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Large Minimal Sets Which Force Long Arithmetic Progressions

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A classic theorem of van der Waerden asserts that for any positive integer k, there is an integer $W(k)$ with the property that if $W \geqslant W(k)$ and the set $\{1, 2, ..., W\}$ is partitioned into r classes $C_1, C_2, ..., C_r$, then some C_i will always contain a k-term arithmetic progression. Let us abbreviate this assertion by saying that $\{1, 2, ..., W\}$ *arrows* $AP(k)$ (written $\{1, 2, ..., W\} \rightarrow AP(k)$). Further, we say that a set X *critically* arrows $AP(k)$ if:(i) X arrows $AP(k)$; (ii) for any proper subset $X' \subset X$, X' does *not* arrow $AP(k)$. The main result of this note shows that for any given k there exist arbitrarily large sets X which critically arrow $AP(k)$.

INTRODUCTION

A fundamental result of van der Waerden (see [13] or [6]) asserts the following:

THEOREM. *In any partition of the set of positive integers $\mathbb{Z}^+ = C_1 \cup \cdots \cup C_r$ into finitely many classes, some class C_i must contain arbitrarily long arithmetic progressions.*

Van der Waerden's theorem formed a key element from which an important component of Ramsey theory developed, through the work of Rado, Deuber, and others (see [11, 3, 5]).

In its (equivalent) finite form, van der Waerden's theorem has the following statement:

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THEOREM. *For all $k, r \in \mathbb{Z}^+$ there exists a least integer $W = W(k, r)$ such that in any partition of*

$$\{1, 2, ..., W\} := [W] = C_1 \cup \cdots \cup C_r,$$

some C_i must contain a k-term arithmetic progression.

The true order of growth of $W(k, r)$ and especially $W(k) := W(k, 2)$ is a subject of great current interest in combinatorics. The best available bounds for $W(k)$ grow like the Ackermann function and consequently, are not even primitive recursive (see [6]). On the other hand, the strongest lower bounds presently known for $W(k)$ are of the form $k \cdot 2^k$ (see [1]).

In this note we investigate the following question: For every $k, r \in \mathbb{Z}^+$, do there exist *arbitrarily large* sets $X = X(k, r) \subseteq \mathbb{Z}^+$ satisfying:

(i) In any partition of $X = C_1 \cup \cdots \cup C_r$, some C_i must contain a k-term arithmetic progression;

(ii) The assertion in (i) does *not* hold if X is replaced by any proper subset of X.

This question is similar in spirit to some of these settled by Nešetřil, Rödl, and others (see [10, 7]) showing the existence, for example, of arbitrarily large graphs G which "critically" force complete graphs K_k for partitions of G's edges into r classes.

We will occassionally revert to the traditional "chromatic" terminology in which classes in a partition are denoted by "colors" and structures contained within a single class are called "monochromatic."

THE MAIN RESULT

Our attention will in fact be focussed on a version of the Hales–Jewett theorem (see [5] for a more detailed description). Very briefly the setting is as follows: For an arbitrary fixed set $A = \{a_1, ..., a_t\}$, a subset X of A^N, the N-fold cartesian product of A, is said to form a (combinatorial) *line* if X can be written for some nonempty $I \subseteq [N]$ in the form

$$X = \{(x_1, ..., x_N): x_i = a, i \in I\} \qquad x_j \in A \text{ fixed}, j \notin I.$$

Thus, X has cardinality $|X| = |A| = t$.

The basic theorem of Hales and Jewett [7] is:

THEOREM. *For all A and $r \in \mathbb{Z}^+$ there exists an $N = N(A, r)$ such that in any partition of $A^N = C_1 \cup \cdots \cup C_r$, some C_i must always contain a line.*

It is well known (and easily shown) that the Hales–Jewett theorem implies not only the theorem of van der Waerden but also its generalizations to higher dimensions (see [14]).

