Classes of Interval Graphs under Expanding Length Restrictions

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ABSTRACT

Let $C(\alpha)$ denote the finite interval graphs representable as intersection graphs of closed real intervals with lengths in $[1, \alpha]$. The points of increase for C are the rational $\alpha \geq 1$. The set $D(\alpha) = [\cap_{\beta>\alpha} C(\beta)] \backslash C(\alpha)$ of graphs that appear as soon as we go past α is characterized up to isomorphism on the basis of finite sets $E(\alpha)$ of irreducible graphs for each rational α. With $\alpha = p/q$ and p and q relatively prime, $|E(\alpha)|$ is computed for all (p, q) with $q \leq 2$ and $p = q + 1$. When $q = 1$, $E(p)$ contains only the bipartite star $K_{1,p+2}$. A lower bound on $|E(\alpha)|$ is given for all rational α.

1. INTRODUCTION

Let $C(\alpha)$ denote the class of nonempty finite interval graphs that have closed-interval representations in which every interval's length is between 1 and $\alpha \geq 1$ inclusive. Thus a reflexive graph $(X, \sim)$ is in $C(\alpha)$ if $0 < |X| < \infty$ and there are $f, \rho: X \to \mathbb{R}$ such that

$$\rho(X) \subseteq [1, \alpha]$$

and, for all x and y in X,

$$x \sim y \Leftrightarrow [f(x), f(x) + \rho(x)] \cap [f(y), f(y) + \rho(y)] \neq \emptyset .$$

Correspondence C is strictly increasing in α. Its smallest class is $C(1)$, the class of finite indifference or unit interval graphs [3, 6]; its upper bound is the class of finite interval graphs.

Our aim is to describe how C increases as α increases. We show first that it is left-continuous for every $\alpha > 1$, i.e.,

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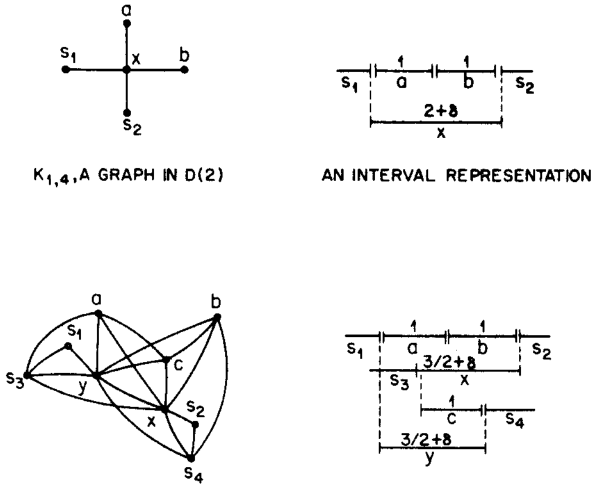

$K_{1,4}$, A GRAPH IN D(2) AN INTERVAL REPRESENTATION

A GRAPH IN D(3/2) AN INTERVAL REPRESENTATION

FIGURE 1

$$C(\alpha) = \bigcup_{\beta<\alpha} C(\beta) \qquad \text{for every } \alpha > 1\,,$$

and that it is right-continuous at every irrational $\alpha > 1$, i.e.,

$$C(\alpha) = \bigcap_{\beta>\alpha} C(\beta) \qquad \text{for every irrational } \alpha > 1\,.$$

Consequently, the points of increase for C are the rational $\alpha \geqslant 1$. The new interval graphs that appear as soon as we pass beyond such an α are those in

$$D(\alpha) = [\bigcap_{\beta>\alpha} C(\beta)]\backslash C(\alpha)\,.$$

Figure 1 pictures graphs in $D(2)$ and $D(\frac{3}{2})$ along with interval representations with minimum length 1. Every point in a graph has a loop (not shown); intervals are displaced vertically for visualization, and unspecified ends of intervals appear without short vertical bars. To minimize the longest length that is needed in a $\rho \geqslant 1$ representation, we divide the points of X into three parts, namely *short primaries* (a, b, c), which get length 1, *long primaries* (x, y), which are assigned the same longest length, and *secondaries* (the s_i), which can have any length between 1 and the long length. It is apparent from the figure that $K_{1,4}$ is not in $C(2)$ but is in $C(2 + \delta)$ for every $\delta > 0$. Similarly, the lower graph is not in $C(\frac{3}{2})$ but is in $C(3/2 + \delta)$ for every $\delta > 0$.

