OLD AND NEW EUCLIDEAN RAMSEY THEOREMS

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INTRODUCTION

Mathematics has sometimes been called the science of order. From this point of view, the guiding principle of Ramsey theory is given by the statement, "Complete disorder is impossible." In a nutshell, Ramsey theory is the study of structure invariant under partitions. Typical examples of results in this subject are:

(i) (Adam [0].) Any rearrangement of the integers $\{1, 2, \ldots, n^2 + 1\}$ always contains a monotone subsequence of length $n + 1$.

(ii) (van der Waerden [18].) For any partition of the set $\mathbb{Z}^+$ of positive integers into finitely many classes, one of the classes must contain arbitrarily long arithmetic progressions.

(iii) (Ramsey [12].) For any partition of the k-element subsets of $\mathbb{Z}^+$ into finitely many classes, some class must contain all the k-element subsets of some infinite subset $X \subseteq \mathbb{Z}^+$.

(iv) (Hindman [9].) For any partition of $\mathbb{Z}^+$ into finitely many classes, some class must contain all the (finite) subset sums of some infinite subset $X \subseteq \mathbb{Z}^+$.

(v) (Graham *et al.* [5]; also see [16].) For any integers k, l, and r and any finite field F, there exists an integer $N = N(k, l, r, F)$ such that if all the k-dimensional subspaces of an n-dimensional vector space G over F are partitioned into r classes, then some class must contain all the k-dimensional subspaces of some l-dimensional subspace of G.

A few remarks may be in order here. First, we should point out that despite their similar appearance, (iv) is significantly more difficult to prove than (iii). The precursor to (iv) was the theorem of I. Schur [13], who in 1916 showed that if the integers $\{1, 2, \ldots, [er!]\}$ are partitioned into r classes, then some class must contain a solution to the equation $x + y = z$. This theme was picked up by Schur's illustrious student R. Rado, who, beginning with his dissertation [11] in 1933, developed a beautiful theory for the "partition regularity" of systems of linear equations. In more recent times, great progress in these (and other) directions has been made by Deuber, Nešetřil, Rödl, and many others. A description of much of this work can be found in [6].

In contrast to (iii) and (iv) there is no infinite analogue to (ii). This can be seen, for example, by the partition

$$\mathbb{Z}^+ = \{x\colon 2^{2k} \le x < 2^{2k+1}\colon k \ge 0\} \cup \{x\colon 2^{2k+1} \le x < 2^{2k+2}\colon k \ge 0\}.$$

Finally, we should remark that in some sense the first Ramsey theorem actually was due to Hilbert [8], who in 1892 proved that for any partition of $\mathbb{Z}^+$ into finitely many classes, some class contains for every m and suitable choices of positive integers $a, a_1, \ldots, a_m$ all 2^m sums

$$a + \sum_{k=1}^{m} \varepsilon_k a_k, \qquad \text{where } \varepsilon_k = 0 \text{ or } 1.$$

[This is a special case of (iv).]

Euclidean Ramsey Theory

For Euclidean Ramsey theory, the fundamental problem is this: Which configurations of points C must always occur, up to some Euclidean motion, in a single class, whenever Euclidean n-space $\mathbb{E}^n$ is partitioned into r classes?

Definition. For such C we will say that $R(C, n, r)$ holds.

For example, it is easy to see that if C_0 is the set of three vertices of some unit equilateral triangle, then $R(C_0, 4, 2)$ holds (by considering the five vertices of a unit simplex in $\mathbb{E}^4$) and $R(C_0, 2, 2)$ does not hold (by partitioning $\mathbb{E}^2$ into two classes of alternating half-open strips of width $\sqrt{3}/2$).

The following is typical of some of the many results available along these lines.

