LINEAR EXTENSIONS OF PARTIAL ORDERS AND THE FKG INEQUALITY

R.L. Graham
Bell Laboratories
Murray Hill, New Jersey 07974

ABSTRACT

Many algorithms for sorting n numbers $\{a_1, a_2, \ldots, a_n\}$ proceed by using binary comparisons $a_i : a_j$ to construct successively stronger partial orders P on $\{a_i\}$ until a linear order emerges (e.g., see Knuth [Kn]). A fundamental quantity in deciding the expected efficiency of such algorithms is $Pr(a_i < a_j \mid P)$, the probability that the result of $a_i : a_j$ is $a_i < a_j$ when all linear orders consistent with P are equally likely. In this talk we discuss various intuitive but nontrivial properties of $Pr(a_i < a_j \mid P)$ and related quantities. The only known proofs of some of these results require the use of the so-called FKG inequality [FKG, SW]. We will describe this powerful result and show how it is used in problems like this.

I. Rival (ed.), Ordered Sets, 213–236.

INTRODUCTION

Many algorithms for sorting n numbers $X = \{x_1,\ldots,x_n\}$ proceed by using binary comparisons $x_i:x_j$ to build successively stronger partial orders P on X until a linear order can be deduced (e.g., see Knuth [Kn]). A fundamental quantity in determining the expected efficiency of such algorithms is $\Pr\{x_i<x_j|P\}$, the probability that the result of $x_i:x_j$ is $x_i < x_j$ when all linear orders consistent with P are equally likely. In this paper we discuss a number of results of this type and show how a fundamental inequality, called the FKG inequality, can be used to prove them as well as a variety of related results.

We begin our discussion with a motivating example. Assume there are two teams of tennis players, say $A = \{a_1,\ldots,a_m\}$ and $B = \{b_1,\ldots,b_n\}$. Suppose that the players are inherently totally ordered by an unknown *linear* ordering which may be any permutation of the players, say $a_{i_1} < a_{i_2} < b_{j_1} < \ldots$, each having probability $\frac{1}{(m+n)!}$ (thus, all are equally likely). Assume further that in a match between two players, the *higher ordered player always wins*. Thus, as we observe the outcomes of various matches, we can learn more about the underlying total order of the players.

Suppose that at some point in time we have seen various matches between A players and have thereby learned some partial ordering $\mathcal{A} = \{a_{i_1}<a_{i_2}, a_{i_3}<a_{i_4},\ldots\}$ of A, and, similarly, we know a partial ordering $\mathcal{B} = \{b_{j_1}<b_{j_2}, b_{j_3}<b_{j_4},\ldots\}$ of B.

Let $\mathcal{C},\mathcal{C}',\ldots$ be observed outcomes of matches between a's and b's, e.g.,

$$\mathcal{C}:\ a_1 < b_5,\ a_1 < b_3,\ a_4 < b_2,\ldots\ .$$

Suppose that in all matches between a's and b's so far, a's always lose to b's. It is certainly reasonable to conjecture that the events $\mathcal{C}$ and $\mathcal{C}'$ are *mutually favorable*, i.e.,

$$\Pr(\mathcal{C}'|\mathcal{A} \text{ and } \mathcal{B} \text{ and } \mathcal{C}) \geq \Pr(\mathcal{C}'|\mathcal{A} \text{ and } \mathcal{B}). \tag{1}$$

Since, by definition,

$$\Pr(C' \mid A \text{ and } B \text{ and } C) = \Pr(A \text{ and } B \text{ and } C \text{ and } C')/\Pr(A \text{ and } B \text{ and } C)$$

then (1) is equivalent to

$$\Pr(C \mid A \text{ and } B \text{ and } C') \geq \Pr(C \mid A \text{ and } B).$$

Indeed, inequality (1), first conjectured in [GYY] was very recently proved by Shepp [Sh]. Shepp's proof employed the so-called FKG inequality, a result which has just begun to be exploited in combinatorics.

The same intuition which leads to (1) also is present if in addition to knowing A and B, we also know that C'' has occurred, i.e., various a's have previously lost to other b's. The occurrence of C'' somehow reinforces the feeling that the a's are generally weaker players than the b's and so, C and C' should still be mutually favorable, i.e.,

$$\Pr(C' \mid A \text{ and } B \text{ and } C \text{ and } C'') \geq \Pr(C' \mid A \text{ and } B \text{ and } C''). \qquad (1')$$

Surprisingly, this is *not* the case, as the following example shows.

Example. $m = n = 2$, $A = B = \phi$

$$C = \{a_2 < b_2\},\ C' = \{a_1 < b_1\},\ C'' = \{a_2 < b_1\}.$$

We show the allowable linear orderings in the table below. Thus,

$$\Pr(C'') = \frac{12}{24} = \frac{1}{2},$$

$$\Pr(C \text{ and } C'') = \frac{8}{24} = \frac{1}{3},$$

$$\Pr(C \text{ and } C' \text{ and } C'') = \frac{5}{24},$$

and so,

$$\Pr(C' \mid C \text{ and } C'') = \frac{5/24}{1/3} = \frac{5}{8} < \frac{2}{3} = \Pr(C' \mid C'').$$

Again, however somewhat unexpectedly, (1') *does* hold if A and B are both *linear* orders. This was first shown by Graham, Yao and Yao [GYY] using basically combinatorial techniques. Subsequently, Shepp [Sh] and Kleitman and Shearer [KS] found much shorter proofs, again using the FKG inequality.