Proofs of the preceding continuity assertions for C appear in the next section. The third section then establishes a precise characterization of $D(\alpha)$ for each rational α in terms of irreducible graphs. The set of irreducible graphs for α is denoted by $E(\alpha)$. The characterization says that $(X, \sim)$ is in $D(\alpha)$ if and only if some induced subgraph of $(X, \sim)$ is isomorphic to a graph in $E(\alpha)$, and no in-

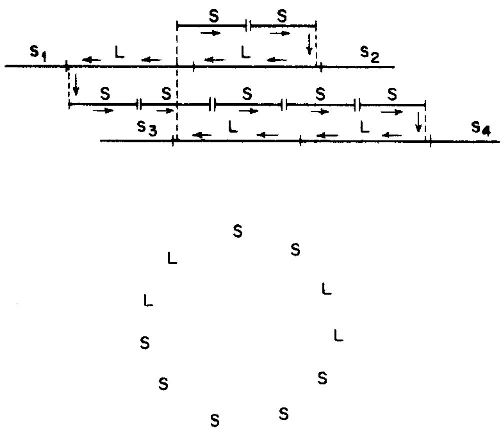

FIGURE 2

duced subgraph of $(X, \sim)$ is isomorphic to a graph in $\cup_{\beta>\alpha} E(\beta)$. The upper and lower graphs of Figure 1 are respectively in $E(2)$ and $E(\frac{3}{2})$.

Each $E(\alpha)$ for rational α is finite. In fact, if α is an integer, then $E(\alpha)$ is a singleton whose only member is the bipartite star $K_{1,\alpha+2}$. More generally, when $\alpha = p/q$ with p and q relatively prime positive integers, each graph in $E(\alpha)$ has $p + q + t$ points for some $2 \leqslant t \leqslant 2q$. The representation of such a graph consists of p short primaries, q long primaries, and t secondaries. The primaries are arranged into alternating runs of shorts and longs that zigzag back and forth along the line, ending approximately back at the beginning. The secondaries project off the ends of the long runs, e.g., s_3 and s_2 for x, and s_1 and s_4 for y in the lower part of Figure 1.

Figure 2 illustrates this for $\alpha = \frac{7}{4}$ with S = short primary, L = long primary, and s = secondary. The S runs go left-to-right; the L runs right-to-left. A run of two S's is followed by a run of two L's, then a run of five S's, and finally a run of two L's. The last L ends just to the left of the left endpoint of the initial S, thus completing a circle of p S's and q L's. The essential pattern of the run arrangement appears in the lower part of the figure. Different graphs in $E(\frac{7}{4})$ for this pattern are obtained from the interval representation by changing the lengths of the two inside secondaries: s_3 can be extended left to intersect s_1, and s_2 can be extended right to intersect s_4. However, because of symmetry, there are three and not four graphs in $E(\frac{7}{4})$ that have the (S, L) pattern shown. The graph for s_3 extended and s_2 not extended is isomorphic to the graph for s_3 not extended and s_2 extended.

The final section of the paper discusses the enumeration of the $E(\alpha)$. We have already mentioned that $|E(\alpha)| = 1$ if α is an integer, and will prove later that

$$\left|E\left(\frac{2n+1}{2}\right)\right| = \binom{n+2}{3} + 1.$$

In addition, we show that two basic (S, L) patterns that are not equivalent under rotation and/or reflection cannot yield isomorphic graphs in $E(\alpha)$. Hence $|E(\alpha)|$ is at least as large as the number of nonequivalent (S, L) patterns. This number is derived from Polya's enumeration theorem. An additional result then shows that

$$\left|E\left(\frac{q+1}{q}\right)\right| = \frac{1}{2}\left[\frac{1}{q+1}\binom{2q}{q} + \binom{q}{\lfloor \frac{1}{2}q \rfloor}\right],$$

which grows exponentially fast as q increases. For example, $|E(2)| = 1$, $|E(\frac{3}{2})| = 2$, $|E(\frac{4}{3})| = 4, \ldots, |E(\frac{11}{10})| = 8524, \ldots$.

2. CONTINUITY

Theorem 1. *C is left-continuous for every $\alpha > 1$ and right-continuous at $\alpha \geq 1$ if and only if α is irrational.*

The proof is based on two lemmas and several auxiliary definitions. We shall say that a finite, asymmetric partially ordered set $(X, <)$ is an *interval order* if there exist f, ρ: $X \to \mathbb{R}$ such that, for all $x, y \in X, \rho(x) > 0$ and

$$x < y \Leftrightarrow f(x) + \rho(x) < f(y).$$

An interval order $(X, <)$ *agrees with* an interval graph $(X, \sim)$ if $\sim$ is the symmetric complement of $<$, i.e., $x \sim y$ if and only if neither $x < y$ nor $y < x$. Every finite interval graph has at least one agreeing interval order—use (f, ρ) for $(X, \sim)$ to define $<$ by the preceding expression—and, since X is finite, the set of interval orders that agree with an interval graph is finite.

