Euclidean Ramsey Theorems on the n-Sphere

R. L. Graham

BELL LABORATORIES, MURRAY HILL, NJ 07974

ABSTRACT

Let us call a finite subset X of a Euclidean m-space $\mathbb{E}^m$ *Ramsey* if for any positive integer r there is an integer $n = n(X;r)$ such that in any partition of $\mathbb{E}^n$ into r classes $C_1, \ldots, C_r$, some C_i contains a set X' which is the image of X under some Euclidean motion in $\mathbb{E}^n$. Numerous results dealing with Ramsey sets have been proved in recent years although the basic problem of characterizing the Ramsey sets remains unsettled. The strongest constraints currently known are: (i) Any Ramsey set must lie on the surface of some sphere; (ii) Any subset of the set of vertices of a rectangular parallelepiped is Ramsey. In this paper we examine the corresponding problem in the case that our underlying spaces are (unit) n-spheres S^n and the allowed motions are orthogonal transformations of S^n onto itself. In particular, we show that for subsets of S^n which are not too "large," results similar to (i) and (ii) hold.

1. INTRODUCTION

Let us call a finite subset X of $\mathbb{E}^m$ *Ramsey* if for any positive integer r there is an integer $n = n(X;r)$ such that in any partition of $\mathbb{E}^n = \cup_{k=1}^{r} C_k$, some C_i contains a set X' which is the image of X under some Euclidean motion in $\mathbb{E}^n$. Numerous results dealing with Ramsey sets in $\mathbb{E}^n$ have been proved in recent years (e.g., see [1], [2], [3], [5], [6], [7], [9], [10], [11]), although the basic problem of characterizing the Ramsey sets remains unsettled. The best results currently known are the following. Let us call a set $Y \subseteq \mathbb{E}^m$ *spherical* if it lies on the surface of some sphere in $\mathbb{E}^m$, i.e., for some $\bar{z} \in \mathbb{E}^m$, all the distances $d(\bar{z},\bar{y})$, $\bar{y} \in Y$, are equal (where d denotes Euclidean distance). Also, we call a set $Y \subseteq \mathbb{E}^m$ a *brick* if it is the set of 2^m vertices of some rectangular parallelepiped in $\mathbb{E}^m$.

Theorem ([1]).

(i) Every brick is Ramsey.

(ii) Every Ramsey set is spherical.

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In this article we examine the analogous question for the case that our underlying spaces are (unit) n-spheres $S^n = \{(x_0, \ldots, x_n): \Sigma_{k=0}^n x_k^2 = 1\} \subseteq \mathrm{E}^{n+1}$ and the allowed motions are orthogonal transformations of S^n onto itself. In this case the unavoidable sets will be termed "sphere-Ramsey." It will turn out that for sets $X \subseteq S^n$ which are not too large (in a sense to be made precise later), a result similar to the preceding Theorem holds. For the remaining cases, only very preliminary results are available, although we suspect that much more is very likely true.

2. NECESSARY CONDITIONS

Theorem 1. Let $X = \{\bar{x}_1, \ldots, \bar{x}_m\}$ be a set of points in E^n such that:

(i) for some nonempty $I \subseteq \{1,2, \ldots, m\} \equiv [m]$, there exist nonzero α_i, $i \in I$, such that

$$\sum_{i \in I} \alpha_i \bar{x}_i = \bar{0};$$

(ii) for all nonempty $J \subseteq I$,

$$\sum_{j \in J} \alpha_j \neq 0.$$

Then there exists $r = r(X)$ such that for any N, there is a partition $S^N = \bigcup_{k=1}^r C_k$ such that no C_i contains a copy of X.

Proof. Consider the homogeneous linear equation

$$\sum_{i \in I} \alpha_i z_i = 0. \tag{*}$$

By (ii), Rado's theorem for the partition regularity of this equation over R^+ (see [8] or [7]) implies that it is *not* regular, i.e., for some r there is an r-coloring $\chi: \mathrm{R}^+ \to [r]$ such that (*) has no monochromatic solution. Color the points of $S_+^N = \{(x_0, \ldots, x_N) \in S^N : x_0 > 0\}$ by

$$\chi^*(\bar{x}) = \chi(\bar{u} \cdot \bar{x}),$$

where $\bar{u}$ denotes the unit vector $(1,0,0, \ldots, 0)$. Thus, the color of $\bar{x} \in S_+^N$ just depends on its distance from the "north pole" of S^N.