Fact. *$R(C, 6, 2)$ holds when C is the set of four vertices of some unit square.*

Proof. Consider the set $S \subseteq \mathbb{E}^6$ defined by $S = \{(x_1, \ldots, x_6)\colon x_i = 1/\sqrt{2}$ for exactly two values of i, and $x_i = 0$ for all other values of $i\}$. Any partition of $\mathbb{E}^6$ into two classes, say $\chi\colon \mathbb{E}^6 \to \{0, 1\}$, also partitions S into two classes. To each point $s = (s_1, \ldots, s_6) \in S$ we can associate a pair $\{i, j\}$ by letting i and j denote the two indices for which $s_k = 1/\sqrt{2}$. Thus, χ induces a partition of the edges of the complete graph on six points into two classes. By a standard result in (Ramsey) graph theory, in any such partition there must be a 4-cycle, say

$$\hookrightarrow c_1 \to c_2 \to c_3 \to c_4 \circlearrowleft$$

in a single class. It is straightforward to check that this 4-cycle corresponds to the four vertices of a unit square in S, which proves our claim. ∎

It is no accident that in the examples presented thus far, a proof that $R(C, n, r)$ holds for a particular C was accomplished in fact by selecting a suitable *finite* subset of $\mathbb{E}^n$ and partitioning it (instead of requiring all of $\mathbb{E}^n$). A standard compactness argument (see [6]) shows that this must always be the case.

Before proceeding to more general considerations, we mention a tantalizing question that, besides being among the most fundamental in the theory, illustrates more than adequately how little we really know about what is going on in this area.

For this example we take for C the set C^* consisting of two points separated by a distance of 1. It is easy to see that $R(C^*, 2, 2)$ holds by considering the three vertices of a unit equilateral triangle. It is somewhat less obvious (but equally true) that $R(C, 2, 3)$ holds. In the graph shown in Figure 1, all edges represent unit distances. This graph has the property that it has chromatic number 4, i.e., any 3-coloring of the vertices must assign the same color to the two endpoints of *some* edge of the graph.

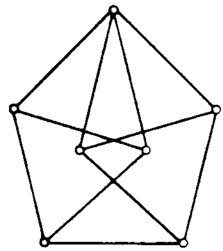

FIGURE 1. The Moser graph.

In the other direction, suppose $\mathbb{E}^2$ is tiled by regular hexagons of diameter 9/10. It is easy to partition these hexagons into seven classes so that no class contains two points at a distance of 1. Thus, $R(C^*, 2, 7)$ does not hold. At present, no one knows the *least* value of r such that $R(C^*, 2, r)$ does not hold. This is also known as the chromatic number of $\mathbb{E}^2$, denoted by $\chi(\mathbb{E}^2)$, since it is the chromatic number of the (uncountable) graph formed by taking each point of $\mathbb{E}^2$ as a vertex and each pair $\{x, y \in \mathbb{E}^2\colon \text{distance}\,(x, y) = 1\}$ as an edge. Thus, the best current bounds are

$$4 \leq \chi(\mathbb{E}^2) \leq 7.$$

More generally, it has recently been shown by Frankl and Wilson [4] that $\chi(\mathbb{E}^n)$, the chromatic number of Euclidean n-space, grows exponentially in n, verifying an earlier conjecture of Erdös. The best available bounds for the general case now are

$$(1 + o(1))(1.2)^n \leq \chi(\mathbb{E}^n) \leq (3 + o(1))^n.$$

RAMSEY SETS

A basic concept in Euclidean Ramsey theory is that of a Ramsey set.

DEFINITION. A configuration C is said to be *Ramsey* if for all r there exists an $N = N(C, r)$ such that $R(C, N, r)$ holds. The most general result for constructing Ramsey sets is given by the following.

THEOREM [1]. *If C_1 and C_2 are Ramsey sets, then the Cartesian product $C_1 \times C_2$ is a Ramsey set.*

Since any two-point set is a Ramsey set, any subset of the vertices of a rectangular parallelepiped (= "brick") is a Ramsey set. The determination of which n-simplexes are subsets of bricks is an interesting open problem. It is certainly necessary that any angle formed by three of the points be less than or equal to 90°. This also turns out to be sufficient for $n = 2$ and 3 but *not* for $n = 4$.

On the other side of the coin, one might well ask if there are any non-Ramsey sets. The simplest example of such a set is given by the following result.

