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ON THE MINIMUM DOMINATING PAIR NUMBER OF A CLASS OF GRAPHS

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1. Introduction

The closed neighbourhood $N_G X$ of the subset X of vertices of a graph G is defined by

$$N_G X = X \cup \{y \mid (x,y) \in E(G) \text{ for some } x \in X\}.$$

The dominating pair number of G, denoted by $DP(G)$, is the maximum taken over all pairs u, v of vertices of G of the cardinality of $N_G\{u,v\}$. The study of this parameter was first suggested by Bollobás. The purpose of this note is to investigate the function $f(\ell)$, which is the minimum value of the dominating pair number for graphs which have 2ℓ vertices and $\binom{\ell}{2}$ edges. The graph $K_\ell \cup \bar{K}_\ell$ shows that $f(\ell) \leq \ell + 1$. In the next section we exhibit a graph F on 22 vertices and $\binom{11}{2}$ edges with $DP(F) = 11$ and then use F to construct an infinite class of graphs with 2ℓ vertices, $\binom{\ell}{2}$ edges and dominating pair number strictly less than ℓ. The exact determination of $f(\ell)$ remains an open question.

2. Results

The following graph F was obtained during the (vain!) attempt to prove that the dominating pair number of any graph with 22 vertices and $\binom{11}{2}$ edges, is at least 12. F is regular of degree 5 and one may verify (tediously!) that. $DP(F) = 11$.

Let $V(F) = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$ (disjoint union), where

$$A_1 = \{1,2\}$$

and

$$A_{i+2} = \{3+5i,\ 4+5i,\ 5+5i,\ 6+5i,\ 7+5i\}$$

for $i = 0,1,2,3$.

The edges of F are as follows:-

(i) The subgraph induced by A_1, A_2 is the complete bipartite graph $K_{2,5}$.

(ii) For $i = 1,2,3$, A_{i+2}, using the given order of vertices induces a cycle C_5.

(iii) Finally, we add the edges of the C_{15} with vertex sequence (8,20,17,19,11,13,10,22,9,16,18,15,12,14,21).

It seemed likely that $f(\ell) \geq \ell$, for all ℓ. This however is false and in fact, using F as the starting point, we now exhibit infinite classes of graphs G with 2ℓ vertices, at least $\binom{\ell}{2}$ edges and $DP(G) = \ell/(1+\alpha/22)$, where α is any positive rational less than the smaller root of $3x^2 - 44x + 44 = 0$. This implies, for example, that there exist an infinite number of values of ℓ for which $f(\ell) \leq .954\ell$.

We first form F_m, the <u>m-expansion of F</u> as follows:-

(i) Each $x \in V(F)$ is replaced by m copies of x, say $x_1, \ldots, x_m$, which form a complete graph K_m in F_m.

(ii) If $[x,y]$ is an edge of F then for all i, j, $[x_i, y_j]$ is an edge of F_m.

The new graph has $22m$ vertices and its size $e(F_m)$ is given by:-

$$e(F_m) = m^2 e(F) + \binom{m}{2} v(F) = 66m^2 - 11m.$$

We now verify that $DP(F_m) = 11m$. We have

$$z_k \in N_{F_m}\{x_i, x_j\} \text{ if and only if } z \in N_F\{x\}.$$

Hence

(1) $$\left|N_{F_m}\{x_i, x_j\}\right| = 6m.$$

Also

$$z_k \in N_{F_m}\{x_i, y_j\} \text{ if and only if } z \in N_F\{x, y\}.$$

Hence

(2) $$\left|N_{F_m}\{x_i, y_j\}\right| = m\left|N_F\{x, y\}\right| \le 11m.$$

From (1) and (2), $DP(F_m) = 11m$ as asserted.

Next, we form the graph F_m^* by adding to F_m a disjoint copy of $K_{\alpha m}$, the complete graph of m vertices, where α is any positive rational less that $\frac{1}{3}(22-4\sqrt{22})$, which is the smaller root of $3x^2 - 44x + 44 = 0$. We restrict ourselves to values of m for which αm is an even integer.

We note that $v(F_m^*) = (22+\alpha)m$, $e(F_m^*) = 66m^2 - 11m + \binom{\alpha m}{2}$ and observe for $x \in V(F_m)$ and $y \in V(K_{\alpha m})$,

$$\left|N_{F_m^*}\{x, y\}\right| = 6m + \alpha m < 11m.$$

It follows that $DP(F_m^*) = 11m$ and if $2\ell = (22+\alpha)m$, then $DP(F_m^*) = \ell/(1+\alpha/22)$.

It remains to show that for sufficiently large m, $e(F_m^*) \ge \binom{\frac{1}{2}v(F_m^*)}{2}$. A little calculation shows that

(3) $$66m^2 - 11m + \binom{\alpha m}{2} - \binom{\frac{m}{2}(22+\alpha)}{2} = \frac{1}{8}\left[(3\alpha^2 - 44\alpha + 44)m^2 - (44+2\alpha)m\right].$$

By defintion of α, $3\alpha^2 - 44\alpha + 44 > 0$. Therefore , for m sufficiently large, the expression on the right-hand side of (3) is positive as required.

BIBLIOGRAPHY

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