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Minimal Decompositions of Hypergraphs into Mutually Isomorphic Subhypergraphs

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INTRODUCTION

By an *r-uniform hypergraph* (or *r-graph*, for short) $H = (V, E)$ we mean a collection $E = E(H) = \{E_1,..., E_m\}$ of r-element subsets (called *edges*) of a set $V = V(H)$, called the *vertices* of H. Let $\mathscr{H} = \{H_1,..., H_k\}$ be a family of r-graphs, each having the same number of edges. By a *U-decomposition* of $\mathscr{H}$ we mean a set of partitions of the edge sets $E(H_i)$ of the H_i, say $E(H_i) = \sum_{j=1}^{t} E_{i,j}$, such that for each j, all the $E_{i,j}$ are *isomorphic* (as hypergraphs). Such decompositions always exist since (by our assumptions) we can always take all the $E_{i,j}$ to be single edges.

Let us define the quantity $U(\mathscr{H})$ as the least possible value of t a U-decomposition of $\mathscr{H}$ can have. Finally, we let $U_k(n, r)$ denote the largest possible value $U(\mathscr{H})$ can assume as $\mathscr{H}$ ranges over all families of k r-graphs, each having n vertices and the same (unspecified) number of edges.

For the value $r = 2$, r-graphs are just ordinary graphs and in this case, the

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functions $U_k(n, 2) = U_k(n)$ have been investigated extensively by the authors and others in [1, 2]. In particular, it is known that

$$U_2(n) = \tfrac{2}{3}n + o(n),$$

and

$$U_k(n) = \tfrac{3}{4}n + o_k(n), \qquad k \geqslant 3.$$

In this paper we continue this study to the much more complex case of $r > 2$. Our basic results are the following (where $c_1, c_2, \ldots,$ denote appropriate positive constants):

$$c_1 n^{4/3} \log\log n / \log n < U_2(n; 3) < c_2 n^{4/3}; \tag{1}$$

$$\text{for any } \varepsilon > 0, \quad c_3 n^{2-2/k-\epsilon} < U_k(n; 3) < c_4 n^{2-1/k}; \tag{2}$$

$$c_5 n^{r/2} < U_2(n; r) < c_6 n^{r/2} \qquad \text{for } r \text{ even}; \tag{3}$$

$$c_7 n^{(r-1)^2/(2r-1)} < U_2(n; r) < c_8 n^{r/2} \qquad \text{for } r \text{ odd}; \tag{4}$$

$$n^{r-1-r/k} \leqslant U_k(n; r) \leqslant n^{r-1-1/k} \qquad \text{for } r \geqslant 3. \tag{5}$$

Preliminaries

We first prove several auxiliary lemmas. Suppose $\mathscr{H} = \{H_1, \ldots, H_k\}$, where each H_i is an r-graph having n vertices and e edges. Let us denote by $c(\mathscr{H})$ the maximum number of edges in any hypergraph H occurring in all the H_i as a common subhypergraph.

Lemma 1.

$$c(\mathscr{H}) \geqslant \frac{e^k}{\binom{n}{r}^{k-1}}.$$

Proof. Let Ω_i denote the set of all one-to-one mappings of $V(H_i)$ into $V(H_1)$. For $\lambda_i \in \Omega_i$, $e_i \in E(G_i)$, $1 \leqslant i \leqslant k$, define

$$\begin{aligned} I_{\lambda_2,\ldots,\lambda_k}(e_1, \ldots, e_k) &= 1 \qquad \text{if } \lambda_i \text{ maps } e_i \text{ onto } e_1, \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

where we say that λ_i maps e_i onto e_1 if $e_1 = \bigcup_{x \in e_i} \lambda_i(x)$. Consider the sum

$$S = \sum_{\substack{e_1 \in E(H_1) \\ \vdots \\ e_k \in E(H_k)}} \sum_{\substack{\lambda_2 \in \Omega_2 \\ \vdots \\ \lambda_k \in \Omega_k}} I_{\lambda_2,\ldots,\lambda_k}(e_1,\ldots,e_k)$$

$$= \sum_{\substack{e_1 \in E(H_1) \\ \vdots \\ e_k \in E(H_k)}} (r!(n-r)!)^{k-1} = e^k (r!(n-r)!)^{k-1}.$$