THEOREM. *Let $C = \{\bar{x}, \bar{y}, \bar{z}\}$ be a set of three equally spaced collinear points (see* FIGURE 2). *Then C is not a Ramsey set.*

Proof. To each $\bar{u} \in \mathbb{E}^2$ assign the color $\chi(u) = [\bar{u} \cdot \bar{u}]$ (mod 4), where $[\alpha]$ denotes the greatest integer not exceeding α.

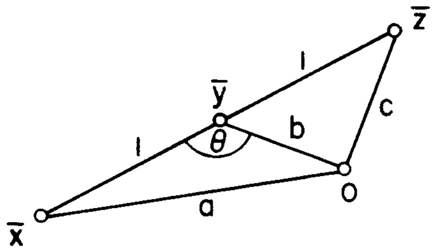

FIGURE 2

Suppose $C = \{\bar{x}, \bar{y}, \bar{z}\}$ occurs "monochromatically," i.e., for some i, there is a copy of C in $\chi^{-1}(i)$. Since

$$a^2 = b^2 + 1 - 2b \cos \theta,$$

$$c^2 = b^2 + 1 + 2b \cos \theta,$$

then

$$a^2 + c^2 = 2b^2 + 2. \tag{1}$$

Since $\chi(\bar{x}) = \chi(\bar{y}) = \chi(\bar{z}) = i$, then

$$a^2 = 4k_a + i + \varepsilon_a, \qquad 0 \le \varepsilon_a < 1,$$

$$b^2 = 4k_b + i + \varepsilon_b, \qquad 0 \le \varepsilon_b < 1,$$

$$c^2 = 4k_c + i + \varepsilon_c, \qquad 0 \le \varepsilon_c < 1,$$

for suitable integers k_a, k_b, k_c. Thus, by **(1)**

$$4(k_a - 2k_b + k_c) - 2 = -\varepsilon_a + 2\varepsilon_b - \varepsilon_c$$

which is, however, (barely) impossible because of the constraints on the ε's. ▮

This argument contains the kernel of an idea which when more fully developed can be used to prove the following result.

THEOREM [1]. *If C is a Ramsey set, then C must lie on the surface of some sphere.*

This is still the strongest restriction known for Ramsey sets.

A fair number of results in this general spirit appear in [1–3, 14, 17]. Rather than repeat these, we will conclude the next section with several very recent results which have not yet appeared in the literature.

EUCLIDEAN RAMSEY THEORY ON THE n-SPHERE

In this section we examine the corresponding questions when the underlying spaces are unit n-spheres

$$S^n = \left\{(x_0, \ldots, x_n) : \sum_{k=0}^{n} x_k^2 = 1\right\} \subseteq \mathbb{E}^{n+1}$$

and the allowed motions are orthogonal transformations of S^n onto itself. The corresponding Ramsey sets will be called "sphere-Ramsey." It will turn out that for sets $X \subseteq S^n$ which are not too large (in a sense to be made precise later), results similar to those preceding hold. For the remaining cases, only very preliminary results are available, although we suspect that much more is very likely true. We begin with a necessary condition.

THEOREM. *Let* $X = \{\bar{x}_1, \ldots, \bar{x}_m\}$ *be a set of points in* $\mathbb{E}^n$ *such that:*

(i) *for some nonempty set* $I \subseteq \{1, 2, \ldots, m\} \equiv [m]$, *there exist nonzero* α_i, $i \in I$, *such that*

$$\sum_{i \in I} \alpha_i \bar{x}_i = \bar{0};$$

(ii) *for all nonempty* $J \subseteq I$,

$$\sum_{j \in J} \alpha_j \neq 0.$$

Then there exists $r = r(X)$ *such that for any* N, *there is a partition* $S^N = \bigcup_{k=1}^r C_k$ *such that no* C_i *contains a copy of* X.