Ordering	C	C'	C''
$a_1 < a_2 < b_1 < b_2$	1	1	1
$a_1 < a_2 < b_2 < b_1$	1	1	1
$a_1 < b_2 < a_2 < b_1$	0	1	1
$a_2 < a_1 < b_1 < b_2$	1	1	1
$a_2 < a_1 < b_2 < b_1$	1	1	1
$a_2 < b_1 < a_1 < b_2$	1	0	1
$a_2 < b_1 < b_2 < a_1$	1	0	1
$a_2 < b_2 < a_1 < b_1$	1	1	1
$a_2 < b_2 < b_1 < a_1$	1	0	1
$b_2 < a_1 < a_2 < b_1$	0	1	1
$b_2 < a_2 < a_1 < b_1$	0	1	1
$b_2 < a_2 < b_1 < a_1$	0	0	1

Table

In the following sections of the paper, we will describe the FKG inequality and its recent numerous generalizations, and illustrate its use in connection with problems involving linear extensions and order preserving maps of partial orders.

THE FKG INEQUALITY AND ITS GENERALIZATIONS

In some sense the FKG inequality has its roots in the old result of Chebyshev which asserts that if f and g are both increasing (or both decreasing) functions on $[0,1]$ then the average value of the product fg is at least as large as the product of the average values of f and g (where the average is taken with respect to some measure μ on $[0,1]$). In symbols,

$$\int_0^1 fg\,d\mu \geq \int_0^1 f\,d\mu \int_0^1 g\,d\mu. \tag{2}$$

In the case that μ is a discrete measure we can restate (2) as follows: If $f(k)$ and $g(k)$ are both increasing (or both decreasing) and $\mu(k) \geq 0$, $k = 1,2,3,\ldots$ then

$$\frac{\sum_k f(k)g(k)\mu(k)}{\sum_k \mu(k)} \geq \frac{\sum_k f(k)\mu(k)}{\sum_k \mu(k)} \cdot \frac{\sum_k g(k)\mu(k)}{\sum_k \mu(k)}$$

i.e.,

$$\sum_k f(k)g(k)\mu(k) \sum_k \mu(k) \geq \sum_k f(k)\mu(k) \sum_k g(k)\mu(k). \qquad (2')$$

The proof of (2') (and (2)) follows once by expanding the inequality

$$\sum_{i,j} (f(i)-f(j))(g(i)-g(j))\mu(i)\mu(j) \geq 0.$$

The FKG inequality, due to Fortuin, Kasteleyn and Ginibre [FKG], was discovered in connection with the proof of certain natural conjectures arising in statistical mechanics (in particular, dealing with correlations of Ising spin systems). They trace rudimentary forms of the inequality back to Griffiths [G] (who was also interested in these questions) and Harris [Ha] (who was investigating the probabilities of certain events in percolation models). It turns out that special cases were also anticipated by Kleitman [K1] (see [K2]) and Marica-Schönheim [MS], among others.

Basically the FKG inequality is an attempt to extend (2') to the case where the underlying set is only *partially ordered*, as opposed to the *totally* ordered index set of integers occurring in (2'). The setting is as follows. Let $(\Gamma,<)$ be a distributive lattice. That is, Γ is a (finite) set, partially ordered by $<$, on which two commutative functions $\wedge$ (greatest lower bound) and $\vee$ (least upper bound) are defined which satisfy the distributive laws:

$$\left.\begin{aligned} x \wedge (y \vee z) &= (x\wedge y) \vee (x\wedge z) \\ x \vee (y\wedge z) &= (x\vee y) \wedge (x\vee z) \end{aligned}\right\} \quad \text{for all } x, y, z \in \Gamma.$$

It is well known that without loss of generality we may assume Γ is a sublattice of the lattice of all subsets of some finite set partially ordered under inclusion with $x \wedge y = x \cap y$ and $x \vee y = x \cup y$.

Let $\mu: \Gamma \to \mathbb{R}_0$, the nonnegative reals, satisfy

$$\mu(x)\mu(y) \leq \mu(x\vee y)\mu(x\wedge y) \text{ for all } x, y \in \Gamma. \tag{3}$$

We call a function $f: \Gamma \to \mathbb{R}$ *increasing* if

$$x \leq y \Rightarrow f(x) \leq f(y) \text{ for } x, y \in \Gamma,$$

(with *decreasing* defined similarly).