To begin with, we need the following definition: For a subset $X \subseteq A^N$, let us write $X \to (line)_r$ if for any partition of $X = C_1 \cup \cdots \cup C_r$, some C_i must contain a line. Similarly, for a family $\mathscr{L}$ of lines in A^N, let us write $\mathscr{L} \to (line)_r$ if for any partition of $X = C_1 \cup \cdots \cup C_r$, some C_i must contain all the points of some line $L \in \mathscr{L}$.

We first need a preliminary result:

LEMMA. *For every A, a, and r with $|A| = t \geqslant 2$ there exists an integer $N_0(A, a, r)$ such that if $N \geqslant N_0(A, a, r)$ then there is a family of lines $\mathscr{L} \subseteq A^N$ with the following properties:*

(a) $\mathscr{L} \to (line)_r$

(b) $\mathscr{L}' \to (line)_2$ *for every* $\mathscr{L}' \subseteq \mathscr{L}$ *with* $|\mathscr{L}'| < a$.

Proof. Define

$$n_1 = N(A, R),$$

$$n_{i+1} = N(A, R^{n_1 n_2 \cdots n_i}), \qquad 1 \leqslant i < R,$$

$$N = \sum_{i=1}^{R} n_i.$$

Let $G = (V, E)$ be a graph with $V = \{0, 1, ..., R\}$, chromatic number $\chi(G) > r$ and without cycles of length less than a (which exists by a result of Erdös [4]). Write

$$A = \{1, 2, ... t\}, \ A' = A \backslash \{t\}.$$

Define the set $X \subseteq A^N$ as follows:

$$x = (x_1, ..., x_N) \in X \qquad \text{iff there exists } i_0 = i_0(x) \leqslant R$$

and for each $i < i_0$, indices $p(i) \in (\sum_{j=1}^{i} n_j, \sum_{j=1}^{i+1} n_j]$ such that:

(i) $x_{p(i)} = t$;

(ii) $x_i \neq t$ for all $i > \sum_{j=1}^{i_0} n_j$.

Using this notation define a system of lines $\mathscr{L}$ in A^N as follows:
A line L belongs to $\mathscr{L}$ iff $L \subseteq X$ and there exists an edge $\{i, j\} \in E$, $i < j$, such that all points x of L with the exception of one satisfy $i_0(x) = i$, while the remaining point satisfies $i_0(x) = j$.

It is routine to prove $\mathscr{L} \to (line)_r$ (using the standard proof of the

Hales–Jewett theorem as given, for example, in [5]). To prove (b), let $\mathscr{L}' \subseteq \mathscr{L}$ with $|\mathscr{L}'| < a$. Thus, the set $\{i_0(x): x \in \cup \mathscr{L}'\}$ may be 2-colored (since G contains no cycle of length less than a), which in turn induces a 2-coloring of $\bigcup \mathscr{L}'$ containing no monochromatic line in $\mathscr{L}'$, as required. ▮

We are now ready to state the main result.

THEOREM. *For every A, a, and r with $|A| = t \geq 3$ there exists $N^*(A, a, r)$ such that if $N \geq N^*(A, a, r)$ then there exists $X \subseteq A^N$ satisfying:*

(i) $X \to (\mathit{line})_r$;

(ii) $X' \nrightarrow (\mathit{line})_r$ *for every* $X' \subseteq X$ *with* $|X'| < a$.

Proof. Without loss of generality let

$$A = \{1, 2,..., t\} \qquad \text{and set } A_0 = A \backslash \{t\}.$$

By the lemma there exist families of lines $\mathscr{L}_i$ satisfying

$$\mathscr{L}_1 \to (\mathit{line})_r \qquad \textit{with } \mathscr{L}_1 \subseteq A_0^{n_1}, \text{ where } n_1 = N_0\left(A_0, \binom{a}{2}, r\right)$$

and, for $1 \leq i \leq r$,

$$\mathscr{L}_{i+1} \to (\mathit{line})_{r_{i+1}} \qquad \text{with } \mathscr{L}_{i+1} \subseteq A_0^{n_{i+1}}$$

where

$$r_{i+1} = r^{t^{n_1 n_2 \cdots n_i}} \qquad \text{and} \qquad n_{i+1} = N_0\left(A_0, \binom{a}{2}, r_{i+1}\right).$$

Set $N = n_1 + n_2 + \cdots + n_{r+1}$.