We refer to (f, ρ) as a *representation* of an interval graph or interval order when it satisfies the requisite interval-intersection or interval-ordering property for all $x, y \in X$. It will always be assumed that $\rho > 0$. Throughout the rest of this section, we assume that $\min\rho(x) = 1$ for each representation.

Lemma 1. Suppose $\alpha > 1$ is the maximum interval length in a particular representation of a finite interval graph. Then there is another representation of the graph with maximum interval length less than α.

Proof. Let (f, ρ) with $\max\rho(x) = \alpha > 1$ be a representation of a finite interval graph $(X, \sim)$. Assume with no loss in generality that $\min f(x) = 0$. Let $\Delta > 0$ be less than the minimum distance between nonintersecting intervals in the representation, and for all x define

$$f'(x) = \frac{f(x)}{1 + \Delta}, \qquad \rho'(x) = \frac{\rho(x) + \Delta}{1 + \Delta}.$$

It is easily checked that (f', ρ') is a representation of $(X, \sim)$ with $\min\rho'(x) = 1$ and $\max\rho'(x) < \alpha$. ∎

Lemma 2. Suppose $(X, \sim)$ is a finite interval graph. Let $\mu = \inf\{\max_x \rho(x)\}$, where the infimum is over all representations of $(X, \sim)$ for which $\min\rho(x) = 1$. Then μ is rational.

Proof. Given the hypotheses of the lemma, let $(X, <)$ be an interval order that agrees with $(X, \sim)$. By Theorem 7.2 in Fishburn [3] there is a set of strict inequalities of the form

$$\sum_{x \in A} \rho(x) < \sum_{x \in B} \rho(x), \qquad \emptyset \neq A \subseteq X, \qquad \emptyset \neq B \subseteq X, \qquad A \cap B = \emptyset$$

called the *ρ-set* of $(X, <)$ which has the following property. There exists an f such that (f, ρ) is a representation of $(X, <)$ if and only if [given $\min\rho(x) = 1$ in the present setting] ρ satisfies the inequalities in the ρ-set of $(X, <)$.

Let $\mu(X, <) = \inf\{\max_x \rho(x)\}$, where the infimum is over all representations of $(X, <)$ for which $\min\rho(x) = 1$. Modify the ρ-set of $(X, <)$ by replacing $<$ with $\leqslant$ throughout, and let ρ^* be a solution to the modified ρ-set that minimizes $\max\rho(x)$, given $\min\rho(x) = 1$. Then it is easily seen that small increases in the values of some $\rho^*(x)$, leaving $\min\rho^*(x)$ at 1, will satisfy the original ρ-set, and therefore $\mu(X, <) = \max\rho^*(x)$. Moreover, because $\min\rho(x) = 1$ and the coefficients in the modified ρ-set are integers, $\max\rho^*(x)$ must be rational.

Since every representation of $(X, \sim)$ is a representation of one of its agreeing interval orders, it follows that

$$\mu = \inf\{\mu(X, <)\colon (X, <) \text{ agrees with } (X, \sim)\}.$$

Since the set of agreeing interval orders is finite, and every $\mu(X, <)$ is rational, μ is rational. ∎

Proof of Theorem 1. Lemma 1 implies that C is continuous from the left for every $\alpha > 1$. For right-continuity, suppose first that α is irrational. If C were not right-continuous at α, then there would be an interval graph in $[\bigcap_{\beta > \alpha} C(\beta)] \setminus C(\alpha)$, so μ as defined in Lemma 2 for this interval graph would

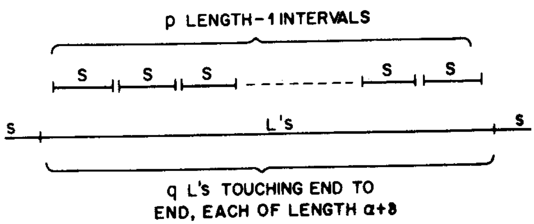

FIGURE 3

equal α. Since this contradicts the conclusion of Lemma 2, C is right-continuous at every irrational $\alpha > 1$.