The FKG inequality: If f and g are both increasing (or both decreasing) functions on a distributive lattice Γ and $\mu: \Gamma \to \mathbb{R}_0$ satisfies (3) then

$$\sum_{x\in\Gamma} f(x)g(x)\mu(x) \sum_{x\in\Gamma} \mu(x) \geq \sum_{x\in\Gamma} f(x)\mu(x) \sum_{x\in\Gamma} g(x)\mu(x). \tag{4}$$

The original proof [FKG] of (4) was not so simple. Several years after (4) appeared, Holley [Ho] found the following generalization of FKG:

Suppose $\alpha, \beta: \Gamma \to \mathbb{R}_0$ satisfy

$$\alpha(x)\beta(y) \leq \alpha(x\vee y)\beta(x\wedge y) \text{ for all } x, y \in \Gamma.$$

Then for any increasing function $\theta: \Gamma \to \mathbb{R}_0$

$$\sum_x \alpha(x)\theta(x) \geq \sum_x \beta(x)\theta(x). \tag{5}$$

Rather recently, Ahlswede and Daykin have given a remarkable strengthening of (4) and (5).

THEOREM [AD2]. *Suppose we are given four functions* $\alpha, \beta, \gamma, \delta: \Gamma \to \mathbb{R}_0$ *which satisfy*

$$\alpha(x)\beta(y) \leq \gamma(x\vee y)\delta(x\wedge y) \text{ for all } x, y \in \Gamma. \tag{6}$$

Then

$$\alpha(X)\beta(Y) \le \gamma(X\vee Y)\delta(X\wedge Y) \text{ for all } X,\ Y \subseteq \Gamma \tag{7}$$

where

$$X \vee Y = \{x\vee y : x\varepsilon X, y\varepsilon Y\},$$

$$X \wedge Y = \{x\wedge y : x\varepsilon X, y\varepsilon Y\} \text{ and}$$

$$\text{for } Z \subseteq \Gamma,\ f(Z) \equiv \sum_{z\varepsilon Z} f(z).$$

The proof of (7) is surprisingly simple. To begin with, it follows from our previous observations that it is enough to prove the following result (where $2^{[N]}$ denotes the family of all subsets of $[N] = \{1,2,\ldots,N\}$): Suppose $\alpha,\beta,\gamma,\delta : 2^{[N]} \to \mathbb{R}_0$ satisfy

$$\alpha(x)\beta(y) \le \gamma(x\cup y)\delta(x\cap y) \text{ for all } x,\ y \ \varepsilon\ 2^{[N]}. \tag{6'}$$

Then

$$\alpha(X)\beta(Y) \le \gamma(X\vee Y)\delta(X\wedge Y) \text{ for all } X,\ Y \subseteq 2^{[N]}. \tag{7'}$$

PROOF: The proof is by induction on N. We first consider the case N = 1. Then $2^{[1]} = \{\phi,\{1\}\}$. Denote $\alpha(\phi)$ by α_0 and $\alpha(\{1\})$ by α_1, with β_0, β_1, etc., defined similarly. The system (6') becomes:

$$\begin{aligned} \alpha_0\beta_0 &\le \gamma_0\delta_0, \\ \alpha_1\beta_0 &\le \gamma_1\delta_0, \\ \alpha_0\beta_1 &\le \gamma_1\delta_0, \\ \alpha_1\beta_1 &\le \gamma_1\delta_1, \end{aligned} \tag{8}$$

It is easily checked that if either X or Y consists of a single element then (7') is an immediate consequence of (8). The only interesting case is $X = \{\phi,\{1\}\} = Y$. In this case, the inequality (7') we are required to prove is

$$(\alpha_0+\alpha_1)(\beta_0+\beta_1) \le (\gamma_0+\gamma_1)(\delta_0+\delta_1). \tag{9}$$

Equation (9) would follow instantly from (8) if one of the two occurrences of $\gamma_1\delta_0$ in (8) were $\gamma_0\delta_1$ instead. As it is, we have to work a little (but not much) harder. If any of α_0, β_0, γ_0 or δ_0 is zero, (9) follows at once. Hence, by a suitable normalization, we can assume

$$\alpha_0 = \beta_0 = \gamma_0 = \delta_0 = 1.$$

The system (8) becomes

$$\alpha_1 \le \gamma_1,\ \beta_1 \le \gamma_1,\ \alpha_1\beta_1 \le \gamma_1\delta_1 \tag{8'}$$

and (9) becomes

$$(1+\alpha_1)(1+\beta_1) \le (1+\gamma_1)(1+\delta_1). \tag{9'}$$

Again (9') is immediate if $\gamma_1 = 0$ so we may assume $\gamma_1 > 0$. Since (9') becomes harder to satisfy as δ_1 decreases, it is enough to prove (9') when δ_1 is as small as possible, i.e., $\delta_1 = \frac{\alpha_1\beta_1}{\gamma_1}$. In this case we need

$$(1+\alpha_1)(1+\beta_1) \le (1+\gamma_1)\left(1 + \frac{\alpha_1\beta_1}{\gamma_1}\right),$$

i.e.,

$$\alpha_1 + \beta_1 \le \gamma_1 + \frac{\alpha_1\beta_1}{\gamma_1}.$$

However, this is an immediate consequence of

$$(\gamma_1-\alpha_1)(\gamma_1-\beta_1) \ge 0$$

which is implied by (8'). This proves the result for N = 1.