For a line $L \in \mathscr{L}_i \subseteq A_0^{n_i}$ we really have $L = L(j), j \in A_0$. In other words, L consists of $|A_0| = t - 1$ points of $A_0^{n_i}$, obtained by letting the "variable" coordinate positions (simultaneously) assume the values $j = 1, 2,..., t-1$.

We now define the final set $X \subseteq A^N$ as follows:

$$x = (x_1, x_2,..., x_N) \in X$$

iff either

(a) $x \in A_0^N$, or

(b) there exists $i_0 \in \{1, 2,..., r\}$ and lines $L_i(j) \in \mathscr{L}_i$ for $1 \leq i \leq i_0$ such that the first $n_1 + n_2 + \cdots + n_{i_0}$ coordinates of x are just $(L_1(t), L_2(t),..., L_{i_0}(t))$ and furthermore, these are the only coordinate positions in which the symbol t occurs.

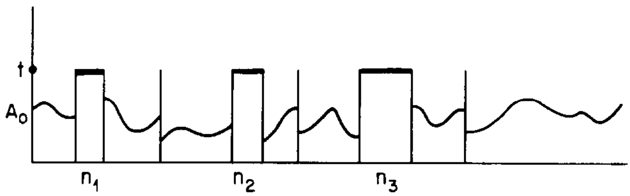

FIG. 1. Schematic representation of $x \in X$ with $i_0(x) = 3$.

Schematically, we have the situation shown in Fig. 1. As before, it is straightforward to prove that $X \to (line)_r$.

To prove (b), fix an arbitrary set $Y \subseteq X$ with $|Y| < a$. Denote by Y_i the set of all restrictions of words of Y to the coordinate positions $(\sum_{j=1}^{i-1} n_j, \sum_{j=1}^{i} n_j]$. Also, let $\bar{A}_0^{n_i}$ denote the restriction of A_0^N to the coordinate positions $(\sum_{j=1}^{i-1} n_j, \sum_{j=1}^{i} n_j]$ and let $Y_{i,0} = Y_i \cap \bar{A}_0^{n_i}$. By the choice of $\mathscr{L}_i$ there exist colorings $c_i\colon \bar{A}_0^{n_i} \to \{0, 1\}$ such that no monochromatic line occurs in $\mathscr{L}_i$. (Here, we use the fact that if $|Y_{i,0}| < a$ then $Y_{i,0}$ must contain fewer than $\binom{a}{2}$ lines).

Finally, define a coloring $c\colon Y \to \{0, 1,..., r-1\}$ as follows: For $y \in Y$, write $y = (\bar{y}_1, \bar{y}_2,..., \bar{y}_{r+1})$, where $\bar{y}_i \in Y_i$:

(i) If $y \in Y \cap A_0^N$ then set

$$c(y) \equiv \sum_{i=1}^{r+1} c_i(\bar{y}_i) \pmod 2.$$

(ii) If $y \in Y \backslash A_0^N$ then

$$y = (L_1(t), L_2(t),..., L_{i_0}(t), \bar{y}_{i_0+1},..., \bar{y}_{r+1}),$$

where $i_0 = i_0(y)$, $L_i(j)$ is a line in $\mathscr{L}_i$, $1 \leqslant i \leqslant i_0$, and the $y_i \in \bar{A}_0^{n_i}$. Define c by

$$c(y) = \begin{cases} \sum_{i=0}^{i_0} c_i(L_i(1)) + \sum_{i>i_0} c_i(\bar{y}_i) \pmod 2 & \text{if } i_0(y) = 1, \\ 1 + \sum_{i=0}^{i_0} c_i(L_i(1)) + \sum_{i>i_0} c_i(\bar{y}_i) \pmod 2 & \text{if } i_0(y) = 2 \text{ or } 3, \\ i_0(y) - 2 & \text{if } i_0(y) \geqslant 4. \end{cases}$$

It is now straightforward to verify that with this coloring c, no line in Y can be monochromatic. This completes the proof of the theorem. ∎

We should note that the assumption $|A| \geqslant 3$ is necessary since the corresponding result for $|A| = 2$ does not hold.

