frac{n}{m} (\log t + 1).$$

This implies that $\log t \geq m - 2$, and, consequently,

$$\Delta(G) \geq t \geq e^{m-2}. \quad \blacksquare$$

We have seen that the independence number of a highly irregular graph can be proportionally close to its order. We now show that every highly irregular graph must have a moderately large independence number.

Theorem 5. Every highly irregular graph G contains an independent set of size at least $\lceil \log(5 \nu(G)) \rceil / \lceil \log 3/2 \rceil$. On the other hand, there are highly irregular graphs G' that have independence number at most $\lceil (4 + o(1)) \log \nu(G') \rceil / \lceil \log 3/2 \rceil$.

Proof. Let G be a highly irregular graph of order n and let

$$1 = d_1 \leq d_2 \leq \dots \leq d_n = \Delta(G)$$

be the degree sequence of G . Since every vertex of degree $\Delta(G)$ is adjacent to exactly one vertex of degree i , for $i = 1, 2, \dots, \Delta(G)$, it follows that each i , $1 \leq i \leq \Delta(G)$, occurs as d_j for at least two values of j .

We now construct an independent set S of vertices of G as follows. First, we place a vertex u_1 of degree 1 in S and delete its neighbor in G . In general, among all the vertices remaining, select one vertex u_i of minimum degree to add to S and delete its neighbors. The degree of u_i in G is no more than half the number of vertices that have deleted so far. Thus, $\deg(u_i) \leq (1/2) \sum_{j=1}^{i-1} (\deg(u_j) + 1) + 1$. It is easy to prove by induction that $\deg(u_i) + 1 \leq 2 \cdot (3/2)^{i-1}$. Since $\sum_{i=1}^{|S|} (\deg(u_i) + 1) \geq \nu(G)$, we have

$$|S| \geq \frac{\log(5\nu(G))}{\log(3/2)}$$

as claimed.

It remains to construct a highly irregular graph G with independence number $(4 + o(1)) \log \nu(G')$. To achieve this we will first form a graph L_m (which is not required to be highly irregular) with independence number $\lceil (1 + o(1)) \log 2m \rceil / \lceil \log 3/2 \rceil$ and with vertices $w_1, \dots, w_{2m}, w_{2m+1}$, where

$$\deg w_{2i} = \deg w_{2i+1} = m - i + 1$$

for $i = 1, 2, \dots, m - 1$ and $\deg w_1 = m + 1$. Such a graph is not difficult to

construct; for example, one can start with disjoint cliques of sizes $\lfloor (2/3)m \rfloor, \lfloor (4/9)m \rfloor, \dots, \lfloor (2/3)^i m \rfloor, \dots$, and put in appropriate additional edges. Similarly, we construct a closely related graph \bar{L}_m with independence number $\lfloor (1 + o(1)) \log 2m \rfloor / \lfloor \log 3/2 \rfloor$ and with vertices $\bar{w}_1, \dots, \bar{w}_{2m}$ where $\deg \bar{w}_{2i-1} = \deg \bar{w}_{2i} = m - i + 1$. Now we construct G_m by combining two copies of L_m , denoted by $L_m^{(1)}$ and $L_m^{(2)}$, and two copies of \bar{L}_m , denoted by $\bar{L}_m^{(1)}$ and $\bar{L}_m^{(2)}$ and additional vertices $x_1, \dots, x_{2m+1}, y_1, \dots, y_{2m+1}, z$ and z' . The edge connection between $L_m^{(1)}, \bar{L}_m^{(1)}$, and x_1, \dots, x_{2m+1}, z can be described as follows: for i odd, x_i is adjacent to $x_j, j \neq i$, to $w_{2j}, 2j \leq 2m + 2 - i$, and to $\bar{w}_{2j+1}, 2j + 1 \leq 2m + 2 - i$; for i even, x_i is adjacent to $x_j, j \neq i$, to $w_{2j+1}, 2j + 1 \leq 2m + 2 - i$, and to $\bar{w}_{2j}, 2j \leq 2m + 2 - i$. Also z is adjacent to x_1 . The edge connection between $L_m^{(2)}, \bar{L}_m^{(2)}$, and y_1, \dots, y_{2m+1}, z' is done just the same way. In addition, x_i is adjacent to $y_i, 1 \leq i \leq 2m + 1$. It is not difficult to check that the resulting graph is highly irregular and has an independence number equal to

$$\frac{(4 + o(1)) \log m}{\log 3/2}$$

as required. ■

5. HIGHLY IRREGULAR TREES

By Fact 4, there exists a highly irregular tree with maximum degree d for every nonnegative integer d . The proof of this result is based on the fact that if we join two vertices of degree d in distinct copies of a highly irregular tree T with maximum degree d , then a highly irregular tree T' with maximum degree $d + 1$ is produced. By noting that K_1 is highly irregular and that if T has order n , then T' has order $2n$, we see that there exists a highly irregular tree with maximum degree d having order 2^d . We show that this is the smallest possible order for such a tree.