Assume now that the assertion holds for $N = n - 1$ for some $n \geq 2$. Let $\alpha,\beta,\gamma,\delta : 2^{[n]} \to \mathbb{R}_0$ satisfy the hypotheses (6') with $N = n$ and let $X, Y \subseteq 2^{[n]}$ be given. We will define new functions α', β', γ', δ' mapping $2^{[n-1]} = T'$ into $\mathbb{R}_0$ as follows:

$$\alpha'(x') = \sum_{\substack{x\in X \\ x'=x\setminus\{n\}}} \alpha(x), \quad \beta'(y') = \sum_{\substack{y\in Y \\ y'=y\setminus\{n\}}} \beta(y)$$

$$\gamma'(z') = \sum_{\substack{z\in X\cup Y \\ z'=z\setminus\{n\}}} \gamma(z), \quad \delta'(w') = \sum_{\substack{w\in X\cap Y \\ w'=w\setminus\{n\}}} \delta(w).$$

Thus, for $x' \in T'$

$$\alpha'(x') = \begin{cases} \alpha(x') + \alpha(x'\cup\{n\}) & \text{if } x' \in X,\ x' \cup \{n\} \in X, \\ \alpha(x') & \text{if } x' \in X,\ x' \cup \{n\} \notin X, \\ \alpha(x'\cup\{n\}) & \text{if } x' \notin X,\ x' \cup \{n\} \in X, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Observe that with these definitions

$$\alpha(X) = \sum_{x\in X}\alpha(x) = \sum_{x'\in T'} \alpha'(x') = \alpha'(T')$$

and

$$\beta(Y) = \beta'(T'), \quad \gamma(X\vee Y) = \gamma'(T'),$$

$$\delta(X\wedge Y) = \delta'(T').$$

Thus, if

$$\alpha'(x')\beta'(y') \leq \gamma'(x'\cup y')\delta'(x'\cap y') \text{ for all } x', y' \in T' \quad (11)$$

holds, then by the induction hypotheses

$$\alpha(X)\beta(Y) = \alpha'(T')\beta'(T') \leq \gamma'(T')\delta'(T') = \gamma(X\vee Y)\delta(X\wedge Y)$$

since $T' \vee T' = T'$, $T' \wedge T' = T'$, which is (7').

It remains to prove (11). However, by (10) this is *exactly* the computation performed for the case $N = 1$ with $x' \leftrightarrow \phi$ and $x' \cup \{n\} \leftrightarrow \{1\}$. Since we have already treated this case then the induction step is completed. This proves the theorem of Ahlswede and Daykin. □

To prove the FKG inequality from Ahlswede-Daykin it suffices as usual to restrict ourselves to the case that $\Gamma = 2^{[N]} = T$. Observe that if A and B are upper ideals in T (i.e., $x, y \in A \Rightarrow x \cup y \in A$) then the indicator functions $f = I_A$ and $g = I_B$ (with $I_A(x) = 1$ iff $x \in A$) are increasing. Taking $\alpha = \beta = \gamma = \delta = \mu$ in (6') and $X = A$, $Y = B$ in (7') we have

$$\mu(A)\mu(B) \leq \mu(A\vee B)\mu(A\wedge B). \quad (12)$$

But

$$\mu(A) = \sum_{x\in A} \mu(x) = \sum_{x\in T} f(x)\mu(x),$$

$$\mu(B) = \sum_{x\in T} g(x)\mu(x),$$

$$\mu(A\wedge B) = \sum_{z\in A\wedge B} \mu(z) = \sum_{z\in T} f(z)g(z)\mu(z)$$

and

$$\mu(A\vee B) = \sum_{z\in A\vee B} \mu(z) \leq \sum_{x\in T} \mu(z).$$

Thus, (12) implies

$$\sum_z f(z)g(z)\mu(z)\sum_z \mu(z) \geq \sum_z f(z)\mu(z)\sum_z g(z)\mu(z)$$

which is just the FKG inequality for this case. The general FKG inequality is proved in just this way by first writing an arbitrary increasing function f on T as $f = \sum_i \lambda_i I_{A_i}$ where $\lambda_i \geq 0$ and A_i are suitable upper ideals in T. That is, for

$$f = \sum_i \lambda_i I_{A_i}, \ \lambda_i \geq 0,$$

we have

$$\sum_{z\varepsilon T} f(z)\mu(z) = \sum_{z\varepsilon T}\sum_i \lambda_i I_{A_i}(z)\mu(z)$$

$$= \sum_i \lambda_i \sum_{z\varepsilon T} I_{A_i}(z)\mu(z), \text{ etc.},$$

and now apply the preceding inequality.

Before concluding this section, we make several remarks concerning the Ahlswede-Daykin result. Setting $\alpha = \beta = \gamma = \delta = 1$ we obtain

$$|X||Y| \leq |X\vee Y||X\wedge Y| \text{ for all } X, Y \subseteq 2^{[N]}. \qquad (13)$$

This was first proved by Daykin [D] and has as immediate corollaries:

(a) *(Marica-Schönheim [MS])*

$$|A| \leq |A \backslash A| \text{ for all } A \subseteq 2^{[N]}.$$

(b) *(Kleitman [K1]). For any upper ideal* U *and any lower ideal* L *of* $2^{[N]}$,

$$|U \cap L| \cdot 2^N \leq |U||L|.$$

(c) *(Seymour [Se]). For any two upper ideals* U_1, U_2 *of* $2^{[N]}$,

$$|U_1||U_2| \leq |U_1 \cap U_2| \cdot 2^N.$$

The special cases (a) and (b) actually appeared in the literature before the FKG inequality was found. We leave as an exercise for the reader to show that the starting inequality (2') also follows from this.