On the other hand, if $\alpha \geqslant 1$ is rational, then $E(\alpha)$ is nonempty, so $D(\alpha)$ is nonempty and C is not right-continuous at α. An explicit description of one member of $E(\alpha)$ is shown by the interval representation in Figure 3. We suppose that the graph has $p + q + 2$ points, where $\alpha = p/q$ with p and q relatively prime positive integers. There are p short primaries S of length 1 each that are separated by very small gaps, q long primaries L of length $\alpha + \delta$ each that touch end-to-end with δ just large enough so that the span of the L's properly includes the span of the S's at both ends, and two secondaries (s) that intersect no S. With the gap length between S's suitably small and δ suitably small, each internal endpoint in the L run falls within an S interval since p and q are relatively prime. It is apparent that the interval graph is not in $C(\alpha)$ but is in $C(\beta)$ for every $\beta > \alpha$. ▮

3. CHARACTERIZATION

In view of Theorem 1, *we assume throughout the rest of the paper that* $\alpha = p/q \geqslant 1$, *with p and q relatively prime positive integers*. We first describe our characterization of $D(\alpha)$, then prove its validity. As noted below, we shall use lengths q and p instead of 1 and $\alpha + \delta$ for short and long primaries.

Given $\alpha = p/q$, consider the family of all circles or "rings" of p S's and q L's (see Figure 2). Two rings are defined to be equivalent if one can be obtained from the other by rotation and/or reflection (change clockwise to counterclockwise). We shall make no distinction between rings that are equivalent in this sense: see Figure 4.

Let $\mathcal{R}(\alpha)$ denote the family of equivalence classes of rings for α. For convenience, we identify a class in $\mathcal{R}(\alpha)$ by an arbitrary ring R in that class. Given $R \in \mathcal{R}(\alpha)$, let R^* be a list of its $p + q$ symbols for one full revolution around the circle that begins with S and ends with L. If we begin at S^* on the left of Figure 4 and proceed clockwise, then $R^* = SSSLLSLLLLLSSSSSSSLLL$.

Given R^* for $R \in \mathcal{R}(\alpha)$, construct intervals for its successive S's and L's as follows. (We now use length q for S, length p for L, and *open* each S interval at

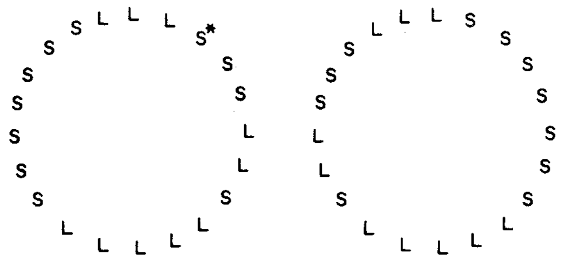

FIGURE 4. Equivalent rings for $\alpha = \frac{11}{10}$.

both ends to avoid intersections.) Assign interval $(0, q)$ to the first S. Suppose assignments have been made through the ith symbol in R^*, $i < p + q$. If symbol i is S with interval $(j, j + q)$, then

$$\text{if } i + 1 \text{ is } S, \text{ assign it } (j + q, j + 2q);$$
$$\text{if } i + 1 \text{ is } L, \text{ assign it } [j + q - p, j + q].$$

If symbol i is L with interval $[k, k + p]$, then

$$\text{if } i + 1 \text{ is } L, \text{ assign it } [k - p, k];$$
$$\text{if } i + 1 \text{ is } S, \text{ assign it } (k, k + q).$$

All endpoints (open or closed) have integer values, every endpoint serves exactly two intervals since p and q are relatively prime, and the left endpoint for the final L is 0 since we went to the right p times with length-q intervals and to the left q times with length-p intervals.

If some other R^* is used in the construction, we obtain either the same set of intervals, uniformly shifted left or right, or a uniformly shifted reflection of the original set. Hence the intersection graph of our $(p + q)$-point interval construction for R is the same regardless of how R^* is formed.

Given the interval construction for R^*, our final step is the placement of secondary intervals. These are closed and have lengths anywhere in $[q, p]$. A secondary interval is put at each end of each run of L intervals; it just touches that end and projects away from the run.

Different intersection graphs may be obtained by changing the lengths of the secondaries. If there are r runs of L's (hence also of S's) in R^*, there will be $2r$ secondary intervals initially. However, some of these may be redundant for a given set of lengths. In particular, the crucial role of a secondary is to prevent the interval for an S that is just inside an end of an L run from having any point outside the interval for that L. Hence, if one secondary touches the right endpoint of one L run *and* the left endpoint of another L run (and does not go beyond those endpoints), then it serves a dual purpose and one of the other initial secondaries can be deleted. Once such deletions have been made, all ends of L runs will still be "blocked" by secondaries, and no remaining secondary will be equivalent to or intersect the same intervals as any other secondary or primary interval.

Let $E(R)$ denote the set of all nonisomorphic intersection graphs thus constructed on the basis of R^* with the possible secondary placements, reduced as just indicated to avoid redundancy. When r is the number of L runs in R^*, each graph in $E(R)$ has at least $p + q + 2 + (r - 1)$ points and no more than $p + q + 2r$.

