Fault-free Tilings of Rectangles

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Imagine that we have an unlimited supply of rectangular tiles of size 2 by 1 and we wish to tile the floor of a rectangular room of size p-by-q. Of course, we must cover all pq square units of floor area. Furthermore, two tiles are never allowed to overlap. As an example, we show in Figure 1 a tiling of a 5-by-6 rectangular floor.

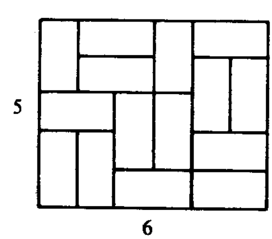

FIGURE 1
A tiling of 5-by-6.

It is easy to see that if such a tiling is to be possible then pq must be even, since the area of each tile is 2. On the other hand, if pq is even, then at least one of p and q must be even, say $p = 2r$. In this case we can place the tiles as shown in Figure 2 to construct the desired tiling.

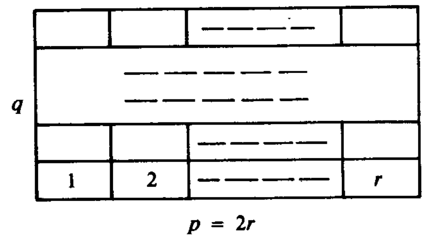

FIGURE 2
A tiling of $2r$-by-q.

Fault-free Tilings

If we examine the tiling shown in Figure 1 more carefully, we notice that it contains a "fault-line", that is, a straight line completely cutting through the rectangle which doesn't cut through any tile. Let us call a tiling which has *no* such fault-lines *fault-free*. If we think of a tiling as a cross-section of a wall built of bricks then it is clear why we might like to avoid fault-lines. In Figure 3 we show a fault-free tiling of a 5-by-6 rectangle.

A curious phenomenon occurs however, in trying to construct a fault-free tiling of a 6-by-6 rectangle. (The reader is encouraged to find one before proceeding further.) The same difficulty occurs for a 4-by-6 rectangle. In fact, there are no fault-free tilings for either of these cases! This leads to a question* that no mathematician can resist asking:

Exactly which p-by-q rectangles have fault-free tilings?

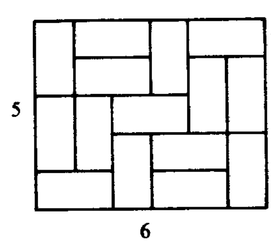

FIGURE 3
A fault-free tiling of 5-by-6.

Answering the Question

To begin with, a rectangle cannot have a fault-free tiling if it has no tiling at all, that is, if its area is an odd number. Stating this another way, a necessary condition for a p-by-q rectangle to have a fault-free tiling is that pq is divisible by 2.

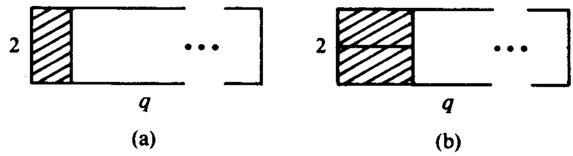

FIGURE 4
A fault-free tiling of 2-by-q?

As we have already seen, however, this is not enough. Suppose for example we try finding a fault-free tiling of a 2-by-q rectangle (where $q \geq 2$). There are just two ways to fill in the left-hand end of the rectangle, which we show in Figure 4. In either case however, (because $q \geq 2$) we must form a (vertical) fault-line. We therefore conclude that such rectangles have no fault-free tilings, even though their area is even.

In a similar spirit, consider what happens in the attempt to avoid fault-lines when tiling a 3-by-q rectangle. There are basically just two ways of tiling the end of the rectangle (see Figure 5).

The start shown in (a) is never any good since if $q = 2$ there is a horizontal fault-line and if $q > 2$ there is a vertical fault-line. What happens in case (b)? We show the various possibilities in Figure 6.

As before, case (a) creates fault-lines and must be discarded. The remaining possibility is to place Tile 1 as shown in (b). However, this forces use to place the additional Tiles 2 and 3 as shown in (c) (nothing else will fit into the gaps created in (b)). But notice that the "profile" of the pattern

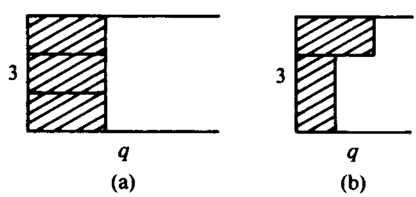

FIGURE 5
Trying to tile 3-by-q.