A good summary of these and related results can be found in [AD1], [D1], [D2]. Far ranging generalizations of the Ahlswede-Daykin Theorem were obtained by the same authors and appear in [AD2].

LINEAR EXTENSIONS OF TWO CHAINS

Suppose $(X,<)$ is a partially ordered set in which $X = A \cup B$ is a disjoint union of two chains $A = \{a_1 < \ldots < a_m\}$ and $B = \{b_1 < \ldots < b_n\}$. Of course, relations such as $a_i < b_j$ and $b_k < a_\ell$ are also allowed. Consider the set Λ of all $(m+n)!$ 1 - 1 mappings of X onto $[m+n] = \{1,2,\ldots,m+n\}$. Assign a probability $\frac{1}{(m+n)!}$ to each $\lambda \in \Lambda$, i.e., assume they are all equally likely. Let Q denote the set of all *linear extensions* λ of X, i.e., $\lambda \in \Lambda$ such that

$$x < y \Rightarrow \lambda(x) < \lambda(y).$$

Let P and P' both be unions of subsets of Λ of the form

$$\{\lambda:\lambda(a_{i_1})<\lambda(b_{j_1}),\lambda(a_{i_2})<\lambda(b_{j_2}),\ldots\}$$

(i.e., a's always lose to b's).

THEOREM 1 (Graham, Yao, Yao [GYY]).

$$\Pr(P|Q\cap P') \geq \Pr(P|Q). \tag{14}$$

PROOF. We will give a proof of this result (due to Shepp [Sh]) based on the FKG inequality. The original proof used explicit combinatorial pairings of certain types of mappings in Λ.

Define a lattice Γ with elements of the form $\bar{x} = \{x_1,\ldots,x_m\}$ with $1 \leq x_1 < \ldots < x_m \leq m+n$. We say that $\bar{x} \leq \bar{x}'$ if $x_i \leq x_i'$ for $1 \leq i \leq m$. Define

$$\bar{x} \vee \bar{x}' = \{\ldots,\max(x_i,x_i'),\ldots\},$$

$$\bar{x} \wedge \bar{x}' = \{\ldots,\min(x_i,x_i'),\ldots\}.$$

It is easily checked that with these definitions Γ is a distributive lattice.

For each $\bar{x} \in \Gamma$ we can associate a unique $\lambda_{\bar{x}} \in \Lambda$ by setting:

$$\lambda_{\bar{x}}(a_i) = x_i,$$

$$\lambda_{\bar{x}}(b_j) = y_j$$

where $[m+n]\backslash\{x_1<\ldots<x_m\} = \{y_1<\ldots<y_n\}$. Finally, define

$$\mu(\bar{x}) = \begin{cases} 1 \text{ if } \lambda_{\bar{x}} \in Q, \\ 0 \text{ otherwise,} \end{cases}$$

$$f(\bar{x}) = \begin{cases} 1 \text{ if } \lambda_{\bar{x}} \in P, \\ 0 \text{ otherwise,} \end{cases}$$

$$f'(\bar{x}) = \begin{cases} 1 \text{ if } \lambda_{\bar{x}} \in P', \\ 0 \text{ otherwise.} \end{cases}$$

To apply FKG, we must check:

$$\mu(\bar{x})\mu(\bar{x}') \leq \mu(\bar{x}\vee\bar{x}')\mu(\bar{x}\wedge\bar{x}'). \tag{15}$$

Suppose $\mu(\bar{x})\mu(\bar{x}') = 1$. Then $\lambda_{\bar{x}} \in Q$, $\lambda_{\bar{x}'} \in Q$. If $a_i < a_j$ in X then

$$\lambda_{\bar{x}}(a_i) = x_i < x_j = \lambda_{\bar{x}}(a_j)$$
$$\lambda_{\bar{x}'}(a_i) = x_i' < x_j' = \lambda_{\bar{x}'}(a_j)$$

and so,

$$\lambda_{\bar{x}\vee\bar{x}'}(a_i) = \max(x_i, x_i') < \max(x_j, x_j') = \lambda_{\bar{x}\vee\bar{x}'}(a_j).$$