Proof. Consider the homogeneous linear equation

$$\sum_{i \in I} \alpha_i z_i = 0. \tag{2}$$

By (ii), Rado's theorem for the partition regularity of this equation over $\mathbb{R}^+$ (see [6] or [11]) implies that it is *not* regular, i.e., for some r there is an r-coloring $\chi: \mathbb{R}^+ \to [r]$ such that **(1)** has no monochromatic solution. Color the points of $S_+^N = \{(x_0, \ldots, x_N) \in S^N: x_0 > 0\}$ by

$$\chi^*(\bar{x}) = \chi(\bar{u} \cdot \bar{x}),$$

where $\bar{u}$ denotes the unit vector $(1, 0, 0, \ldots, 0)$. Thus, the color of $\bar{x} \in S_+^N$ just depends on its distance from the "north pole" of S^N.

For each nonempty subset $J \subseteq I$, consider the equation

$$\sum_{j \in J} \alpha_j z_j = 0. \tag{2$_J$}$$

Of course, by (ii) this also fails to satisfy the (necessary and sufficient) condition of Rado for partition regularity. Hence, there is a coloring χ_J of $\mathbb{R}^+$ (using r_J colors) so that **(2)$_J$** has no monochromatic (under χ_J) solution. As before, we can color S_+^N by giving $\bar{x} \in S_+^N$ the color

$$\chi_J^*(\bar{x}) = \chi_J(\bar{x} \cdot \bar{u}).$$

Now, form the *product* coloring $\hat{\chi}$ of S_+^N by defining for $\bar{x} \in S_+^N$,

$$\hat{\chi}(\bar{x}) = (\chi_I(\bar{x}), \ldots, \chi_J(\bar{x}), \ldots),$$

where the sequence has length $2^{|I|} - 1$ and the indices of the χ_J range over all nonempty subsets $J \subseteq I$. The number of colors required by the coloring $\hat{\chi}$ is at most $\prod_{\varnothing \neq J \subseteq I} r_J \equiv R$.

An important property of $\hat{\chi}$ is this. Suppose we extend $\hat{\chi}$ to $S_0^N \equiv \{(x_0, \ldots, x_N) \in S^N: x_0 \geq 0\}$ by assigning *all* R colors to any point in $S_0^N \backslash S_+^N$, i.e., having $x_0 = 0$. Then the *only* monochromatic solution to **(2)** in $\mathbb{R}^+ \cup \{0\}$ is $z_i = 0$ for all $i \in I$.

Next, construct a similar coloring $\check{\chi}$ on $S_-^N = \{-\bar{x}: \bar{x} \in S_+^N\}$, but use R completely different colors. This assures that any set X that hits both hemispheres S_+^N and S_-^N cannot be monochromatic. Finally, we have to color the equator

$$S^{N-1} = \{\bar{x} \in S^N: x_0 = 0\}.$$

By our construction, any copy of X that is not contained entirely in S^{N-1} cannot be monochromatic. Hence, it suffices to color S^{N-1} avoiding monochromatic copies of X where we may use any of the $2R$ colors previously used in the coloring of $S_+^N \cup S_-^N$. By induction, this can be done provided we can so color S^1. However, since $m > 1$, S^1 can in fact always be 3-colored without a monochromatic copy of X (in fact, of any 2-element subset of X since the corresponding graph has maximum degree 2). This proves the theorem. ▌

Note that if X is a constant distance $d \neq 90°$ from some point $\bar{t} \in S^n$, then X cannot satisfy both (i) and (ii). For

$$\sum_{i \in I} \alpha_i \bar{x}_i = \bar{0}$$

implies

$$\begin{aligned} 0 = \bar{t} \cdot \left(\sum_{i \in I} \alpha_i \bar{x}_i\right) &= \sum_{i \in I} \alpha_i \bar{t} \cdot \bar{x}_i \\ &= (\cos d) \cdot \sum_{i \in I} \alpha_i, \end{aligned}$$

i.e.,

$$\sum_{i \in I} \alpha_i = 0$$

since $\cos d \neq 0$.

However, these are not the only sets not ruled out from being possible Ramsey sets by the theorem. Another such example is given by the three-point set

$$T = \left\{(1, 0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\right\} = \{t_1, t_2, t_3\}.$$

Their linear dependence is given by

$$t_1 - t_2 - t_3 = \bar{0}$$

which does not satisfy (ii).

