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ON GRAPHS WHICH CONTAIN ALL SPARSE GRAPHS

L. BABAI
Department of Algebra, Eötvös University, Budapest, Hungary

F. R. K. CHUNG
Bell Laboratories, Murray Hill, NJ 07974, USA

P. ERDŐS
Mathematical Institute, Hungarian Academy of Sciences, Budapest, Hungary

R. L. GRAHAM
Bell Laboratories, Murray Hill, NJ 07974, USA

J. H. SPENCER
State University of New York, Stony Brook, NY 11794, USA

Dedicated to Professor A. Kotzig on the occasion of his sixtieth birthday

1. Introduction

Let $\mathscr{H}_n$ denote the class of all graphs with n edges and denote by $s(n)$ the minimum number of edges a graph G can have which contains all $H \in \mathscr{H}_n$ as subgraphs. In this paper we establish the following bounds on $s(n)$:

Theorem 1.

$$\frac{cn^2}{\log^2 n} < s(n) < \frac{c'n^2 \log \log n}{\log n}$$

for n sufficiently large and c, c' some constants.

We also consider the problem of determining the minimum number of edges, denoted by $s'(n)$, a graph can have which contains every planar graph on n edges as a subgraph. We prove:

Theorem 2. $s'(n) < cn^{3/2}$.

In [1, 2, 3], two of the authors investigated the problem of determining the minimum number of edges a graph or a tree could have which contains all trees on n edges as subgraphs. For a brief survey on these 'universal' graphs the reader is referred to [4].

2. A lower bound for $s(n)$

Let G be a graph which contains all graphs on n edges. Suppose G has t edges. Thus G contains at most $\binom{t}{n}$ different subgraphs on n edges.

On the other hand, G contains all graphs on n edges and $\lfloor n/\log n \rfloor$ vertices where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. There are at least

$$\left(\binom{\lfloor n/\log n\rfloor}{2} \atop n\right) \cdot \frac{1}{\lfloor n/\log n\rfloor !}$$

different graphs with n edges and $\lfloor n/\log n \rfloor$ vertices (see [5]). Therefore we have

$$\binom{t}{n} \geqslant \left(\binom{\lfloor n/\log n\rfloor}{2} \atop n\right) \cdot \frac{1}{\lfloor n/\log n\rfloor !}.$$

By a straightforward calculation, this implies

$$t \geqslant cn^2/\log^2 n$$

for some constant c.

Hence we have shown $s(n) > cn^2/\log^2 n$.

3. An upper bound for $s(n)$

We will prove (by the probability method) that there exists a graph with $cn^2 \log\log n/\log n$ edges[1] that contains all graphs with at most n edges. The existence of such a graph will follow from the following sequence of observations.

Claim 1. *Given positive integers a and b where $a < b < a \log a$ and $\log\log\log a \geqslant 1$, there is a bipartite graph H with vertex set $A \cup B$ where $|A| = a$ and $|B| = b$, which satisfies the following conditions:*

(i) *H has no more than abp edges where $p = \log\log a/\log a$;*

(ii) *For any k disjoint subsets of B, say, $S_1, \ldots, S_k$, each with cardinality at most p^{-1}, and $2kp^{-2} < a$, we have*

$$\left|\bigcup_{i=1}^{k} N(S_i)\right| \geqslant kp^{-2}$$

where

$$N(S_i) \equiv \{v \in A : v \text{ is adjacent to all vertices in } S_i\}.$$

Proof. We consider the set of all bipartite graphs on a and b vertices with abp edges. For a set $S_i \subset B$, $|s_i| < d = p^{-1}$, the probability of a vertex v in A being in $N(S_i)$, is at least p^d. Therefore the probability of v not being in any $N(S_i)$ is at most $(1-p^d)^k$. The

[1] Strictly speaking, we should use $3n[\log\log n/\log n]$ or $[3n \log\log n/\log n]$ since $|A|$ is an integer. However, we will usually not bother with this type of detail since it has no significant effect on the arguments or results.