For each nonempty subset $J \subseteq I$, consider the equation

$$\sum_{j \in J} \alpha_j z_j = 0. \tag{*}_J$$

Of course, by (ii) this also fails to satisfy the (necessary and sufficient) condition of Rado for partition regularity. Hence, there is a coloring χ_J of $\mathbf{R}^+$ (using r_J colors) so that $(*)_J$ has no monochromatic (under χ_J) solution. As before, we can color S_+^N by giving $\bar{x} \in S_+^N$ the color

$$\chi_J^*(\bar{x}) = \chi_J(\bar{x} \cdot \bar{u}).$$

Now, form the *product* coloring $\hat{\chi}$ of S_+^N by defining for $\bar{x} \in S_+^N$,

$$\hat{\chi}(\bar{x}) = (\chi_J(\bar{x}), \ldots, \chi_J(\bar{x}), \ldots),$$

where the sequence has length $2^{|I|} - 1$ and the indices of the χ_J range over all nonempty subsets $J \subseteq I$. The number of colors required by the coloring $\hat{\chi}$ is at most $\Pi_{\emptyset \neq J \subseteq I} r_j \equiv R$.

An important property of $\hat{\chi}$ is this. Suppose we extend $\hat{\chi}$ to $S_0^N \equiv \{(x_0, \ldots, x_N) \in S^N: x_0 \geq 0\}$ by assigning *all* R colors to any point in $S_0^N \setminus S_+^N$ i.e., having $x_0 = 0$. Then the *only* monochromatic solution to $(*)$ in $\mathbf{R}^+ \cup \{0\}$ is $z_i = 0$ for all $i \in I$.

Next, construct a similar coloring $\check{\chi}$ on $S_-^N = \{-\bar{x}: \bar{x} \in S_+^N\}$, but using R completely different colors. This assures that any set X which hits both hemispheres S_+^N and S_-^N cannot be monochromatic. Finally, we have to color the equator

$$S^{N-1} = \{\bar{x} \in S^N: x_0 = 0\}.$$

By our construction, any copy of X which is not contained entirely in S^{N-1} cannot be monochromatic. Hence, it suffices to color S^{N-1} avoiding monochromatic copies of X where we may use any of the $2R$ colors previously used in the coloring of $S_+^N \cup S_-^N$. By induction, this can be done provided we can so color S^1. However, since $m > 1$, then S^1 can in fact always be 3-colored without a monochromatic copy of X (in fact, of any 2-element subset of X since the corresponding graph has maximum degree 2). This proves the theorem. ■

Note that if X is a constant distance $d \neq 90°$ from some point $\bar{t} \in S^n$, then X cannot satisfy both (i) and (ii). For

$$\sum_{i \in I} \alpha_i \bar{x}_i = \bar{0}$$

implies

$$0 = \bar{t} \cdot \left(\sum_{i \in I} \alpha_i \bar{x}_i \right) = \sum_{i \in I} \alpha_i \bar{t} \cdot \bar{x}_i = (\cos d) \cdot \sum_{i \in I} \alpha_i$$

i.e.,

$$\sum_{i\in I} \alpha_i = 0$$

since $\cos d \neq 0$.

However, these are not the only sets not ruled out from being possible Ramsey sets by Theorem 1. Another such example is given by the 3-point set

$$T = \left\{(1,0), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\right\} = \{t_1, t_2, t_3\}$$

(corresponding to the three cube roots of unity). Their linear dependence is given by

$$t_1 - t_2 - t_3 = \bar{0}$$

which does not satisfy (ii).

We restate Theorem 1 in its positive form.

Theorem 1. If X is sphere-Ramsey, then for any linear dependence $\Sigma_{i\in I}\alpha_i\bar{x}_i = \bar{0}$ there must exist a nonempty $J \subseteq I$ such that $\Sigma_{j\in J}\alpha_j = 0$.