Since $|\Omega_i| = n!$ for all i then for some choice of $\bar{\lambda}_2 \in \Omega_2,\ldots,\bar{\lambda}_k \in \Omega_k$,

$$\sum_{\substack{e_1 \in E(H_1) \\ \vdots \\ e_k \in E(H_k)}} I_{\bar{\lambda}_2,\ldots,\bar{\lambda}_k}(e_1,\ldots,e_k) \geqslant \frac{S}{(n!)^{k-1}} = \frac{e^k}{\binom{n}{r}^{k-1}}.$$

Consequently, the $\bar{\lambda}_i$, $2 \leqslant i \leqslant k$, determine a subhypergraph H common to all of the H_i which has at least $e^k/\binom{n}{r}^{k-1}$ edges. ∎

LEMMA 2. *Let H be an r-graph with $|E(H)| \geqslant rab + 1$. Suppose* $\deg v = |\{e \in E(H): v \in e\}| < a$ *for all vertices $v \in V(H)$. Then H contains b disjoint edges.*

Proof. Suppose F is a maximal set of disjoint edges. If $|F| < b$, the number of edges containing *some* element of F must be at most $|F|\, ar < |E(H)|$, contradicting the maximality of F. ∎

LEMMA 3. *If $r = 3$,*

$$c(\mathscr{H}) \geqslant \sqrt{\frac{e}{5n}}.$$

Proof. It suffices to prove there is a *star* with $t = \lceil \sqrt{e/5n} \rceil$ edges contained in each H_i. By a star S we mean a collection of edges e_i such that for some point x, $e_i \cap e_j = \{x\}$ for all $i \neq j$. Suppose H has n vertices and e edges and does not contain S. Consider the set P of disjoint pairs of vertices of $V(H)$ defined as follows:

(i) Select v_1 with $\deg_H(v_1) \geqslant \deg_H(v)$ for any $v \in V(H)$. Let v_1^* be a vertex in H_{x_1} of maximum degree and define $P_1 = \{v_1, v_1^*\}$ (where, for $x \in V(H)$, H_x denotes the ordinary (2-) graph with edge set $\{\{y,z\}: \{x,y,z\} \in E(H)\}$).

(ii) Suppose now that $P_1,\ldots,P_i$ have been defined. We form P_{i+1} as follows. Choose v_{i+1} so that:

(a) $v_{i+1} \notin P_j$ for $1 \leqslant j \leqslant i$;

(b) v_{i+1} has maximum degree in the subhypergraph induced on $V(H) - X_i$, where

$$X_i = \bigcup_{j=1}^{i} P_j.$$

Let v_{i+1}^* be the vertex the graph $H_{v_{i+1}} - X_i$ having maximum degree at least one in $H_{v_{i+1}}$. Define $P_{i+1} \equiv \{v_{i+1}, v_{i+1}^*\}$. We continue this process as long as possible. The final set of pairs P is defined to be $\bigcup_{i \geqslant 1} P_i$.

Let d_i denote the degree of v_i in the hypergraph induced in $V(H) - X_i$ (with X_0 taken to be $\varnothing$). Since H does not contain a copy of S, we have

$$|\{\bar{e} \in E(H): P_i \subseteq \bar{e}\}| \geqslant d_i/2t$$

for all $i \geqslant 1$. Let d_i^* denote the degree of v_i^* in the hypergraph induced on $V(H) - X_{i-1} - \{v_i\}$. Then $d_i \geqslant d_i^*$ and $\sum_i (d_i + d_i^*) \geqslant e$. Therefore, $\sum_i d_i \geqslant e/2$.