We restate the theorem in its positive form.

THEOREM′. *If X is sphere-Ramsey, then for any linear dependence $\sum_{i \in I} \alpha_i \bar{x}_i = \bar{0}$ there must exist a nonempty $J \subseteq I$ such that $\sum_{j \in J} \alpha_j = 0$.*

The next result gives a sufficient condition for a set to be a sphere-Ramsey set. Let us call an m-dimensional brick with edge lengths $\lambda_1, \lambda_2, \ldots, \lambda_m$ *small* if

$$\sum_{i=1}^{m} \lambda_m^2 \leq 2. \tag{3}$$

THEOREM. *Every small brick is sphere-Ramsey.*

Proof. We sketch the proof (which has the same basic structure as that of the Hales–Jewett theorem given in [7]). Let a fixed number r of colors be given. For $m = 1$, the theorem is immediate: We simply consider the $r + 1$ points

$$\overbrace{\begin{matrix}(\beta_1, 0, 0, \ldots, 0, \gamma)\\(0, \beta_1, 0, \ldots, 0, \gamma)\\(0, 0, \beta_1, \ldots, 0, \gamma)\\ \vdots \\(0, 0, 0, \ \ldots, \beta_1, \gamma),\end{matrix}}^{r+1}$$

where $\beta_1 = \lambda_1/\sqrt{2} \leq 1$ and $\gamma^2 + \beta_1^2 = 1$. These $r + 1$ points are on S^{r+1}. Since they are r-colored, some pair must have the same color. This pair has distance $\beta_1\sqrt{2} = \lambda_1$, which is the desired conclusion.

In general, for a $\lambda_1 \times \cdots \times \lambda_m$ brick B, the set S of points we consider is of the form

$$(\overbrace{0, \ldots, \beta_m, \ldots, 0}^{N_m},\ \overbrace{0, \ldots, \beta_{m-1}, \ldots, 0}^{N_{m-1}},\ \ldots\ 0, \ldots\ldots 0,\ \overbrace{0, \ldots, \beta_1, \ldots, 0}^{N_1}, \gamma).$$

That is, S consists of $(N_m + N_{m-1} + \cdots + N_1 + 1)$-tuples in which exactly one of the entries in the jth block (of length N_j) is $\beta_j = \lambda_j/\sqrt{2}$ and all other entries are 0, with the exception of the last entry

$$\gamma = \left(1 - \sum_{j=1}^{m} \beta_j^2\right)^{1/2},$$

chosen so that all points are a unit N-sphere with $N = N_m + N_{m-1} + \cdots + N_1$. The hypothesis (3) guarantees that γ is real. The key to this construction is (as usual) in the choice of the N_j's. Needless to say, for the proof to work they must grow very rapidly.

As an example, we consider the case $m = 2$. Choose $N_1 = r + 1$, $N_2 = r^{r+1} + 1$. An r-coloring χ of S induces an r^{r+1}-coloring χ' of the set

$$S' = \{(\overbrace{0, \ldots, \beta_2, \ldots, 0}^{N_2}, \gamma)\}$$

by

$$\chi'(s_1') = \chi'(s_i'), \qquad s_1', s_2' \in S',$$

iff

$$\chi(s_1'\, t) = \chi(s_2'\, t)$$

for all

$$t \in \{(\overbrace{0, \ldots, \beta_1, \ldots, 0}^{N_1}, \gamma)\} = T_1,$$

where the concatenation $s'_1\, t$ has the obvious interpretation of being an element of S. Since $|S'| = N_2 = r^{r+1} + 1$ and S' is r^{r+1}-colored, some pair of points $s'_1, s'_2 \in S'$ has $\chi'(s'_1) = \chi'(s'_2)$, i.e., $\chi(s'_1\, t) = \chi(s'_2\, t)$ for all $t \in T_1$. Since χ is an r-coloring and $|T_1| = N_1 = r + 1$, some pair of points $t, t' \in T_1$ has