3. SUFFICIENT CONDITIONS—SMALL BRICKS

Let us call an m-dimensional brick with edge lengths $\lambda_1, \lambda_2, \ldots, \lambda_m$ *small* if

$$\sum_{i=1}^{m} \lambda_m^2 \leq 2. \tag{1}$$

Theorem 2. Every small brick is sphere-Ramsey.

Proof. We sketch the proof (which has the same basic structure as that of the Hales–Jewett theorem given in [6]). Let a fixed number r of colors be given. For $m = 1$, the theorem is immediate: we simply consider the $r + 1$ points

$$\overbrace{\begin{matrix}(\beta_1, 0, 0, \ldots, 0, \gamma)\\(0, \beta_1, 0, \ldots, 0, \gamma)\\(0, 0, \beta_1, \ldots, 0, \gamma)\\ \vdots \\(0, 0, 0, \ldots, \beta_1, \gamma)\end{matrix}}^{r+1}$$

where $\beta_1 = \lambda_1/\sqrt{2} \leq 1$ and $\gamma^2 + \beta_1^2 = 1$. These $r+1$ points are on S^{r+1}. Since they are r-colored, then some pair must have the same color. This pair has distance $\beta_1\sqrt{2} = \lambda_1$, which is the desired conclusion.

In general, for a $\lambda_1 \times \ldots \times \lambda_m$ brick B, the set S of points we consider is of the form

$$\underbrace{(0, \ldots, \beta_m, \ldots, 0,}_{N_m}\ \underbrace{0, \ldots, \beta_{m-1}, \ldots, 0,}_{N_{m-1}}\ \cdots\cdots\ 0,\ldots 0,\ \underbrace{0, \ldots, \beta_1, \ldots, 0,}_{N_1} \gamma)$$

That is, S consists of $(N_m + N_{m-1} + \cdots + N_1 + 1)$-tuples in which exactly one of the entries in the jth block (of length N_j) is $\beta_j = \lambda_j/\sqrt{2}$ and all other entries are 0, with the exception of the last entry

$$\gamma = \left(1 - \sum_{j=1}^{m} \beta_j^2\right)^{1/2},$$

chosen so that all points are a unit N-sphere with $N = N_m + N_{m-1} + \cdots + N_1$. The hypothesis (1) guarantees that γ is real. The key to this construction is (as usual) in the choice of the N_j's. Needless to say, for the proof to work, they must grow very rapidly.

As an example, we consider the case $m = 2$. Choose $N_1 = r + 1$, $N_2 = r^{r+1} + 1$. An r-coloring χ of S induces an r^{r+1}-coloring χ' of the set

$$S' = \{(\overbrace{0, \ldots, \beta_2, \ldots, 0,}^{N_2} \gamma)\}$$

by

$$\chi'(s_1') = \chi'(s_2'), \qquad s_1', s_2' \in S'$$

iff

$$\chi(s_1't) = \chi(s_2't)$$

for all

$$t \in \{(\overbrace{0, \ldots, \beta_1, \ldots, 0,}^{N_1} \gamma)\} = T_1$$

where the concatenation $s_1't$ has the obvious interpretation as an element of S. Since $|S'| = N_2 = r^{r+1} + 1$ and S' is r^{r+1}-colored, then some pair of points $s_1', s_2' \in S'$ have $\chi'(s_1') = \chi'(s_2')$, i.e., $\chi(s_1't) = \chi(s_2't)$ for all $t \in T_1$. Since χ is an r-coloring and $|T_1| = N_1 = r + 1$, then some pair of points $t, t' \in T_1$ have

$$\chi(s_1't) = \chi(s_1't').$$

Of course, this implies

$$\chi(s_1't) = \chi(s_1't') = \chi(s_2't) = \chi(s_2't').$$

But

$$d(s_1't,\ s_1't') = \beta_1\sqrt{2} = \lambda_1 = d(s_2't,\ s_2't')$$
$$d(s_1't,\ s_2't) = \beta_2\sqrt{2} = \lambda_2 = d(s_1't',\ s_2't')$$

so that these 4 points form the desired monochromatic $\lambda_1 \times \lambda_2$ brick.