Similarly, if $b_i < b_j$ in X then

$$\lambda_{\bar{x}\vee\bar{x}'}(b_i) < \lambda_{\bar{x}\vee\bar{x}'}(b_j).$$

On the other hand, if $a_i < b_j$ in X then

$$\lambda_{\bar{x}}(a_i) = x_i < y_j = \lambda_{\bar{x}}(b_j)$$

$$\lambda_{\bar{x}'}(a_i) = x_i' < y_j' = \lambda_{\bar{x}'}(b_j)$$

i.e.,

$$x_i \le i + j - 1,\ x_i' \le i + j - 1.$$

Thus,

$$\lambda_{\bar{x}\vee\bar{x}'}(a_i) = \max(x_i, x_i') \le i + j - 1$$

so that

$$\lambda_{\bar{x}\vee\bar{x}'}(a_i) < \lambda_{\bar{x}\vee\bar{x}'}(b_j).$$

The argument for $b_i < a_j$ is similar. This shows that $\lambda_{\bar{x}\vee\bar{x}'} \in Q$, i.e., $\mu(\bar{x}\vee\bar{x}') = 1$. In this same way it follows that $\mu(\bar{x}\wedge\bar{x}') = 1$. Therefore, we have shown that

$$\mu(\bar{x})\mu(\bar{x}') = 1 \Rightarrow \mu(\bar{x}\vee\bar{x}')\mu(\bar{x}\wedge\bar{x}') = 1$$

and consequently (15) always holds.

The final condition to check before applying FKG is that f and f' are decreasing. To see this, suppose $\bar{x} < \bar{x}'$ and $f(\bar{x}') = 1$. Then by definition, $\lambda_{\bar{x}'} \in P = \bigcup_k P_k$ where

$$P_k = \{\lambda : \lambda(a_{i_1}) < \lambda(b_{j_1}), \dots\}.$$

Thus, for some k, the elements $\bar{x}_i'$ of $\bar{x}'$ satisfy all the constraints $x_{i_1}' \le i_1 + j_1 - 1, \dots,$ imposed by P_k. But since $\bar{x} \le \bar{x}'$, i.e., $x_i \le x_i'$ for all i, then $x_{i_1} \le x_{i_1}' \le i_1 + j_1 - 1, \dots,$ as well. This implies that $\lambda_{\bar{x}} \in P_k \subseteq P$ and $f(x) = 1$, i.e., f is decreasing.

The FKG inequality can now be applied to the functions we have defined, yielding:

$$\sum_{\bar{x}\in\Gamma} f(\bar{x})f'(\bar{x})\mu(\bar{x}) \sum_{\bar{x}\in\Gamma} \mu(\bar{x}) \ge \sum_{\bar{x}\in\Gamma} f(\bar{x})\mu(\bar{x}) \sum_{\bar{x}\in\Gamma} f'(\bar{x})\mu(\bar{x}) \qquad (16)$$

Interpreting (15) in terms of Q, P and P', we obtain

$$|P\cap P'\cap Q|\,|Q| \geq |P\cap Q|\,|P'\cap Q| \tag{17}$$

i.e.,

$$\Pr(P\cap P'|Q) \geq \Pr(P|Q)\Pr(P'|Q)$$

which implies (14). □

As the example at the beginning of the paper shows, (14) does not necessarily hold if A and B are not linearly ordered. A somewhat different example [GYY] showing this is the following

Example:

$$A = \{a_1<a_2\},\ B = \{b_1<b_3, b_2<b_3\}.$$

In addition, in $X = A \cup B$ we have $b_1 < a_2$ and $a_1 < b_2$. Let P denote the event $\{a_1<b_1\}$ and P' denote the event $\{a_2<b_3\}$. An easy calculation shows

$$\Pr(P|P'\cap Q) = \frac{3}{5} < \frac{5}{8} = \Pr(P|Q).$$

ORDER PRESERVING MAPS

In this section we show how the FKG inequality theorem can be used to prove a broader class of similar results for the class of order preserving maps on a partially ordered set. We say that a mapping $\rho:X \to X'$ between partially ordered sets $(X,<)$ and $(X',<)$ is *order preserving* if $x, y \in X$ with $x < y \Rightarrow \rho(x) < \rho(y)$. Note that ρ is not required to be 1 - 1. (The use of the same symbol < for both partial orders should cause no confusion). Let $R = R(X,X')$ denote the set of order preserving maps of X into X'. As before, we will assume that all $\rho \in R$ are equally likely.

We need one further definition. For $x \in X$ define *range*(x) to be $\{\rho(x):\rho\in R\}$.

THEOREM 2. Suppose X is the disjoint union of $A = \{a_1,\ldots,a_m\}$ *and* $B = \{b_1,\ldots,b_n\}$. *Let P and P' both be unions of subsets of* *the form* $\{\rho:X\to X':\rho(a_i)<\rho(b_j),\rho(a_k)<\rho(b_\ell),\ldots\}$ *(i.e., certain* a's *are always mapped below certain* b's*).*

Suppose for all $a \in A$, $b \in A$ *which are related by* $<$,

$$\text{range}(a) \cap \text{range}(b) = \emptyset. \tag{18}$$

Then

$$\Pr(P\cap P'|R) \geq \Pr(P|R)\Pr(P'|R). \tag{19}$$

PROOF: Define a lattice $(\Gamma,<)$ by taking the points of Γ to be sequences $\bar{z} = (x_1,\ldots,x_m;y_1,\ldots,y_n)$ where x_i, $y_j \in X'$ and $\bar{z} \leq \bar{z}'$ means $x_i \leq x_i'$ and $y_j \geq y_j'$ for all i, j. Further, define

$$\bar{z} \vee \bar{z}' = (\ldots,\max(x_i,x_i'),\ldots;\ldots,\min(y_j,y_j'),\ldots)$$

$$\bar{z} \wedge \bar{z}' = (\ldots,\min(x_i,x_i'),\ldots;\ldots,\max(y_j,y_j'),\ldots).$$

It is easy to check that with these operations, Γ is distributive.