$$\chi(s'_1\, t) = \chi(s'_1\, t').$$

Of course, this implies

$$\chi(s'_1\, t) = \chi(s'_1\, t') = \chi(s'_2\, t) = \chi(s'_2\, t').$$

But

$$d(s'_1\, t, s'_1\, t') = \beta_1 \sqrt{2} = \lambda_1 = d(s'_2\, t, s'_2\, t),$$
$$d(s'_1\, t, s'_2\, t) = \beta_2 \sqrt{2} = \lambda_2 = d(s'_1\, t', s'_2\, t')$$

so that these four points form the desired monochromatic $\lambda_1 \times \lambda_2$ brick.

The general result follows by the same techniques where, in general, we choose $N_1 = r + 1$ and $N_{j+1} = 1 + r^{N_1 N_2 \cdots N_j}$ for $j \geq 1$. Specifically, we think of S as $S(m) \times T(m)$, where $S(m)$ consists of the N_m N_m-tuples $(0, \ldots, \beta_m, \ldots, 0)$ and $T(m)$ consists of the $N_1 N_2 \cdots N_{m-1}$ complementary $(N_1 + \cdots + N_{m-1} + 1)$-tuples

$$(\overbrace{0, \ldots, \beta_{m-1},}^{N_{m-1}}\ \overbrace{\ldots, \beta_{m-2}, \ldots,}^{N_{m-2}}\ \ldots\ldots, \overbrace{\ldots, \beta_1, \ldots,}^{N_1} \gamma).$$

The initial r-coloring χ of S induces an $r^{N_1 \cdots N_{m-1}}$-coloring χ' of $S(m)$ by

$$\chi'(s'_1) = \chi'(s'_2), \qquad s'_1, s'_2 \in S(m),$$

iff

$$\chi(s'_1\, t) = \chi(s'_2\, t) \qquad \text{for all } t \in T(m).$$

Since

$$|S(m)| = N_m = 1 + r^{N_1 \cdots N_{m-1}},$$

there exists a pair of points, say $s_1, s_2 \in S(m)$, such that

$$\chi'(s'_1) = \chi'(s'_2).$$

Also, there is induced r-coloring $\hat{\chi}$ of $T(m)$ by

$$\hat{\chi}(t) = \chi(s'_1\, t), \qquad t \in T(m).$$

By induction, there is a monochromatic $\lambda_1 \times \cdots \times \lambda_m$ brick under the coloring $\hat{\chi}$ of $T(m)$. By the definition of $\hat{\chi}$ and χ', this extends to a monochromatic $\lambda_1 \times \cdots \times \lambda_m$ brick in the original coloring of S. ∎

By suitable manipulations, it can be shown that the N_m satisfy

$$N_m \leq \overbrace{(r+2)^{(r+2)^{\cdot^{\cdot^{\cdot^{(r+2)}}}}}}^{m} = (r+2)\uparrow\uparrow m.$$

Bricks that have a main diagonal of length exceeding 2 seem much less tractable, although we expect that any $\lambda_1 \times \cdots \times \lambda_m$ brick with

$$\lambda_1^2 + \cdots + \lambda_m^2 < 4$$

is sphere-Ramsey. We can only prove this in the case $m = 1$.

THEOREM. *Let B be the set $\{-\lambda/2, \lambda/2\}$, where $0 < \lambda < 2$. Then B is sphere-Ramsey.*

Proof. It is enough to show that the graph G_n with vertex set S^n and edge set $\{\{\bar{x}, \bar{y}\}: d(\bar{x}, \bar{y}) = \lambda\}$ has chromatic number tending to infinity as n tends to infinity (where d denotes Euclidean distance). To prove this, we use the following recent result of Frankl and Wilson (suggested by Bárány, Füredi, and Pach).