The general result follows by the same techniques where, in general, we choose $N_1 = r + 1$ and $N_{j+1} = 1 + r^{N_1N_2\ldots N_j}$ for $j \geq 1$. Specifically, we think of S as $S(m) \times T(m)$, where $S(m)$ consists of the N_m N_m-tuples $(0, \ldots, \beta_m, \ldots, 0)$ and $T(m)$ consists of the $N_1N_2 \ldots N_{m-1}$ complementary $(N_1 + \cdots + N_{m-1} + 1)$-tuples

$$(\overbrace{0, \ldots, \beta_{m-1}, \ldots}^{N_{m-1}}, \overbrace{\ldots, \beta_{m-2}, \ldots}^{N_{m-2}}, \ldots, \overbrace{\ldots, \beta_1, \ldots}^{N_1}, \gamma).$$

The initial r-coloring χ of S induces an $r^{N_1\ldots N_{m-1}}$-coloring χ' of $S(m)$ by

$$\chi'(s_1') = \chi'(s_2'), \qquad s_1', s_2' \in S(m)$$

iff

$$\chi(s_1't) = \chi(s_2't) \quad \text{for all } t \in T(m).$$

Since

$$|S(m)| = N_m = 1 + r^{N_1\ldots N_{m-1}},$$

then there exists a pair of points, say $s_1, s_2 \in S(m)$, such that

$$\chi'(s_1') = \chi'(s_2').$$

Also, there is induced r-coloring $\hat{\chi}$ of $T(m)$ by

$$\hat{\chi}(t) = \chi(s_1't), \qquad t \in T(m).$$

By induction, there is a monochromatic $\lambda_1 \times \cdots \times \lambda_m$ brick under the coloring $\hat{\chi}$ of $T(m)$. By the definition of $\hat{\chi}$ and χ', this extends to a monochromatic $\lambda_1 \times \cdots \times \lambda_m$ brick in the original coloring of S. ■

By suitable manipulations, it can be shown that the N_m satisfy

$$N_m \leq \overbrace{(r+2)^{(r+2)^{\cdot^{\cdot^{\cdot^{(r+2)}}}}}}^{m}$$
$$= (r+2) \uparrow\uparrow m.$$

Large Bricks. Bricks which have a main diagonal of length exceeding 2 seem much less tractable, although we expect that any $\lambda_1 \times \cdots \times \lambda_m$ brick with

$$\lambda_1^2 + \cdots + \lambda_m^2 < 4$$

is sphere-Ramsey. We can only prove this in the case $m = 1$.

Theorem 3. Let B be the set $\{-\lambda/2, \lambda/2\}$ where $0 < \lambda < 1$. Then B is sphere-Ramsey.

Proof. It is enough to show that the graph G_n with vertex set S^n and edge set $\{[\bar{x},\bar{y}] : d(\bar{x},\bar{y}) = \lambda\}$ has chromatic number tending to infinity as n tends to infinity. To prove this, we use the following recent result of Frankl and Wilson (which was suggested by I. Bárány, Z. Füredi, and J. Pach).

Theorem [4]: Let $\mathscr{F}$ be a family of k-sets of $[n]$ such that for some prime power q,

$$|F \cap F'| \not\equiv k \pmod q$$

for all $F \neq F'$ in $\mathscr{F}$. Then

$$|\mathscr{F}| \leq \binom{n}{q-1}.$$

For a fixed r, choose a prime power q so that

$$\binom{2(1+\varepsilon)q}{(1+\varepsilon)q} > r\binom{2(1+\varepsilon)q}{q-1}, \qquad (2)$$

where $\lambda = 2\beta\sqrt{2q}$, and $\varepsilon > 0$ and α are chosen so that

$$\alpha^2 + 2(1+\varepsilon)q\beta^2 = 1$$

and $N = (1 + \varepsilon)q$ is an integer. Consider the set

$$S = \left\{ (s_0, \ldots, s_{2N}) \colon s_0 = \alpha,\ s_i = \pm\beta,\ \sum_{i=1}^{2N} s_i = 0 \right\}.$$