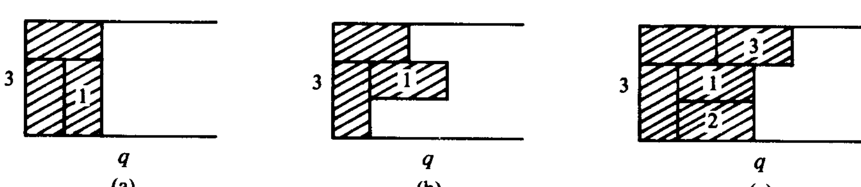

FIGURE 6
Still trying to tile 3-by-q.

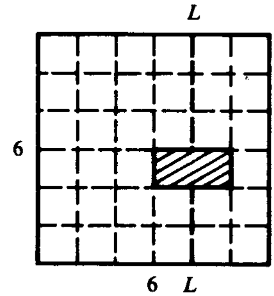

FIGURE 7
Potential fault-lines in 6-by-6.

in Figure 6 (c) is exactly the same as that in Figure 5 (b). Thus, we have only *delayed* the problem of filling the indentation created in Figure 5 (b). Eventually we must face the issue of completing the tiling (q *is* finite, after all) and this can only be done by finally filling this annoying indentation with a vertical tile as in Figure 6 (a). However, as soon as this happens we have formed a fault-line!

So, we conclude that no 3-by-q rectangle has a fault-free tiling.

In a similar way (but with a few more cases) it follows that no 4-by-q rectangle has a fault-free tiling as well. (Naturally, from the symmetry of the situation this means that no p-by-4 rectangle has a fault-free covering either.) Of course, one must resist the temptation to generalize at this point by trying to show that the same holds for 5-by-q rectangles, etc. After all, we do have a fault-free tiling of a 5-by-6 rectangle.

What we have shown at this point is that another necessary condition for a p-by-q rectangle to have a fault-free tiling is that both p and q must be at least 5.

This still leaves the nonexistence of a fault-free tiling of the 6-by-6 square unexplained. This gap can be filled by the following stunning argument of S. W. Golomb and R. I. Jewett.

Suppose for the moment that we have managed to find a hypothetical fault-free tiling of the 6-by-6 square. In the square there are 5 vertical and 5 horizontal fault-lines which, we assume, must all be broken by tiles (see Figure 7). Notice that each tile breaks *exactly one* potential line. Furthermore (and this is the crucial observation), if any fault-line (say L in Figure 7) is broken by just a *single* tile, then the remaining regions on either side of it must have an odd area, since they consist of 6-by-t rectangles with a single unit square removed. However, such regions are impossible to tile by 2-by-1 tiles. Each of the 10 potential fault-lines must be broken by at least two tiles. Since no tile can break more than one fault-line, then at least 20 tiles will be necessary for the tiling. But the area of the 6-by-6 square is only 36 while the area of the 20 tiles is 40! Thus, we have reached a contradiction. No such tiling of a 6-by-6 square can exist.

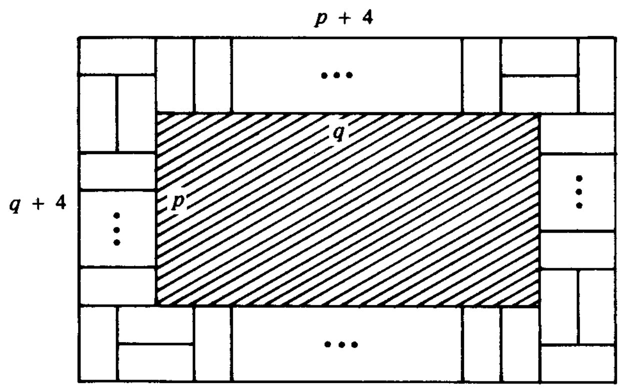

FIGURE 8
Extending fault-free tilings.

We can summarize what we know up to this point as follows: *Necessary* conditions for the existence of a fault-free tiling of a p-by-q rectangle are:

1 pq is divisible by 2;

2 $p \geq 5$, $q \geq 5$;

3 $(p,q) \neq (6,6)$.