Finally, let $E(\alpha)$ be the set of all nonisomorphic graphs in the $E(R)$ for all $R \in \mathcal{R}(\alpha)$.

Theorem 2. A finite interval graph is in $D(\alpha)$ if and only if it has an induced subgraph in $E(\alpha)$ and has no induced subgraph in $\cup\{E(\beta): \beta > \alpha, \beta \text{ rational}\}$. In addition, every proper induced subgraph of a graph in $E(\alpha)$ is in $C(\alpha)$.

Let $C^*(\alpha)$ be the class of all nonempty finite interval orders that have closed-interval representations in which every interval's length is between 1 and α inclusive, or between q and p under uniform rescaling. The proof of Theorem 2 is based on two lemmas for C^*. The first, from Fishburn [2] (also [3, Theorem 8.1]), derives from Hanlon's study [4] of interval graphs.

Lemma 3. A finite interval graph is in $C(\alpha)$ if and only if every interval order that agrees with the interval graph is in $C^*(\alpha)$.

The other lemma, from Fishburn [1] (also [3, Theorem 8.4]), uses compositions of the ordering relation $<$ and its symmetric complement $\sim$ for an interval order $(X, <)$. Let $<^c$ and $\sim^c$ for $c \in \{1, 2, \ldots\}$ be respectively the c-fold compositions of $<$ and $\sim$. Thus, $x <^c(\sim^c)y$ if there are $z_1 = x, z_2, \ldots, z_c, z_{c+1} = y$ such that

$$z_1 < (\sim)z_2 < (\sim) \cdots < (\sim)z_c < (\sim)z_{c+1}\,.$$

Then, given positive integers $a_1, b_1, a_2, b_2, \ldots, a_r, b_r$,

$$<^{a_1} \sim^{b_1} <^{a_2} \sim^{b_2} \ldots <^{a_r} \sim^{b_r}$$

is the composite composition that contains (x, y) if there are $x_1 = x, y_1, x_2, y_2, \ldots, x_r, y_r, y$ such that

$$x_1 <^{a_1} y_1 \sim^{b_1} x_2 <^{a_2} y_2 \sim^{b_2} \cdots x_r <^{a_r} y_r \sim^{b_r} y\,.$$

The composite $\sim^{b_r} <^{a_r} \sim^{b_{r-1}} <^{a_{r-1}} \ldots \sim^{b_1} <^{a_1}$ is defined analogously.

Lemma 4. A finite interval order $(X, <)$ is in $C^*(\alpha)$ if and only if, for all $r \in \{1, \ldots, q\}$ and all integer vectors $(a_1, b_1, \ldots, a_r, b_r) \geq (2, 2, \ldots, 2, 1)$ for which $\Sigma a_i = p + r$ and $\Sigma b_i = q + r - 1$,

$$<^{a_1} \sim^{b_1} \ldots <^{a_r} \sim^{b_r} \subseteq <, \qquad \sim^{b_r} <^{a_r} \ldots \sim^{b_1} <^{a_1} \subseteq <.$$

Proof of Theorem 2. Suppose interval order $(X, <)$ is *not* in $C^*(\alpha)$. Then, by Lemma 4, it has a restriction on $p + q + 2r$ or fewer points for some $r \leq q$ which violates one of the composition-inclusion conditions at the end of the lemma. Let $(Y, \sim)$ be the interval graph for such a restriction. Then, by Lemma 3, $(Y, \sim)$ is not in $C(\alpha)$.

Alternatively, suppose an interval graph $(X, \sim)$ is *not* in $C(\alpha)$. Let $(X, <)$ be any one of its agreeing interval orders. By Lemma 3, $(X, <)$ is not in $C^*(\alpha)$.

Hence, by the preceding paragraph, $(X, \sim)$ has an induced subgraph $(Y, \sim)$ with $|Y| \leq p + q + 2r$ for some $r \leq q$ such that *all* (Lemma 3) of the interval orders $(Y, <)$ that agree with $(Y, \sim)$ violate one of the composition-inclusion conditions in Lemma 4.

It follows that the interval graphs that are not in $C(\alpha)$ are precisely those that have induced subgraphs on $p + q + 2r$ or fewer points for some $r \leq q$ such that one (and hence all) of the interval orders that agrees with the induced subgraph violates a condition of Lemma 4. At the same time, since $D(\alpha) = [\bigcap_{\beta>\alpha} C(\beta)]\backslash C(\alpha)$, the interval graphs in $D(\alpha)$ are precisely those which have induced subgraphs as just described *and* do not have any induced subgraph whose agreeing interval orders violate some composition-inclusion condition of Lemma 4 applied to all rational $\beta > \alpha$.