Theorem 6. The order of a highly irregular tree with maximum degree d is at least 2^d .

Proof. The smallest highly irregular tree with maximum degree 0, 1, or 2 is K_1, K_2 , or P_4 , respectively; thus, the result is true if $d \leq 2$. Hence we assume that $d \geq 3$.

Let v be a vertex of degree k in a highly irregular tree T with $\Delta(T) = d \geq 3$. Let u be a vertex of T adjacent with v , and if $k > 1$, let v_1, v_2, \dots, v_{k-1} be the remaining vertices of T adjacent with v , where the vertices v_i are labeled so that $\deg v_i \geq i$.

Let $f(k)$ denote the minimum number of edges in the subtree of T that contains uv as well as the component of $T - uv$ containing v . We show that $f(k) \geq 2^{k-1}$. This is certainly the case for $k = 1$ and $k = 2$. Assume that

$f(j) \geq 2^{j-1}$ for $2 \leq j \leq k$. Thus,

$$f(k) \geq 1 + \sum_{i=1}^{k-1} f(i) \geq 1 + \sum_{i=1}^{k-1} 2^{i-1} = 2^{k-1},$$

as desired.

To complete the proof, consider a vertex x with $\text{deg } x = \Delta(T) = d$. Necessarily, x has a neighbor x' with $\text{deg } x' = d$. By the above claim, the total number of edges in T is at least $2f(d) - 1$ (where the subtraction takes into account the fact that the edge xx' was counted twice). Therefore, the order n of T satisfies

$$n \geq 2f(d) \geq 2^d. \quad \blacksquare$$

Next, we discuss the independence number β of highly irregular trees. Unlike the situation for highly irregular graphs in general, there is no highly irregular tree almost all of whose vertices are independent. This fact is verified next.

Theorem 7. If T is a highly irregular tree of order $n \geq 2$, then $\beta(T) \leq 9n/14$.

Proof. Let X be a set of $\beta(T)$ independent vertices of T and let $Y = V(T) \setminus X$. Furthermore, let $e(X, Y)$ denote the number of edges joining X and Y . Choose $y_1 \in Y$ so that y_1 has the maximum number of neighbors in X ; denote by $N(y_1)$ the set of neighbors of y_1 in X , and let $n(y_1) = |N(y_1)|$. Next choose $y_2 \in Y$ so that y_2 has the maximum number of neighbors in $X \setminus N(y_1)$. Denote by $N(y_2)$ the set of neighbors of y_2 in $X \setminus N(y_1)$, and let $n(y_2) = |N(y_2)|$. Continuing in this manner, we produce a sequence y_1, y_2, \dots, y_k of vertices of Y so that for every $y \in Y \setminus \{y_1, y_2, \dots, y_k\}$ the neighborhood of y in X is contained in $\cup_{i=1}^k N(y_i)$. Thus,

$$\sum_{i=1}^k n(y_i) = |X|.$$

Since T is highly irregular, the sum of the degrees of the $n(y_i)$ vertices of $N(y_i)$ is at least

$$\sum_{j=1}^{n(y_i)} j = \binom{n(y_i) + 1}{2}.$$

Also, for $1 \leq i \leq k$, none of the vertices of $N(y_i)$ is adjacent to any y_j for $j < i$. It therefore follows that

$$e(X, Y) \geq \sum_{i=1}^k \binom{n(y_i) + 1}{2}.$$

Suppose now, to the contrary, that $|X| > 9n/14$. Then

$$|X| > \frac{9}{5}|Y|.$$

Since $|X| > |Y|$, $n(y_i) \geq 2$. We employ the fact that

$$|X| > \frac{3}{2}|Y|$$

to show that $n(y_i) \geq 3$. Suppose that $n(y_i) = 2$. Let a denote the number of vertices y_i for which $n(y_i) = 2$, and let b denote the number of vertices y_i for which $n(y_i) = 1$. Thus $a + b = k$. Observe that

$$\frac{3}{2}|Y| < |X| = 2a + b = a + k \leq a + |Y|$$

so that $a > (1/2)|Y|$. Then

$$e(X, Y) = \sum_{x \in X} \deg x \geq |X| + a > \frac{3}{2}|Y| + \frac{1}{2}|Y| = 2|Y|$$

so that $e(X, Y) > 2|Y|$. Since $n(y_i) = 2$, there are at most two edges joining y and X for every $y \in Y$. Consequently, $e(X, Y) \leq 2|Y|$, which produces a contradiction. Therefore $n(y_i) \geq 3$.