To each $\bar{z} \in \Gamma$, associate a map $\rho_{\bar{z}}:X \to X'$ by defining

$$\rho_{\bar{z}}(a_i) = x_i,\ \rho_{\bar{z}}(b_j) = y_j.$$

Finally, define the three functions:

$$\mu(\bar{z}) = \begin{cases} 1 \text{ if } \rho_{\bar{z}} \in R, \\ 0 \text{ otherwise,} \end{cases}$$

$$f(\bar{z}) = \begin{cases} 1 \text{ if } \rho_{\bar{z}} \in P, \\ 0 \text{ otherwise,} \end{cases}$$

$$f'(\bar{z}) = \begin{cases} 1 \text{ if } \rho_{\bar{z}} \in P', \\ 0 \text{ otherwise.} \end{cases}$$

In order to apply the FKG inequality, we must check that f and f' are both decreasing (this is basically the same argument as before) and that μ satisfies the "log supermodularity" condition

$$\mu(\bar{z})\mu(\bar{z}') \leq \mu(\bar{z} \vee \bar{z}')\mu(\bar{z} \wedge \bar{z}') \text{ for all } \bar{z}, \bar{z}' \in \Gamma.$$

Again the argument is quite close to the previous one with one exception. In showing that $\mu(\bar{z})\mu(\bar{z}') = 1 \Rightarrow \mu(\bar{z} \vee \bar{z}')\mu(\bar{z} \wedge \bar{z}') = 1$ we must know (for example) that

$$\min(x_i, x_i') > \max(y_j, y_j')$$

if $x_i < y_j$ and $x_i' < y_j'$. While this is not ordinarily always the case, with the assumption (18) in force, it does indeed hold. Using this observation several times, the verification that μ, f and f' satisfy the hypotheses of FKG is accomplished. The conclusion of FKG now implies (similar to (16) ⇒ (17))

$$|P \cap P' \cap R|\,|R| \geq |P \cap R|\,|P' \cap R|$$

which in turn implies (19). □

Whether or not the "range" condition (18) is needed is not currently known.

LINEAR EXTENSIONS OF PAIRS OF GENERAL PARTIAL ORDERS

It is natural to try and extend Theorem 1 to linear extensions of more general partial orders. However, numerous examples like those previously presented show that *some* restrictions on the partial order will be necessary. One such extension, conjectured by Graham, Yao and Yao [GYY], and proved by Shepp [Sh], is the following.

THEOREM 3. Suppose $(X,<)$ *is a partial order where* X *is a disjoint union of two partial orders* $A = \{a_1,\dots,a_m\}$ *and* $B = \{b_1,\dots,b_n\}$, *i.e.,* a_i *and* b_j *are all pairwise incomparable. Let* R *denote the set of linear extensions of* X *onto* $[m+n]$ *and let* P *and* P' *be sets of maps* λ *of* X *onto* $[m+n]$ *which are unions of sets of the form*

$$\{\lambda:\lambda(a_{i_1})<\lambda(b_{j_1}),\lambda(a_{i_2})<\lambda(b_{j_2}),\dots\}.$$

If all elements of R *are given equal probability then* P *and* P' *are positively correlated, i.e.,*

$$\Pr(P|P'\cap R) \geq \Pr(P|R). \tag{20}$$

PROOF: The proof is similar to that of Theorem 2 but with an extra complication. To begin with, we choose a large fixed integer N and define $\Gamma = \Gamma_N$ to be the set of all sequences $\bar{z} = (x_1,\dots,x_m;y_1,\dots,y_n)$ partially ordered by setting $\bar{z} \leq \bar{z}'$ iff $x_i \leq x_i'$ and $y_j \geq y_j'$ for all i and j where $1 \leq x_i$, $y_j \leq N$. As before define

$$\bar{z} \wedge \bar{z}' = (\dots,\min(x_i,x_i'),\dots;\dots,\max(y_j,y_j'),\dots)$$

$$\bar{z} \vee \bar{z}' = (\dots,\max(x_i,x_i'),\dots;\dots,\min(y_j,y_j'),\dots).$$

This makes Γ into a distributive lattice. For $\bar{z} \in \Gamma$, define a mapping $\rho_{\bar{z}}:\Gamma \to [N]$ by $\rho_{\bar{z}}(a_i) = x_i$, $\rho_{\bar{z}}(b_j) = y_j$. We will assume all N^{m+n} such mappings are equally likely.

Let $\bar{R}$, $\bar{P}$, $\bar{P}'$ be the subsets of mappings $\rho_{\bar{z}}$, $\bar{z} \in \Gamma$, which preserve the order given by R, P and P', respectively, i.e., $\lambda(x) < \lambda(y) \Rightarrow \rho_{\bar{z}}(x) < \rho_{\bar{z}}(y)$, with μ, f and f' defined to be the corresponding characteristic functions.