In view of the constructions developed earlier, it is not hard to see that $D(\alpha)$ is characterized as in Theorem 2. Because the interval graphs that correspond to an interval order and its dual are identical, it suffices to consider the first condition at the end of Lemma 4. Suppose this condition is violated, say by

$$x <^{a_1} y_1 \sim^{b_1} x_2 <^{a_2} y_2 \sim^{b_2} \cdots x_r <^{a_r} y_r \sim^{b_r} y,$$

and either $y < x$ or $y \sim x$. Then, in order for the corresponding interval graph to be in $C(\beta)$ for every $\beta > \alpha$ [it is not in $C(\alpha)$, as just proved], it must have an interval representation that adheres to the very tight construction developed previously. In particular, we must have $y \sim x$ and not $y < x$, and, in any $\sim^b$ path such as $y_1 \sim z_2 \sim \cdots \sim z_b \sim x_2$, all points must be distinct and $\sim$ holds between no other two distinct points in the path. The secondaries referred to earlier are the turning points $x, y_1, x_2, y_2, \ldots, x_r$, and y_r, but not y since the final $\sim$ path extends to $y_r \sim^{b_r+1} x$ *with* $b_r + 1 \geq 2$. Some secondaries may appear twice in the composite composition, but all other points (the primaries) are distinct. The short primaries are the points internal to the $<^a$ compositions, and the long primaries are the points internal to the $\sim^b$ compositions. Because $a_i \geq 2$ and $b_i \geq 2$ (replace b_r by $b_r + 1$) in Lemma 4, there are r nonempty runs of shorts and r nonempty runs of longs. Moreover, there are exactly p short primaries and q long primaries.

To get an interval representation with the desired length properties whose corresponding interval graph is not in $C(\alpha)$ but is in $C(\beta)$ for every $\beta > \alpha$, it is necessary to use the zigzag pattern for the short and long runs, with secondaries projecting off the ends of the long runs. If some secondaries serve a dual purpose as described previously, others can be deleted without changing the fact that the intersection graph is in $D(\alpha)$, and such deletions are made whenever possible.

All this translates into the construction described earlier in the section. The possible patterns of short runs and long runs (up to the circular equivalence) are given by the rings in $\mathcal{R}(\alpha)$ and, for each $R \in \mathcal{R}(\alpha)$, the graphs in $E(R)$ are those obtainable from different feasible length assignments to the secondaries with redundant deletions.

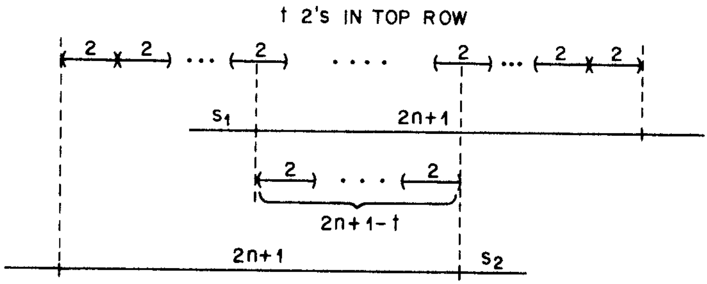

FIGURE 5

The final sentence of Theorem 2 says that the graphs in each $E(R)$ are irreducible; i.e., every proper subgraph thereof will be in $C(\alpha)$ instead of not in $C(\alpha)$. It is easily seen from the construction of $E(R)$ that if any point in a graph in $E(R)$ is removed, then slight adjustments of intervals show that the remainder is in $C(\alpha)$. ∎

4. ENUMERATION

We conclude with remarks on the cardinalities of the irreducible $E(\alpha)$ families. Exact counts are given only for $q \leqslant 2$ and $p = q + 1$. A lower bound for other q's is noted later in the section. Each graph that is counted is nonisomorphic to every other graph that is counted.

Theorem 3. For every positive integer n,

$$|E(n)| = 1 \qquad \text{and} \qquad \left|E\left(\frac{2n+1}{2}\right)\right| = 1 + \binom{n+2}{3}.$$

Proof. Clearly, $E(n/1) = \{K_{1,n+2}\}$. Henceforth, $\alpha = \frac{1}{2}(2n + 1)$. One ring in $\mathfrak{R}(\alpha)$ has the two L's together, with $R^* = S \ldots SLL$. This has $|E(R)| = 1$ since neither secondary intersects a short primary.