Suppose that $n(y_i) \geq 3$ for $1 \leq i \leq s$ and $n(y_i) \leq 2$ for $i > s$, and let $t = |Y| - s$. Then

$$\sum_{i>s} n(y_i) \leq \frac{3}{2}t$$

by the argument given earlier in the proof. Let

$$w = \sum_{i=1}^s [n(y_i) - 3].$$

Then

$$\sum_{i=1}^s n(y_i) = 3s + w.$$

Hence

$$|X| = \sum_{i=1}^s n(y_i) + \sum_{i>s} n(y_i) \leq 3s + w + \frac{3}{2}(|Y| - s)$$

implying that

$$s \geq \frac{2}{3}(|X| - w - \frac{3}{2}|Y|).$$

Consequently,

$$|Y| - t \geq \frac{2}{3}(|X| - w - \frac{3}{2}|Y|)$$

so that

$$t \leq 2|Y| - \frac{2}{3}|X| + \frac{2}{3}w.$$

Since T is a tree, $n > e(X, Y)$. Therefore,

$$n > e(X, Y) \geq \sum_{i=1}^k \binom{n(y_i) + 1}{2} = \sum_{i=1}^s \binom{n(y_i) + 1}{2} + \sum_{i>s} \binom{n(y_i) + 1}{2}.$$

Observe that

$$\sum_{i=1}^s \binom{n(y_i) + 1}{2} \geq 6s + 4w.$$

Let t_j ($j = 1, 2$) denote the number of vertices y_i for which $n(y_i) = j$, so that $t_1 + t_2 = t' \leq t$. Then $|X| = 3s + w + t_1 + 2t_2$ so that $t_2 = |X| - 3s - w - t'$ and $t_1 = 2t' + 3s + w - |X|$. Therefore,

$$\begin{aligned} n &> 6s + 4w + t_1 + 3t_2 \\ &= 6s + 4w + (2t' + 3s + w - |X|) + 3(|X| - 3s - w - t') \\ &= 2w + 2|X| - t' \\ &\geq 2w + 2|X| - t \\ &\geq 2w + 2|X| - 2|Y| + \frac{2}{3}|X| - \frac{2}{3}w \\ &= \frac{4}{3}w + \frac{14}{3}|X| - 2n, \end{aligned}$$

from which it follows that

$$3n > \frac{14}{3}|X| + \frac{4}{3}w.$$

Hence $|X| < 9n/14$, contrary to our assumption. ■

By a similar but more complicated argument, it can be shown that $\beta(T) \leq 12n/19$ for every highly irregular tree T of order $n \geq 2$, which provides a slight improvement over the result presented in Theorem 7. Although we do not believe $12n/19$ is the best possible bound for $\beta(T)$, where T is a highly irregular

tree of order n , this bound cannot be far from best possible since there exist highly irregular trees of order n with approximately $13n/21$ independent vertices, which we next show.

Consider the trees A and B shown in Fig. 5. (Note the vertices u and u' in A and vertices v and v' in B). The tree A is highly irregular, has order 26, and

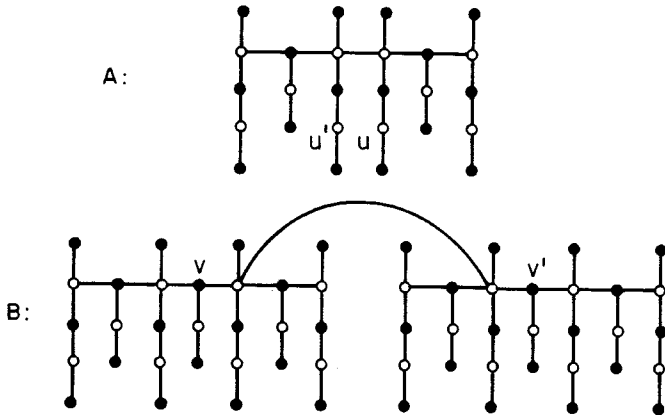


FIGURE 5. A highly irregular tree and a tree that is not highly irregular.

contains 16 independent vertices (indicated by solid circles). The tree B is *not* highly irregular since each vertex of degree 4 or 5 is adjacent to two vertices of degree 3. The tree B has order 58 and 36 independent (solid) vertices. For $k \geq 1$, we construct a highly irregular tree T_k by taking $k + 1$ copies A_1, A_2, \dots, A_{k+1} of A, k copies B_1, B_2, \dots, B_k of B and for $i = 1, 2, \dots, k$, joining v in B_i to u in A_i and joining v' in B_i to u' in A_{i+1} . Then the tree T_k has order $84k + 26$ and $52k + 16$ independent vertices. Consequently, the ratio of the number of independent vertices of T_k to its order is for large k

$$\frac{52k + 16}{84k + 26} \approx \frac{13}{21}$$

It would be quite interesting to know what the “truth” is in this case.

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