Now, for any $v \in V(H)$, define $\alpha(v)$ by

$$\alpha(v) \equiv |\{\bar{e} \in V(H): \bar{e} = \{v\} \cup P_i \text{ for some } i\}|.$$

Thus,

$$\begin{aligned} \sum_v \alpha(v) &\geqslant \sum_i |\{\bar{e} \in E(H): P_i \in \bar{e}\}| \\ &\geqslant \sum_i d_i/2t \\ &\geqslant e/4t. \end{aligned} \tag{6}$$

However, the assumption that $S \nsubseteq H$ implies $\alpha(v) \leqslant t - 1$. Therefore, by (6)

$$(t-1)n \geqslant e/4t,$$

which clearly contradicts the hypothesis that $t \leqslant \sqrt{e/5n}$. ∎

In a similar way we can prove the following.

LEMMA 4. *Let $\mathscr{H}$ be a family of r-graphs, each with e edges. Then*

$$c(\mathscr{H}) \geqslant \sqrt{\frac{ce}{n^{r-2}}},$$

where c is a constant depending on k.

Bounds on $U_2(n; 3)$

The main result of this section is the following.

THEOREM 1.

$$c_1 n^{4/3} \log\log n/\log n < U_2(n; 3) < c_2 n^{4/3}.$$

Proof. We first prove the upper bound. Let G_1 and G_2 be two 3-graphs, each with n vertices and e edges. We will successively remove isomorphic subgraphs H from the G_i, thereby decreasing the number e of edges currently remaining in each of the original graphs. The subgraph $H = H(e)$ removed will depend on the current value of e. We distinguish two ranges for e.

(i) $e \geqslant n^{5/3}$. In this case we repeatedly remove a common subgraph $H(e)$ having at least $e^2/\binom{n}{3}$ edges. The existence of such an $H(e)$ is guaranteed by Lemma 1. If e_i denotes the number of edges remaining in each hypergraph after i such subgraphs have been removed then

$$e_{i+1} \leqslant e_i - \frac{e_i^2}{\binom{n}{3}}. \tag{7}$$

Letting $\alpha_i = e_i/\binom{n}{3}$ we have

$$\alpha_{i+1} \leqslant \alpha_i - \alpha_i^2.$$

Since $\alpha_i < 1$ and $i^{-1} - i^{-2} < (i+1)^{-1}$, it follows by induction that $\alpha_i < i^{-1}$ for all i. Thus, after $n^{4/3}$ steps, the remaining graphs have at most $n^{5/3}$ edges.

(ii) $e < n^{5/3}$. For this range, we repeatedly apply Lemma 3. Let e_0 denote the number of edges each graph has at the beginning of this process. In general, if e_i denotes the number of edges remaining after i applications of Lemma 3, then

$$e_{i+1} \leqslant e_i - \sqrt{\frac{e_i}{5n}}. \tag{8}$$

Setting $\alpha_i = 5e_i n$, we have $\alpha_{i+1} \leqslant \alpha_i - \sqrt{\alpha_i}$. By hypothesis, $\alpha_0 \leqslant 5n^{8/3}$. Suppose

$$\alpha_i \leqslant (\sqrt{5}\, n^{4/3} - i/2)^2$$

for some $i \geqslant 0$. Then,

$$\begin{aligned}\alpha_{i+1} &\leqslant (\sqrt{5}\, n^{4/3} - i/2)^2 - \sqrt{5}\, n^{4/3} + i/2 \\ &\leqslant (\sqrt{5}\, n^{4/3} - (i+1)/2)^2.\end{aligned}$$

Therefore, after at most $2\sqrt{5}\,n^{4/3}$ steps, all edges in each graph will have been removed. Since, the total number of steps required in (i) and (ii) is at most $(2\sqrt{5}+1)\,n^{4/3}$ then we have proved

$$U_2(n;3) \leqslant c_2 n^{4/3}$$

as required.

The lower bound is obtained by proving the existence of two hypergraphs G_1 and G_2 with $cn^{5/3}$ edges with the property that any common subgraph has at most $c'n^{1/3}\log n/\log\log n$ edges.

Let G_1 consist of the disjoint union of $n^{2/3}$ copies of complete 3-graphs on $n^{1/3}$ vertices. We remark here that although $n^{2/3}$ and $n^{1/3}$ may not be integers, such statements are always made with the implicit understanding that the hypergraphs (and quantities) involved may have to be adjusted slightly by adding or deleting (asymptotically) trivial subgraphs (and amounts) so as to make stated inequalities true.