It is straightforward to check that the hypotheses of the FKG inequality are satisfied. The potential difficulty for μ which required the range disjointness condition (18) does not occur here since A and B are incomparable by hypothesis.

Thus, by the FKG inequality we obtain

$$|\bar{P}\cap\bar{P}'\cap\bar{R}|\,|\bar{R}| \geq |\bar{P}\cap\bar{R}|\,|\bar{P}'\cap\bar{R}|. \tag{21}$$

Define $\hat{R}$ to be the subset of R in which all components of $\bar{z} \in \bar{R}'$ are *distinct* (with $\hat{P}$ and $\hat{P}'$ defined similarly). Since

$$|\hat{R}| = \frac{N!}{(N-m-n)!}$$

then

$$\lim_{N\to\infty} \frac{|\hat{R}|}{|\bar{R}|} = 1.$$

Each $\rho_{\bar{z}}$, $\bar{z} \in \hat{R}$, determines a *linear extension* λ of $X \to [m+n]$ in the obvious way, with the relative ordering of the $\rho_{\bar{z}}(a_i)$ and $\rho_{\bar{z}}(b_j)$ determining their ranks in $[m+n]$. Furthermore, for $N \geq m + n$, each linear extension $\lambda: X \to [m+n]$ is generated exactly the same number of times by the $\rho_{\bar{z}}$, $\bar{z} \in \hat{R}$. Of course, the preceding comments also apply to $\hat{P}$ and $\hat{P}'$. Thus,

$$\Pr(P|R) = \lim_{N\to\infty} \frac{|\hat{P}\cap\hat{R}|}{|\hat{R}|} = \lim_{N\to\infty} \frac{|\bar{P}\cap\bar{R}|}{|\bar{R}|}$$

and

$$\Pr(P\cap P'\cap R) = \lim_{N\to\infty} \frac{|\bar{P}\cap\bar{P}'\cap\bar{R}|}{|\bar{R}|}.$$

The desired conclusion (20) now follows and the theorem is proved. □

CONCLUDING REMARKS

We close the paper with a discussion of a number of questions dealing with linear extensions of partial orders and potential applications of the FKG inequality.

1. It would be quite interesting to know under exactly what conditions two events (= subsets of linear extensions of a partially ordered set) are mutually favorable. Theorems 1

and 3 give some conditions under which this occurs but they certainly do not tell the whole story. Is it true, for example, that Theorem 2 holds for linear extensions rather than order preserving maps?

2. Recently, Stanley [St2] proved the following result (a strengthening of a conjecture of Rivest [R] which was conjectured by Chung, Fishburn and Graham [CFG]). Let X be a partially ordered set with n elements and let $x \in X$ be a fixed element. Let N_i denote the number of linear extensions λ of X onto [n] for which $\lambda(x) = i$.

THEOREM. The sequence N_i *is logarithmically concave.*

This result had been established earlier in [CFG] for the case that X can be covered by two chains (= linearly ordered sets). Stanley's proof required the use of the so-called Aleksandrov-Fenchel inequalities from the theory of mixed volumes (see [B] or [Fe]) similar to those for mixed discriminants which have recently been used by Egoritsjev [E], [vL] to prove the infamous van der Waerden permanent conjecture. It is known that the FKG inequality can be used very naturally to prove the log concavity of various sequences of combinatorial interest (e.g., see [SW]). Can Stanley's result be proved using the FKG inequality (or the Ahlswede-Daykin theorem?) Is the analog of Stanley's theorem for *order preserving* maps true as well?

3. Fishburn [Fi1], [Fi2] has recently studied the following problem. If x and y are two elements of a partially ordered set X on n-elements, let us say that x *dominates* y (written $x \to y$) if more linear extensions $\lambda : X \to [n]$ have $\lambda(x) > \lambda(y)$ than $\lambda(y) > \lambda(x)$. He has shown that there are partial orders X for which the cycle $x \to y \to z \to x$ is possible (the smallest such X known has 31 elements). More generally he has constructed partial orders X for which every vertex of this (directed) domination graph D(X) has outdegree at least one. What is the least X such that D(X) has a cycle? Is it possible to characterize those directed graphs which occur as D(X) for some X? What if we require

$$|\{\lambda : \lambda(x) > \lambda(y)\}| > \alpha |\{\lambda : \lambda(y) > \lambda(x)\}|$$

for some (fixed) $\alpha > 1$? What is the largest α for which the corresponding domination graph can have a cycle?

4. The following nice problem of Stanley (see [St1]) is still open. Let X be a partially ordered set on n elements and let $(a_1, a_2, \ldots)$ be a sequence of nonnegative integers satisfying $\sum_k a_k = n$. Define $f_X(a_1, a_2, a_3, \ldots)$ to be the number of order preserving maps $\rho: X \to \{1,2,3,\ldots\}$ such that

$$|x \in X: \rho(x) = i| = a_i.$$

Is it possible for two different partially ordered sets X and X' to have

$$f_X(a_1, a_2, a_3, \ldots) = f_{X'}(a_1, a_2, a_3, \ldots)$$

for all $(a_1, a_2, a_3, \ldots)$? Stanley has shown that if such X, X' exist then $n \geq 7$.

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