There are n rings in $\mathfrak{R}(\alpha)$ with the L's separated. They are the R_t with

$$R_t^* = (S, t \text{ times})L(S, 2n + 1 - t \text{ times})L$$

for $t = 2n, 2n - 1, \ldots, n + 1$. Figure 5 pictures the full construction of the preceding section for R_t^*. There are two outside secondaries (unmarked), two inside secondaries (s_1 and s_2), and all four are essential since no redundancies can arise. It is easily checked that each of s_1 and s_2 can intersect up to $t - n - 1$ other intervals that it does not intersect when it has the shortest length 2. So $t - n$ choices are available for each s_i.

Because of symmetry about the midpoint of the primary span in Figure 5, it follows that

$$|E(R_t)| = \frac{1}{2}[(t - n)^2 + (t - n)] = \binom{t - n + 1}{2}.$$

Moreover, since the maximal independent sets of graphs in different $E(R_t)$ have different cardinalities, there is no inter-$E(R_t)$ isomorphism. Hence

$$|E(\alpha)| = 1 + \sum_{t=n+1}^{2n} |E(R_t)| = 1 + \sum_{v=0}^{n-1} \binom{v + 2}{2} = 1 + \binom{n + 2}{3}. \quad \blacksquare$$

The lower graph in Figure 1 is the one for $\binom{n+2}{3}$ when $\alpha = \frac{3}{2}$. Our further counts will be based on

$$|E(\alpha)| = \sum_{R \in \mathcal{R}(\alpha)} |E(R)|,$$

which is a direct consequence of

Theorem 4. If $R, R' \in \mathcal{R}(\alpha)$ and $R \neq R'$, then $E(R) \cap E(R') = \emptyset$.

Proof. Let G be a graph in $E(R)$ for $R \in \mathcal{R}(\alpha)$, with interval representation H constructed on the basis of R^* as described early in the preceding section. In view of that construction and the proof of Theorem 2, G could be in $E(R')$ for some $R' \neq R$ in $\mathcal{R}(\alpha)$ only if the intervals in H could be relabeled to yield a different pattern (R') of p short and q long primaries, with the remaining intervals (the new secondaries) touching ends of the new L runs in the prescribed manner. In this relabeling, as in the original labeling, each endpoint of a primary must serve exactly two primary intervals and have an integer value, the left endpoint and the leftmost L must be the same as the open left endpoint of the leftmost S, all intervals in S runs fit tightly together, and all adjacent intervals in L runs must just touch at the ends. [If this could be done, then the new secondaries could be adjusted to give other interval graphs in $E(R')$.]

We shall show that this cannot be done, i.e., any "new" labeling of the type described must be the same as the original labeling of H. Clearly, no original S can be an L under relabeling, nor can it be a secondary under relabeling since it is open at both ends and therefore could not intersect the end of any relabeled L run at a single point. Hence the p S's in H must be the S's in any relabeling that has the properties described above.

It follows that the only feasible candidates for different labels are the original long primaries (L) and secondaries (s). Any s that is relabeled L must have length p and have one endpoint at the same point as *exactly one* of the S intervals. Moreover, it must have the proper orientation: for right-end coincidence it goes left from that point; for left-end coincidence it goes right from that point. However, any length-p interval of this type (at the *end* of a run of adjacent S's, projecting back across some of those S's) is in fact one of the original L intervals. Consequently, the q L's in H must be the L's in any feasible relabeling. ∎

As remarked previously, two rings each consisting of p S's and q L's are considered equivalent if one can be obtained from the other by rotation and/or reflection. These operations correspond exactly to the action of the dihedral group D_n acting on the set of rings for $\alpha = p/q$ with $n = p + q$. Thus, to count the number of nonequivalent rings, i.e., $|\mathfrak{R}(\alpha)|$, we need only apply Polya's enumeration theorem (see [5]) to this situation.

This application requires the *cycle index* of D_n, which is given by

$$Z(D_n) = \frac{1}{2n}\sum_{k|n}\phi(k)s_k^{n/k} + \begin{cases}\frac{1}{2}s_1s_2^{(n-1)/2}, & n \text{ odd},\\ \frac{1}{4}(s_2^{n/2} + s_1^2s_2^{(n-2)/2}), & n \text{ even},\end{cases}$$

where $\phi(k)$ is the Euler phi function, i.e., the number of integers in $[1, k]$ that are relatively prime to k. If c_k is the number of nonequivalent rings with exactly k S's (and $n - k$ L's), then, by Polya's theorem, the generating function

$$C(x) = \sum_k c_k x^k$$

satisfies

$$C(x) = \frac{1}{2n}\sum_{k|n}\phi(k)(1 + x^k)^{n/k} + \begin{cases}\frac{1}{2}(1 + x)(1 + x^2)^{(n-1)/2}, & n \text{ odd},\\ \frac{1}{4}[(1 + x^2)^{n/2} + (1 + x)^2(1 + x^2)^{(n-1)/2}], & n \text{ even}.\end{cases}$$