G_2 will be a 3-graph having the following properties:

(a) There is a point v_1 such that $v_1 \in \bar{e}$ for all $\bar{e} \in E(G_2)$;

(b) Consider the ordinary (2-) graph G' with $V(G') = \{v_2, \ldots, v_n\}$ and $E(G') = \{\bar{e} - \{v_1\} : \bar{e} \in E(G_2)\}$. Then G' has $\binom{n^{1/3}}{3} n^{2/3}$ edges.

(c) Any induced subgraph of G' on $n^{1/3}$ points has at most $n^{1/3}\log n/\log\log n$ edges.

The existence of such a G_2 follows from the following probability argument.

Consider the set $\mathscr{F}$ of all ordinary (2-) graphs with n vertices and $e = \binom{n^{1/3}}{3} n^{2/3}$ edges. A graph $F \in \mathscr{F}$ is said to be *bad* if there exists a set of $n^{1/3}$ points such that the induced subgraph on these vertices has at least $n^{1/3}\log n/\log\log n$ edges. The number of such bad graphs $F \in \mathscr{F}$ is bounded above by

$$A = \binom{n}{n^{1/3}} \binom{n^{2/3}}{n^{1/3}\log n/\log\log n} \binom{\binom{n}{2} - n^{1/3}\log n/\log\log n}{e - n^{1/3}\log n/\log\log n}.$$

A straightforward calculation shows

$$\frac{A}{\binom{\binom{n}{2}}{e}} \leqslant \left\{ \frac{n}{n^{1/3}/e} \left(\frac{n^{2/3}}{\dfrac{n^{1/3}\log n}{e\log\log n}} \cdot \frac{n^{5/3}}{n^2} \right)^{\log n/\log\log n} \right\}^{n^{1/3}} < 1.$$

Thus,

$$A < \binom{\binom{n}{2}}{e} = |F|$$

so that *some* graph $G' \in \mathscr{F}$ is *not* bad.

Now, let us consider a common subgraph of G_1 and G_2. H must be connected since all edges in G_2 contain the common vertex v_1. Also, $|V(H)| \leqslant n^{1/3}$ since any connected component of G_1 has at most $n^{1/3}$ vertices. Finally, property (c) of G_2 implies

$$|E(H)| \leqslant n^{1/3} \log n/\log\log n.$$

Since G_1 and G_2 each have at least $n^{5/3}/10$ edges then

$$U(\{G_1, G_2\}) \geqslant c_1 n^{4/3} \log\log n/\log n.$$

This completes the proof. ∎

Bounds on $U_k(n; 3)$

In this section we consider U-decompositions of $k \geqslant 3$ 3-graphs. As might be expected, our bounds are not as tight as in the case $k = 2$.

THEOREM 2. *For any* $\varepsilon > 0$,

$$c_3 n^{2-2/k-\epsilon} < U_k(n; 3) < c_4 n^{2-1/k}.$$

Proof. Again, we first attack the upper bound. Let $G_1, G_2, \ldots, G_k$ be k 3-graphs with n vertices and e edges. There are two possibilities.

(i) $e \geqslant n^{3-2/k}$. In this case repeatedly remove a common subgraph (guaranteed by Lemma 1) having at least $e^k/\binom{n}{3}^{k-1}$ edges. Let e_i denote the number of edges remaining in each graph after i such subgraphs have been removed. Then

$$e_{i+1} \leqslant e_i - \frac{e_i^k}{\binom{n}{3}^{k-1}}.$$

Letting $\alpha_i = e_i/\binom{n}{3}$ we obtain

$$\alpha_{i+1} \leqslant \alpha_i - \alpha_i^k.$$

Since $\alpha_i = e/\binom{n}{3} < 1$ and

$$(i)^{-1/(k-1)} - (i)^{-k/(k-1)} \leqslant (i+1)^{-1/(k-1)}$$

then it follows by induction that

$$\alpha_i \leqslant (i)^{-1/(k-1)} \qquad \text{for all } i,$$

i.e.,

$$e_i \leqslant (i)^{-1/(k-1)} \binom{n}{3}.$$

After $n^{2-1/k}$ such subgraphs have been removed, the number of edges remaining in each graph is at most

$$(n^{2-1/k})^{-1/(k-1)} \binom{n}{3} \leqslant n^{3-(2-1/k)(1/(k-1))} < n^{3-2/k}.$$