Surprisingly, it turns out that these conditions are also *sufficient*. That is, if p and q satisfy **1**, **2** and **3** then the p-by-q rectangle will always have a fault-free tiling. One way this can be proved is by starting with small fault-free tilings, such as 5-by-8, 6-by-8 and our earlier 5-by-6, and building up larger fault-free tilings from these. For example, if we have a fault-free tiling of a p-by-q rectangle then in Figure 8 we show how to form a fault-free tiling of a $(p + 4)$ by $(q + 4)$ rectangle. A similar construction can be used to form a fault-free tiling of a $(p + 2r + 2)$ by $(q + s + 2)$ rectangle from the original p-by-q for any positive integers r and s.

Other Tiles

It is only natural to wonder about the possibilities of fault-free tilings when other-sized tiles are used; for example, 3-by-1 or 7-by-5. At first, an exact characterization of just which p-by-q rectangles can be appropriately tiled appears to be a hopelessly difficult problem. However, it turns out that there is a surprisingly beautiful answer to this question. Suppose we are to use tiles of size a-by-b where we can assume by changing our units if necessary, that a and b have no common factor greater than 1. (We can also assume that we don't have $a = b = 1$ since otherwise *no* fault-free tilings are possible, except for the trivial tiling consisting of a single tile!) As in the previous result, there are two basic necessary conditions: one dealing with a *divisibility* condition and one dealing with a *size* condition.

Regarding divisibility, the area pq of the rectangle must be divisible by the area ab of the tile if any tiling, fault-free or not, is to be possible. Other requirements are necessary as well. By *coloring* the tile and the rectangle with ab colors in an appropriate cyclic manner, it can be argued that it is necessary that

1′ Both a and b each divide at least one of p and q.(This is stronger than just requiring that ab divides pq).

Regarding size, there should be an analog in the general case to the earlier condition **2** which required $p, q \geq 5$. The corresponding condition is unexpected:

2′ Each of p and q must be able to be expressed as a sum $xa + yb$ with positive integers x and y in at least two ways.

Basically, this guarantees that there is enough freedom in placing tiles across the sides of the rectangle; that is, we don't always have to place the same numbers of tiles horizontally and vertically. For the case $a = 2, b = 1$, **2′** reduces to the earlier condition **2**, since 2, 3 and 4 do not have two representations as $x \cdot 2 + y \cdot 1$, $x, y > 0$, while any integer ≥ 5 does; for example, $5 = 1 \cdot 2 + 3 \cdot 1 = 2 \cdot 2 + 1 \cdot 1$, etc.

Curiously enough, the impossibility of a fault-free tiling of a 6×6 square with 2 by 1 tiles remains as the unique anomalous exception. In all other cases it is possible by construction techniques similar to those mentioned before to produce the required fault-free tilings whenever p and q satisfy the necessary conditions. The general result is summarized in the following statement.

THEOREM A fault-free tiling of a p-by-q rectangle with a-by-b tiles exists (where we assume $pq > ab$ and $(a,b) = 1$) if and only if

1′ Each of a and b divides p or q;

2′ Each of p and q can be expressed as $xa + yb$, $x, y > 0$, in at least two ways;

3′ For $\{a,b\} = \{1,2\}$, $(p,q) \neq (6,6)$.

The actual proof of this result is not difficult and is left to the energetic reader. We remark that m can be expressed as $xa + yb$ in at least two ways if and only if $m - ab$ can be expressed as $xa + yb$ in at least one way (assuming $(a,b) = 1$). In general, such integers do not form an interval as they do for $a = 2$, $b = 1$. For example, by using 3-by-2 tiles, it is possible to find fault-free tilings for 11-by-18 and 14-by-15 rectangles but it is not possible for a 12-by-12 rectangle (in particular, $11 = 3 \cdot 3 + 2 \cdot 1 = 3 \cdot 1 + 2 \cdot 4$ but $12 = 3 \cdot 2 + 2 \cdot 3$ is all).

What Next?

It is typical of this business that one answer leads to n more questions. For example, how many fault-free tilings does a rectangle have? What if we can use two sizes of tiles instead of just one? What about these same questions in 3 (or more) dimensions? What if we require each fault-line to be broken by at least 2 tiles? At least n? We have reached the frontier of our current knowledge on this topic. We encourage the interested reader to explore this fascinating byway of geometry and discover for himself the gems which must surely lie waiting to be discovered.

* Another more general question is: How many different fault-free tilings does a p by q rectangle have? We don't confront this question here.