probability that there are $a-kd^2$ vertices in A not in any $N(S_i)$ is at most

$$\binom{a}{kd^2}(1-p^d)^{k(a-kd^2)} \leqslant 2^a e^{-p^d ka/2}.$$

Since there are at most b^{dk} choices for S_i, $1 \leqslant i \leqslant k$, the probability for a bipartite graph to be 'bad' is at most

$$b^{dk} \cdot 2^a e^{-p^d ka/2} < (a \log a)^{p^{-1}k} \cdot 2^a e^{-p^d ka/2}$$
$$< (a \log a)^{a \log\log a/\log a} 2^a e^{-a^2 p^{d-2}/4} < 1'$$

Therefore the required bipartite graph exists as claimed.

Claim 2. *Given positive integers a and b where $a<b<a \log a$ and $\log\log\log a \geqslant 1$, there is a bipartite graph H with vertex set $A \cup B$ where $|A|=a$ and $|B|=b$ satisfying the following conditions:*

(i) *H has no more than abp edges where $p=\log\log a/\log a$.*

(ii) *Let H' be a bipartite graph with vertex set $X \cup Y$ where $|X| \leqslant \frac{1}{2}a$, $|Y|=b$ and maximum degree p^{-1}. Then H' can be embedded in H in the strong sense, i.e. any one-to-one map $\lambda: Y \to B$ can be extended to $\bar{\lambda}: X \cup Y \to A \cup B$ such that $\bar{\lambda}(u)$ is adjacent to $\bar{\lambda}(v)$ in H if u is adjacent to v in H'.*

Proof. We take H to be the graph in Claim 1. The mapping λ will be extended to $\bar{\lambda}: X \cup Y \to A \cup B$ in the following way:

For a vertex x in X, we define

$$S(x)=\{b \in B : b=\lambda(y) \text{ and } y \text{ is adjacent to } x\},$$

$$M(x)=N(S(x))=\{v \in A : v \text{ is adjacent to all vertices in } S(x)\}.$$

The existence of $\bar{\lambda}$ is equivalent to a system of distinct representatives for $\{M(x)\}_{x \in X}$.

It suffices to show that for any set $X' \subseteq X$ we have

$$\left| \bigcup_{x \in X'} M(x) \right| \geqslant |X'|.$$

This is clearly true for $|X'| \leqslant (\log a/\log\log a)^2$ by property (ii) of H.

Now suppose $|X'| > (\log a/\log\log a)^2$. Since H' is of bounded degree $d= \log a/\log\log a$, for each x there are at most d^2 vertices x' in X with $S(x) \cap S(x') \neq \emptyset$. Thus there is a subset X'' of X where $|X''| \geqslant |X'|/d^2$ such that all $S(x)$, $x \in X''$, are mutually disjoint. Therefore,

$$\left| \bigcup_{x \in X'} M(x) \right| \geqslant \left| \bigcup_{x \in X''} M(x) \right| \geqslant \frac{|X'| p^{-2}}{d^2} \geqslant |X'|.$$

This completes the proof of Claim 2.

Claim 3. *There exists a graph $\bar{H}$ with $4n^2 \log\log n/\log n$ edges which contains all graphs with n vertices and degree at most $\log n/\log\log n = d$.*

Proof. We will construct a d-partite graph $\bar{H}$ as follows:

(i) $\bar{H}$ has vertex set $A_1 \cup A_2 \cup \cdots \cup A_{d+1}$ with $|A_i| = 2n/d$ for each i;

(ii) For each i, no $u, v \in A_i$ are adjacent;

(iii) The edges between A_i and $A_1 \cup A_2 \cup \cdots \cup A_{i-1}$ form a graph described in Claim 2.

It can be easily seen that $\bar{H}$ has at most $4n^2 \log\log n/\log n$ edges. It suffices to prove that any graph G with degree d can be embedded in $\bar{H}$. A nice result of Hajnal and Szemerédi [6] states that any graph with degree at most d can be colored by $d+1$ colors in such a way that the sizes of the color classes differ by at most 1. Suppose G has color classes $C_1, \ldots, C_{d+1}$. We will then embed C_1 into A_1, C_2 into A_2, and so on, as guaranteed by Claim 2.