(ii) $e < n^{3-2/k}$. In this case we repeatedly apply Lemma 3. Let e_i denote the number of edges each (hyper) graph has after i applications of Lemma 3 (with e_0 denoting the initial number of edges on each graph at the beginning of this step). Thus,

$$e_{i+1} \leqslant e_i - \sqrt{\frac{e_i}{5n}}.$$

As in the proof of Theorem 1, it can be shown that this implies

$$5e_i n \leqslant (\sqrt{5}\, n^{2-1/k} - i/2)^2.$$

Therefore, after at most $2\sqrt{5}\, n^{2-1/k}$ steps all edges have been removed from all G_i.

Taking count of the number of subgraphs removed in each of the two ranges for e_1 we conclude

$$U_k(n; 3) \leqslant c_4 n^{2-1/k}$$

as required.

The lower bound on $U_k(n; 3)$ will be proved using probability arguments. More precisely, we claim that for all $\varepsilon > 0$, there exist 3-graphs $G_1, G_2, \ldots, G_k$ with n vertices and $n^{3-2/k}$ edges such that any subgraph common to all of them has at most $n^{1+\epsilon}$ edges, provided n is sufficiently large. Elementary counting arguments show that the number of k-sets of 3-graphs with n vertices and $n^{3-2/k}$ edges which contain a common subgraph with at least $n^{1+\epsilon}$ edges is less than

$$B = \binom{\binom{n}{3}}{n^{1+\epsilon}} (n!)^k \binom{\binom{n}{3} - n^{1+\epsilon}}{n^{3-2/k} - n^{1+\epsilon}}^k.$$

Since

$$\frac{B}{\binom{\binom{n}{3}}{n^{3-2/k}}^k} \leqslant \frac{n^{3n^{1+\epsilon}} n^{kn} n^{(3-2/k)k \cdot n^{1+\epsilon}}}{n^{(1+\epsilon)n^{1+\epsilon}} e^{kn} n^{3kn^{1+\epsilon}}} < 1$$

for n sufficiently large then

$$B < \left(\frac{\binom{n}{3}}{n^{3-2/k}} \right)^k$$

and so, there exists a k-set of such graphs $G_1, G_2, \ldots, G_k$ with any common subgraph having at most $n^{1+\epsilon}$ edges. Thus,

$$U_k(n; 3) \geqslant U(\{G_1, \ldots, G_k\}) \geqslant \frac{e}{n^{1+\epsilon}} \geqslant n^{2-2/k-\epsilon}$$

for n sufficiently large.

This completes the proof of Theorem 2. ▮

Bounds on $U_2(n; r)$

This section will investigate bounds for general r-graphs. There are two cases, depending on the parity of r.

THEOREM 3. *For r even,*

$$c_5 n^{r/2} < U_2(n; r) < c_6 n^{r/2}.$$

Proof. Let G_1 and G_2 be two r-graphs, each with n vertices and e edges. There are two possibilities.

(i) $e \geqslant n^{r/2}$. For this case we apply Lemma 1 repeatedly, removing common subgraphs having at least $e^2/\binom{n}{2}$ edges. If e_i denotes the current number of edges remaining after i steps, then it can be shown by methods similar to those used in Theorems 1 and 2 that $e_i \leqslant \binom{n}{r}/i$. Thus, after at most $n^{r/2}$ steps there are at most $n^{r/2}$ edges left.

(ii) $e < n^{r/2}$. In this case we simply remove one edge at a time.