As immediate corollaries we have the following results:

COROLLARY 1. *For every A, a, and r with $|A| \geqslant 3$ there exists $\bar{N}(A, a, r)$ such that if $N \geqslant \bar{N}(A, a, r)$ then there exists $X \subseteq A^N$ satisfying*

(i) $|X| > a$;

(ii) $X \to (line)_r$;

(iii) $X' \nrightarrow (line)_r$ *for any proper subset* $X' \subset X$.

COROLLARY 2. *For any a, r, and $t \geqslant 3$ there exists $X \subseteq \mathbb{Z}^+$ satisfying*

(i) $|X| > a$.

(ii) *In any partition of X into r classes, some class must contain a t-term arithmetic progression.*

(iii) *The assertion in* (ii) *does not hold if X is replaced by any proper subset $X' \subset X$.*

(iv) *X contains no $(t+1)$-term arithmetic progression.*

Proof. We apply Corollary 1 with $A = \{0, 1, ..., t-1\}$ and with the association

$$x = (x_1, ..., x_N) \leftrightarrow \sum_{i=1}^{N} x_i T^i$$

for a sufficiently large integer T. ∎

REFERENCES

1. E. BERLEKAMP, A construction for partitions avoiding long arithmetic progressions, *Canad. Math. Bull.* **11** (1968), 409–414.
2. S. A. BURR, R. J. FAUDREE, AND R. H. SCHELP, On Ramsey-minimal graphs, *in* "Proc. 8th Southeastern Conf. on Comb., Graph Theory and Computing," pp. 115–124, Utilitias *Math. Winnipeg*, 1977.
3. W. DEUBER, Partitionen and Lineare Gleichungssysteme, *Math. Z.* **133** (1973), 109–123.
4. P. ERDÖS, Graph theory and probability, *Canad. J. Math.* **11** (1959), 34–38.
5. R. L. GRAHAM, "Rudiments of Ramsey Theory," CBMS Regional Conference in Mathematics Vol. 45, Amer. Math. Soc., Providence, R.I., 1981.
6. R. L. GRAHAM, B. L. ROTHSCHILD, AND J. H. SPENCER, "Ramsey Theory," Wiley, New York, 1980.
7. A. HALES AND R. I. JEWETT, Regularity and positional games, *Trans. Amer. Math. Soc.* **106** (1963), 222–229.
8. F. HARARY, "Graph Theory," Addison–Wesley, Reading, Mass. 1969.
9. J. NĚSETŘIL AND V. RÖDL, Van der Waerden's theorem for sequences of integers not containing an arithmetic progression of k terms, *Comment. Math. Univ. Carolin.* **17** (1976), 675–688.

10. J. NEŠETŘIL AND V. RÖDL, The structure of critical Ramsey graphs, *Colloq. Internat. C.N.R.S.* **260** (1978), 307–308.
11. R. RADO, Studien zur Kombinatorik, *Math. Z.* **36** (1933), 424–480.
12. J. H. SPENCER, Restricted Ramsey configurations, *J. Combin. Theory Ser. A* **19** (1975), 267–286.
13. B. L. VAN DER WAERDEN, Beweis einer Baudet'schen Vermutung, *Nieuw Arch. Wisk.* **15** (1927), 212–216.
14. E. WITT, Ein kombinatorischen Satz der Elementargeometrie, *Math. Nachr.* **6** (1951), 261–262.