Claim 4. *There exists a graph $F(n)$ with $Cn^2 \log\log n \log n$ edges which contains all graphs on n edges where C is an absolute constant.*

Proof. We will construct the graph $F(n)$ as follows:

(i) The vertex set is the disjoint union of A and B where $|A| = 2n \log\log n/\log n$ and $|B| = 2n$.

(ii) Every vertex v in A is adjacent to all vertices in $V(F(n)) - \{v\}$.

(iii) The subgraph of $F(n)$ induced by B is the graph, as described in Claim 3, which has $4n^2 \log\log n/\log n$ edges and contains all graphs with $2n$ vertices and degree at most d.

It is easy to see that $F(n)$ has at most $10n^2 \log\log n/n^2$ edges. Let G be an arbitrary graph on n edges. G has at most $2n \log\log n/\log n$ vertices with degree more than $\log n/\log\log n$. These vertices will be embedded in A. The remaining part of the graph will then be embedded in B as guaranteed by Claim 3.

This completes the proof of Claim 4.

Remark. If instead of using the result of Hajnal and Szemerédi, we use the simple fact that a graph on n vertices and maximum degree d can be $2(d+1)$ colored so that each color class has size at most n/d, then the resulting bound will differ from the one presented by a constant factor.

4. Universal graphs for planar graphs

We will use the following theorem to give an upper bound of $n^{3/2}$ for the universal graphs which contain all planar graphs on n edges.

Separator Theorem (Lipton and Tarjan [6]). *Let G be any planar graph with n vertices. The vertices of G can be partitioned into three sets, A, B, C such that no edge joins a vertex in B with a vertex in C, neither B and C contain more than $n/2$ vertices, and A contains no more than $2\sqrt{2n}/(1-\sqrt{2/3})$ vertices.*

Let $G(m)$ denote the graph constructed as shown in Fig. 1.

The vertices of $G(n)$ can be partitioned into three parts, X, Y and Z where $|X| =$

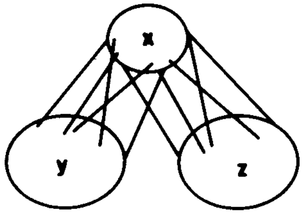

Fig. 1.

$2\sqrt{2n}/(1-\sqrt{2/3})=c_1\sqrt{n}$, $|Y|=|V(G(\lfloor n/2\rfloor))|$ and $|Z|=|V(G(\lfloor n/2\rfloor))|$. Any vertex in X is adjacent to any vertex in $G(n)$ except itself. The induced subgraph on Y is $G(\lfloor n/2\rfloor)$ and the induced subgraph on Z is $G(\lfloor n/2\rfloor)$.

It is rather straightforward to see that any planar graph with n vertices can be embedded in $G(n)$ since we can partition any planar graph into three parts, A, B and C as described in the Separator Theorem, and we can embed A in X, B in Y and C in Z.

We also note that $G(n)$ has fewer than $c_2 n$ vertices since

$$|V(G(n))|<2|V(G(n/2))|+c_1\sqrt{n}$$

and we can prove by induction on n that

$$|V(G(n))|\leqslant\frac{c_1\sqrt{2}}{\sqrt{2}-1}\,n\left(1-\frac{1}{\sqrt{2n}}\right).$$

Now, by the construction of $G(n)$, we know that

$$|E(G(n))|\leqslant|V(G(n))|\cdot c_1\sqrt{n}+2|E(G(n/2))|.$$

It follows by induction that $G(n)$ has fewer than $cn^{3/2}$ edges where $c=c_1^2\sqrt{2}/(\sqrt{2}-1)=19.7607\ldots$. Therefore we have

$$s'(n)<cn^{3/2}$$

and Theorem 2 is proved.

We note that the obvious lower bound for $s'(n)$ is $\frac{1}{2}n\log n$ which is the lower bound for the number of edges in graphs which contains all trees on n edges (see [2]). At present we do not know any better lower bound than $cn\log n$.

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