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Mathematics

Isometric embeddings of graphs

(metric space/isometry)

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ABSTRACT **We prove that any finite undirected graph can be canonically embedded isometrically into a maximum cartesian product of irreducible factors.**

With each finite connected undirected graph¶ $G = (V, E)$ one can associate a *metric* d_G: $V \times V \to \mathbf{N}$ (the set of nonnegative integers) by defining $d_G(x, y)$ to be the number of edges in a shortest path between x and y, for all $x, y \in V$, the vertex set of G. If (M, d_M) is an arbitrary metric space we say that an embedding λ: $V \to M$ is *isometric* if for all $x, y \in V$,

$$d_M(\lambda(x), \lambda(y)) = d_G(x, y).$$

We denote this by λ: $G \xrightarrow{I} M$. Note that λ isometric implies that λ is injective.

Isometric embeddings of graphs into various metric and semi-metric spaces have been studied extensively in recent years (e.g., see refs. 2–6). In particular, attention has been focused on metric spaces which are formed as cartesian products of other spaces with the induced l_1 metric—that is, for a finite family of spaces (M_i, d_{M_i}), $1 \le i \le r$, the product space $(\Pi_{i=1}^r M_i, d_\pi)$ is defined by setting

$$\prod_{i=1}^{r} M_i = \{\bar{x} = (x_1, \ldots, x_r): x_k \in M_k\}$$

and

$$d_\pi(\bar{x}, \bar{x}'): = \sum_{i=1}^{r} d_{M_i}(x_i, x_i').$$

For a graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$, define the distance matrix $D(G) = (d_{ij})$ of G by setting $d_{ij} = d_G(v_i, v_j)$. Since $D(G)$ is real and symmetric, it has all real eigenvalues. Let $n_+(G)$ and $n_-(G)$ denote the number of positive and negative eigenvalues, respectively, of $D(G)$.

As an example, let (S, d_S) be given by taking $S = \{0, 1, \alpha\}$ and defining||

$$d_S(x, x') = \begin{cases} 1 \text{ if } x = 0, x' = 1 \text{ or } x = 1, x' = 0 \\ 0 \text{ otherwise.} \end{cases}$$

It is known (1) that in this case there always exists a least integer $N(G)$ so that

$$G \xrightarrow{I} \prod_{i=1}^{N(G)} S = S^{N(G)}.$$

Furthermore,

(*i*) $N(G) \geq \max\{n_+(G), n_-(G)\}$ (6)

(*ii*) $N(G) \leq |V| - 1$ (7)

with equality holding in (*ii*) for trees, complete graphs, odd cycles, and many other classes of graphs.

In this note we announce a number of basic results for isometric embeddings of graphs into cartesian products of *graphs*. In particular, we show how a natural concept of isometric dimension of a graph G can be defined, and we construct a canonical product graph G^*, which enjoys an attractive universal property for isometric embeddings of G into cartesian products. We also determine the isometric dimension of G in the important case that G embeds isometrically into an n-cube (see ref. 5). Finally, we give a generalization of the tree determinant formula (6, 8):

$$\det(D(T_n)) = (-1)^n n \cdot 2^{n-1}$$

for any tree T_n with n edges.

The details needed for proofs of our claims are somewhat lengthy and will appear elsewhere.

The Main Results

For a connected graph $G = (V, E)$ define a relation θ on E as follows:

$$\text{If } e = \{x, y\} \in E \text{ and } e' = \{x', y'\} \in E,$$

then $e\ \theta\ e'$ if

$$d_G(x, x') + d_G(y, y') \neq d_G(x, y') + d_G(x', y).$$

θ is easily seen to be well-defined, reflexive, and symmetric; let $\hat{\theta}$ be its transitive closure and let E_i, $1 \le i \le r$, be the equivalence classes of $\hat{\theta}$. Thus, $E = \cup_{i=1}^r E_i$. For $1 \le i \le r$, let G_i denote the graph $(V, E\backslash E_i)$ and let $C_i(1), C_i(2), \ldots, C_i(m_i)$ denote the connected components of G_i. Finally, form the graphs $G_i^* = (V_i^*, E_i^*)$, $1 \le i \le r$, by letting $V_i^* = \{C_i(1), \ldots, C_i(m_i)\}$ and taking $\{(C_i(j), C_i(j')\}$ to be an edge of G_i^* iff some edge in E_i joins a vertex in $C_i(j)$ to a vertex in $C_i(j')$. For $v \in C_i(j)$, denote by α_i: $V \to V_i^*$ the natural contraction $v \to C_i(j) \in V_i^*$.

We next define an embedding α: $G \to \Pi_{i=1}^r G_i^*$, which we will call the *canonical* embedding of G, by

$$\alpha(v) = (\alpha_1(v), \alpha_2(v), \ldots, \alpha_r(v)).$$

THEOREM 1. *The canonical embedding* α: $G \xrightarrow{I} \Pi_{i=1}^r G_i^*$ *is isometric.*

Let us call an embedding β: $G \to \Pi_{i=1}^s H_i$ *irredundant* if for all i and all $h \in H_i$, h occurs as a coordinate value in $\beta(g)$ for some $g \in G$ (where we always assume $|H_i| > 1$). We will say that G is *irreducible* if $G \xrightarrow{I} \Pi_{i=1}^s H_i$ implies $G \xrightarrow{I} H_i$ for some i.

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¶In general, we follow the terminology of ref. 1.
||Since d_s fails to satisfy the triangle inequality, (S, d_s) is actually only a semi-metric space.

COROLLARY 1. *G is irreducible if and only if E has one $\hat{\theta}$-equivalence class.*

COROLLARY 2. *The fraction of graphs on n vertices which are irreducible tends to 1 as $n \to \infty$.*

THEOREM 2. *The canonical embedding is irredundant, has irreducible factors, and has the largest possible number of factors among all irredundant isometric embeddings of G.*

We call this number of factors the *isometric dimension* of G, denoted by $\dim_I(G)$.

THEOREM 3. *The only irredundant isometric embedding of G into a cartesian product of $\dim_I(G)$ factors is the canonical one. Each factor H_i of an irredundant isometric embedding $G \xrightarrow{I} \Pi_{i=1}^{s} H_i$ embeds canonically into a product of G_j^*s.*

THEOREM 4. *Suppose $G \xrightarrow{I} K_2^m = \Pi_{i=1}^{m} K_2$, where K_2 is the graph with two vertices and one edge. Then $\dim_I(G) = n_-(G)$.*

Let us say that a family of subsets S of $\{1, 2, \ldots, n\}$ is *full-dimensional* if the corresponding characteristic vectors $S \leftrightarrow (\chi_1(S), \ldots, \chi_n(S)) \subseteq \mathbf{R}^n$ span a set of positive n-dimensional volume (where $\chi_i(S) = 1$ if $i \in S$, and 0 otherwise).

Suppose μ is a discrete measure on the subsets of $\{1, 2, \ldots, n\}$: $= [n]$, i.e.,

$$\mu(k) \geq 0, \; 1 \leq k \leq n,$$
$$\mu(X) = \sum_{x \in X} \mu(x), \; X \subseteq [n].$$

Our final result generalizes the tree determinant theorem mentioned earlier (6, 8).

THEOREM 5. *Suppose $\{S_0, S_1, \ldots, S_n\}$ is a full-dimensional family of subsets of [n] and μ is a discrete measure on the subsets of [n]. Then*

$$\det\,(\mu(S_i \Delta S_j)) = (-1)^n 2^{n-1} \sum_{k=1}^{n} \mu(k) \prod_{k=1}^{n} \mu(k),$$

where $X \Delta Y$ denotes the symmetric difference of X and Y.

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