For our application we take $k = p$ with $\alpha = p/q$, p and q relatively prime, and $n = p + q$. Then n and p are relatively prime, so the only term of the sum in $C(x)$ that contributes to the count is the $k = 1$ term, i.e., $\phi(1)(1 + x)^n/2n$, which generates the term $\binom{n}{p}x^p/2n$. A simple computation based on the parities of n and p shows that the x^p term from the final piece of $C(x)$ is just $\frac{1}{2}\binom{\lfloor (n-1)/2\rfloor}{\lfloor p/2\rfloor}x^p$. Thus

$$|\mathfrak{R}(\alpha)| = c_p = \frac{1}{2n}\binom{n}{p} + \frac{1}{2}\binom{\lfloor\frac{1}{2}(n-1)\rfloor}{\lfloor\frac{1}{2}p\rfloor}.$$

Hence, in view of Theorem 4, we have

Theorem 5.

$$|E(\alpha)| \geqslant \frac{1}{2(p + q)}\binom{p + q}{p} + \frac{1}{2}\binom{\lfloor\frac{1}{2}(p + q - 1)\rfloor}{\lfloor\frac{1}{2}p\rfloor}.$$

Since this lower bound counts only the nonequivalent ring patterns for α, and takes no account of changes in the lengths of secondaries, it will be just a fraction of $|E(\alpha)|$ in most cases. For example, when $(p, q) = (2n + 1, 2)$, the lower bound in Theorem 5 reduces to $n + 1$, so, by Theorem 3, the ratio of that bound to $|E[\frac{1}{2}(2n + 1)]|$ is $6(n + 1)/[6 + (n + 2)(n + 1)n]$.

In one case, however, equality obtains in Theorem 5, and that occurs when $p = q + 1$.

Theorem 6. If $p = q + 1$, then $|E(R)| = 1$ for every $R \in \mathcal{R}(\alpha)$.

Corollary. $$\left|E\left(\frac{q+1}{q}\right)\right| = \frac{1}{2}\left[\frac{1}{q+1}\binom{2q}{q} + \binom{q}{\lfloor \frac{1}{2}q \rfloor}\right].$$

By Stirling's formula, $|E(\frac{q+1}{q})| \approx K4^q/q^{3/2}$ with $K = 1/(2\sqrt{\pi})$, so even in this case $|E(\alpha)|$ increases exponentially fast.

We conclude with remarks on the proofs of Theorem 6 and the Corollary. The latter follows directly from Theorems 4 and 6 along with our calculation of $|\mathcal{R}(\alpha)|$.

Theorem 6 is verified by showing that, in our standard interval assignment for $\alpha = (q + 1)/q$ presented early in Section 3, no new intersections can be obtained by increasing the lengths of secondaries from q to $q + 1$. That is, if the length of a secondary is changed from q to $q + 1$, this will not create a new intersection with a short primary or a long primary or a secondary (of any length). Since the full proof of this is straightforward, we shall do only one case to illustrate the procedure.

Suppose L_1 and L_2 are long primaries with L_2 to the right of L_1. Suppose further that L_1 is at the right end of an L run, so it has a secondary s projecting from its right endpoint toward L_2. A new intersection of s with L_2 is created when s increases from length q to $q + 1$ if and only if the left endpoint of L_2 is $q + 1$ units to the right of the right endpoint of L_1. Suppose this is the case, and, starting from L_1 and going backward over its L run, . . . , until we reach L_2 (part of a revolution around the ring), suppose that we encounter a S's and b L's (other than L_1) before we get to L_2. For convenience, suppose L_1's left endpoint is at 0. Then its right endpoint is at $q + 1$, and L_2's left endpoint is at $qa - (q + 1)(b + 1)$. Thus, with the presumed spread of $q + 1$ units between these points, we have $q + 1 = [qa - (q + 1)(b + 1)] - (q + 1)$, which reduces to

$$(q + 1)/q = a/(b + 3).$$

The only way that this can hold is to have $a = q + 1$, i.e., all S's are met in going from L_1 to L_2. But this is impossible since the S that has the same right endpoint as L_1 is not between L_1 and L_2 in the direction traveled. Alternatively, since the ratio requires $b + 3 = q$, exactly one L other than L_1 and L_2 is missed in going from L_1 to L_2, and since all S's are supposedly encountered and one full revolution around the ring must return to the start, the missing L must cover the $(q + 1)$-unit gap between L_1 and L_2. But then L_1 would not be at the right end of an L run.

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