Combining the two processes, the decomposition requires at most $2n^{r/2}$ and so,

$$U_2(n; r) \leqslant c_6 n^{r/2}.$$

The lower bound is established by constructing two hypergraphs G_1 and G_2 with $cn^{r/2}$ edges for which the largest common subgraph has a single edge. To begin with, let G_1' be the (hyper)graph defined by $V(G_1') = \{v_1, \ldots, v_n\}$ and $E(G_1') = \{\{v_1, \ldots, v_{r/2}\} \cup \bar{e} : \bar{e} \subseteq \{v_{r/2+1}, \ldots, v_n\},\ |\bar{e}| = r/2\}$. G_1 will be formed by selecting an arbitrary set of $c_5 n^{r/2}$ edges from G_1'.

G_2 will be an r-graph with $\binom{n}{r}/\binom{r}{r/2}\binom{n}{r/2}$ edges having the property that any two edges of G_2 intersect in at most $r/2 - 1$ vertices. The existence of such a G_2 is guaranteed by the following considerations. Let S be an arbitrary r-subset of $\{1, 2, \ldots, n\}$. The number of r-sets which intersect S in i elements is $\binom{r}{i}\binom{n-r}{r-i}$. The total number of r-sets which intersect S in more than $r/2 - 1$ elements is $\sum_{j=0}^{r/2} \binom{r}{j}\binom{n-r}{j}$. Therefore, there must exist a family $\mathscr{F}$ of r-sets such that:

(a) any two r-sets in $\mathscr{F}$ intersect in at most $r/2 - 1$ elements;

(b) $|\mathscr{F}| \geqslant \dfrac{\binom{n}{r}}{\sum_{j=0}^{r/2} \binom{r}{j}\binom{n-r}{j}} \geqslant \dfrac{\binom{n}{r}}{\binom{r}{r/2}\binom{n}{r/2}} \geqslant c_5 n^{r/2}.$

Note that any two edges in G_1 intersect in at least $r/2$ elements. Thus, the largest common subgraph of G_1 and G_2 has just one edge. This implies

$$U_2(n; r) \geqslant U(\{G_1, G_2\}) \geqslant c_5 n^{r/2}.$$

and the proof of Theorem 3 is complete. ∎

THEOREM 4. *For r odd,*

$$c_7 n^{(r-1)^2/(2r-3)} \frac{\log\log n}{\log n} < U_2(n; r) < c_8 n^{r/2}.$$

Proof. The upper bound proof follows the same lines as the corresponding result in Theorem 3.

For the lower bound, we consider the following two r-graphs G_1 and G_2 on n vertices. G_1 consists of $n^{(r-1)/(2r-3)}$ disjoint copies of complete r-graphs on $n^{(r-2)/(2r-3)}$ vertices. Observe that G_1 has $c' n^{(r^2-r-1)/(2r-3)}$ edges. For G_2 we will take a hypergraph satisfying the following properties:

(a) There is a vertex v_1 which belongs to all edges of G_2;

(b) G_2 has $c' n^{(r^2-r-1)/(2r-3)}$ edges;

(c) Consider the $(r-1)$-graph G' given by $V(G') = V(G) - \{v_1\}$ and $E(G') = \{\bar{e} - \{v_1\} : \bar{e} \in E(G_2)\}$. Then any induced subhypergraph of G' on $n^{(r-2)/(2r-3)}$ points has at most $n^{(r-2)/(2r-3)} \log n/\log\log n$ edges.

The (probabilistic) proof that such a graph G_2 exists is very similar to that used in Theorem 1 and is omitted.

Any common subgraph of G_1 and G_2 must be connected and has at most $n^{(r-2)/(2r-3)}$ vertices. Thus, it has at most $n^{(r-2)/(2r-3)} \log n/\log\log n$ edges. It follows from this that

$$U_2(n; r) > c_7 n^{(r-1)^2/(2r-3)}. \quad ∎$$

CONCLUDING REMARKS

We close this section with the final result of the paper. Its proof uses no new techniques and will not be included.

THEOREM 5. *For all $r \geqslant 3$ and all k,*

$$n^{r-1-r/k} \leqslant U_k(n; r) \leqslant n^{r-1-1/k},$$

for n sufficiently large.

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