To each $s \in S$ associate the subset

$$F(s) = \{ i \in [2N] \colon s_i = \beta \}.$$

Thus, the family

$$\mathcal{F} = \{ F(s) \colon s \in S \}$$

consists of the $\binom{2N}{N}$ N-element subsets of $[2N]$. If $F, F' \in \mathcal{F}$, $F \neq F'$, then

$$|F \cap F'| \equiv N \pmod{q}$$

iff

$$|F \cap F'| = N - q = \varepsilon q.$$

If the elements of $\mathcal{F}$ are r-colored, then some color class must contain at least

$$\frac{1}{r} |\mathcal{F}| = \frac{1}{r} \binom{2N}{N} > \binom{2N}{q-1}$$

elements of $\mathcal{F}$. However, by Frankl–Wilson, if $|F \cap F'| = \varepsilon q$ *never* occurs, then the number of N-sets must be at most $\binom{2N}{q-1}$, which is a contradiction. Thus, some monochromatic pair F, F' must have

$$|F \cap F'| = \varepsilon q.$$

This means that the corresponding points $s, s' \in S$ must (up to a permutation of coordinate positions) look like

$$s = (\alpha, \overbrace{\beta, \ldots, \beta}^{\varepsilon q}, \overbrace{\beta, \ldots, \beta}^{q}, \overbrace{-\beta, \ldots, -\beta}^{\varepsilon q}, \overbrace{-\beta, \ldots, -\beta}^{q}),$$

$$s' = (\alpha, \underbrace{\beta, \ldots, \beta}_{\varepsilon q}, \underbrace{-\beta, \ldots, -\beta}_{q}, \underbrace{\beta, \ldots, \beta}_{\varepsilon q}, \underbrace{-\beta, \ldots, -\beta}_{q}).$$

Note that

$$d(s,s') = \sqrt{8q\beta^2} = \lambda$$

and

$$d(s,0) = d(s',0) = \alpha^2 + 2(1+\varepsilon)q\beta^2 = 1,$$

i.e., $s,s' \in S^{2N}$. This proves the theorem. ■

As remarked previously, one would expect that the corresponding result should hold for any $\lambda_1 \times \cdots \times \lambda_m$ brick provided $\lambda_1^2 + \cdots + \lambda_m^2 < 4$. However, we are unable to prove this for even the case $m = 2$.

4. SOME REMARKS ON EDGE COLORINGS

Instead of coloring the points of E^n, we could color the line segments in E^n and, as before, look for monochromatic copies of some fixed structure C (again, up to some Euclidean motion). A set C of line segments which must always occur monochromatically in an r-coloring of E^n, provided only that n is sufficiently large as a function of r (and C), $r = 1,2,3,\ldots$, is said to be *line-Ramsey*. Several results on line-Ramsey sets were mentioned in [1], such as the fact that any line-Ramsey set must have all edges the same length (which we can assume is 1).

For a configuration C of unit line segments L_i, let $V(C)$ denote the set of endpoints of the L_i. Form a graph $G(C)$ with vertex set $V(C)$ and having all the L_i as its edges.

Theorem 4. Suppose C is a configuration of unit line segments such that:

(i) $V(C)$ is not spherical;
(ii) $G(C)$ is not bipartite.

Then C is not line-Ramsey.

Proof. Since $V(C)$ is not spherical, then by the previously mentioned necessary condition for $V(C)$ to be Ramsey, there exists an r and, for each N, an r-coloring χ_N of E^n so that $V(C)$ does not occur monochromatically. Let us color the unit line-segments $\{x,y\}$ of E^n by $\chi^*(\{x,y\}) = \{\chi(x),\chi(y)\}$.

Consider a fixed copy C' of C. Since $V(C')$ is not monochromatic, there are two points of $V(C')$, say x' and y' with $\chi(x') \neq \chi(y')$. Suppose C' is monochromatic under χ^*. Then *all* edges of $G(C')$ must have color $\{\chi(x'),\chi(y')\}$ since both colors $\chi(x')$ and $\chi(y')$ occur in the coloring $V(C')$. By (ii), $G(C')$ is not bipartite, and so, contains an odd cycle. However, it is easy to see that this results in a contradiction since an odd cycle cannot have all its edges with color $\{\chi(x'),\chi(y')\}$. ■

By the same technique, we can show that if $V(C)$ does not lie on two concentric spheres then C cannot be line-Ramsey, even when $G(C)$ is bipartite.

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