THEOREM [4]. *Let $\mathscr{F}$ be a family of k-sets of $[n]$ such that for some prime power q,*

$$|F \cap F'| \not\equiv k \pmod{q}$$

for all $F \neq F'$ in $\mathscr{F}$. Then

$$|\mathscr{F}| \leq \binom{n}{q-1}.$$

For a fixed r, choose a prime power q so that

$$\binom{2(1+\varepsilon)q}{(1+\varepsilon)q} > r\binom{2(1+\varepsilon)q}{q-1},$$

where $\lambda = 2\beta\sqrt{2q}$, and choose $\varepsilon > 0$ and α so that

$$\alpha^2 + 2(1+\varepsilon)q\beta^2 = 1$$

and $N = (1+\varepsilon)q$ is an integer. Consider the set

$$S = \left\{(s_0, \ldots, s_{2N}): s_0 = \alpha,\ s_i = \pm\beta,\ \sum_{i=1}^{2N} s_i = 0\right\}.$$

To each $s \in S$ associate the subset

$$F(s) = \{i \in [2N]: s_i = \beta\}.$$

Thus, the family

$$\mathscr{F} = \{F(s): s \in S\}$$

consists of the $\binom{2N}{N}$ N-element subsets of $[2N]$. If $F, F' \in \mathscr{F}$, $F \neq F'$, then

$$|F \cap F'| \equiv N \pmod{q}$$

iff

$$|F \cap F'| = N - q = \varepsilon q.$$

If the elements of $\mathscr{F}$ are r-colored, then some color class must contain at least

$$\frac{1}{r}|\mathscr{F}| = \frac{1}{r}\binom{2N}{N} > \binom{2N}{q-1}$$

elements of $\mathscr{F}$. However, by Frankl and Wilson, if $|F \cap F'| = \varepsilon q$ *never* occurs, then the number of N-sets must be at most $\binom{2N}{q-1}$, which is a contradiction. Thus, some monochromatic pair F, F' must have

$$|F \cap F'| = \varepsilon q.$$

This means that the corresponding points s, $s' \in S$ must (up to a permutation of coordinate positions) look like

$$s = (\overbrace{\alpha, \beta, \ldots, \beta,}^{\varepsilon q} \; \overbrace{\beta, \ldots, \beta,}^{q} \quad \overbrace{-\beta, \ldots, -\beta,}^{\varepsilon q} \; \overbrace{-\beta, \ldots, -\beta}^{q}),$$

$$s' = (\underbrace{\alpha, \beta, \ldots, \beta,}_{\varepsilon q} \; \underbrace{-\beta, \ldots, -\beta,}_{q} \quad \underbrace{\beta, \ldots, \beta,}_{\varepsilon q} \; \underbrace{-\beta, \ldots, -\beta}_{q}).$$

Note that

$$d(s, s') = \sqrt{8q\beta^2} = \lambda$$

and

$$d(s, 0) = d(s', 0) = \alpha^2 + 2(1 + \varepsilon)q\beta^2 = 1,$$

i.e., s, $s' \in S^{2N}$. This proves the theorem. ▮

As remarked previously, one would expect that the corresponding result should hold for any $\lambda_1 \times \cdots \times \lambda_m$ brick provided $\lambda_1^2 + \cdots + \lambda_m^2 < 4$. However, we are unable to prove this for even the case $m = 2$.

We conclude with a final observation. Many of the results in (Euclidean) Ramsey theory assert that the desired structure will occur whenever the dimension N of the space is sufficiently large. The proofs of these results typically end up constructing N's that are quite large. In this domain, bounds such as $N_m \leq (r + 2)\uparrow\uparrow m$ given before are considered very small. For the finite form of van der Waerden's theorem, which asserts for a suitable integer $W(k)$, any 2-coloring of $[W(k)]$ must contain a monochromatic arithmetic progression of length k, the only known upper bounds on $W(k)$ are not even primitive recursive (see [6]) (they grow like the Ackermann function). The best lower bound grows like $k \cdot 2^k$. Whether these gigantic upper bounds are a reflection of the truth or just of our inability to find the right proofs of these results is not currently known. Some evidence for the former is given by recent results of Paris, Harrington, and others (e.g., see [6, 10]) who give examples of combinatorial theorems of this general type which have *lower* bounds that grow this fast (and even much faster; see [15] for an amusing account).

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