SIAM J. ALG. DISC. METH.
Vol. 1, No. 3, September 1980


0196-5212/80/0103-0001 $01.00/0

# SOME MONOTONICITY PROPERTIES OF PARTIAL ORDERS*

R. L. GRAHAM†, A. C. YAO‡ AND F. F. YAO§

**Abstract.** A fundamental quantity which arises in the sorting of $n$ numbers $a_1, a_2, \cdots, a_n$ is $\Pr(a_i < a_j|P)$, the probability that $a_i < a_j$ assuming that all linear extensions of the partial order $P$ are equally likely. In this paper, we establish various properties of $\Pr(a_i < a_j|P)$ and related quantities. In particular, it is shown that $\Pr(a_i < b_j|P') \geqq \Pr(a_i < b_j|P)$ if the partial order $P$ consists of two disjoint linearly ordered sets $A = \{a_1 < a_2 < \cdots < a_m\}$, $B = \{b_1 < b_2 < \cdots < b_n\}$ and $P' = P \cup \{\text{any relations of the form } a_k < b_l\}$. These inequalities have applications in determining the complexity of certain sorting-like computations.

**1. Introduction.** Many algorithms for sorting $n$ numbers $\{a_1, a_2, \cdots, a_n\}$ proceed by using binary comparisons $a_i : a_j$ to build successively stronger partial orders $P$ on $\{a_i\}$ until a linear order emerges (see, e.g., Knuth [2]). A fundamental quantity in deciding the expected efficiency of such algorithms is $\Pr(a_i < a_j|P)$, the probability that the result of $a_i : a_j$ is $a_i < a_j$ when all linear orders consistent with $P$ are equally likely. In this paper, we prove some intuitive but nontrivial properties of $\Pr(a_i < a_j|P)$ and related quantities. These results are important, for example, in establishing the complexity of selecting the $k$th largest number [4].

We begin with a motivating example. Suppose that tennis skill can be represented by a number, so that player $x$ will lose to player $y$ in a tennis match if $x < y$. Imagine a contest between two teams $A = \{a_1, a_2, \cdots, a_m\}$ and $B = \{b_1, b_2, \cdots, b_n\}$, where within each team the players are already ranked as $a_1 < a_2 < \cdots < a_m$ and $b_1 < b_2 < \cdots < b_n$. If the first match of the contest is between $a_1$ and $b_1$, what is the probability $p$ that $a_1$ will lose? Supposing that the two teams have never met before, it is reasonable to assume that all relative rankings among players of $A \cup B$ are equally likely, provided they are consistent with $a_1 < a_2 < \cdots < a_m$ and $b_1 < b_2 < \cdots < b_n$. It is easy to show by a simple calculation that $p = m/(m+n)$. Consider now a different situation when the two teams did compete before with results $a_{i_1} < b_{j_1}, a_{i_2} < b_{j_2}, \cdots, a_{i_t} < b_{j_t}$; in other words, the team $B$ players always won. Let $p'$ be the probability for $a_1 < b_1$ assuming that all orderings of elements in $A \cup B$, consistent with the known constraints, are equally likely. One would certainly expect that $p' \geqq p$, as the additional information indicates that the players on team $B$ are better than those on team $A$. However, the proof of this does not seem to be so trivial. The purpose of this paper is to establish several general theorems concerning such monotone properties.

We now give a proof[1] that $p' \geqq p$ in the preceding example. It establishes the result even when $A$ and $B$ are themselves only partially ordered, provided that $a_1$ and $b_1$ are the unique minimum elements in $A$ and $B$, respectively. Let us denote by $P'$ the partial order obtained by adding the relations $\{a_{i_1} < b_{j_1}, a_{i_2} < b_{j_2}, \cdots, a_{i_t} < b_{j_t}\}$ to $P = A \cup B$. We will show that $\Pr(a_1 < b_1|P')/\Pr(b_1 < a_1|P') \geqq m/n$, from which it follows that $\Pr(a_1 < b_1|P') \geqq m/(m+n) = \Pr(a_1 < b_1|P)$.

* Received by the editors October 5, 1979, and in final revised form November 12, 1979.

† Bell Laboratories, Murray Hill, New Jersey 07974.

‡ Computer Science Department, Stanford University, Stanford, California 94305. The work of this author was supported in part by the National Science Foundation under Grant MCS-77-05313.

§ Xerox Palo Alto Research Center, Palo Alto, California 94304.

[1] The proof given here, simplifying our original proof, is due to D. Knuth.

Consider the sets $S_0$ of all $(m+n-1)!/(m-1)!(n-1)!$ possible sequences of 0's and 1's with one element "underlined", where

(i) the sequence is of length $m+n$, with $m$ 0's and $n$ 1's,
(ii) the first character is 0,
(iii) one of the 1's is underlined.

Define the set $S_1$ similarly but with first character 1 and with one of the 0's underlined. We get a 1–1 correspondence between $S_0$ and $S_1$ by complementing both the first character and the underlined character. If $x_0 \in S_0$ corresponds to $x_1 \in S_1$, then $x_0 < x_1$ in the partial order $<$ defined on (0, 1)-sequences as follows: Say that $x < y$ if we can transform $x$ into $y$ by one or more replacements of '01' by '10'; or, equivalently, $x < y$ if $x$ and $y$ have the same number of 0's and for all $k$, the position of the $k$th 0 of $x$ is no further to the right than the $k$th 0 of $y$. List all the pairs of the correspondence as $x_0 \leftrightarrow x_1$, $y_0 \leftrightarrow y_1, \cdots$.

For a partial order $Q$ on a set $X$, we say that a 1–1 mapping $\lambda: X \to \{1, 2, \cdots, n\}$ is a *linear extension* of $Q$ if $\lambda(x) < \lambda(y)$ whenever $x < y$ in $Q$. Let $\lambda_{x_1}$ be a linear extension of $P'$ which places elements of $A$ into the positions where $x_1$ has a 0, and elements of $B$ into the positions where $x_1$ has a 1. The correspondence $x_0 \leftrightarrow x_1$ naturally associates to $\lambda_{x_1}$ a linear extension $\lambda_{x_0}$ of $P'$ in which the relative order of the $a_i$ and also the relative order of the $b_j$ are both unchanged. We therefore obtain a list of inequalities $N(x_1) \leqq N(x_0)$, $N(y_1) \leqq N(y_0), \cdots$, where $N(x_i)$ denotes the number of all linear extensions $\lambda_{x_i}$ defined above. (For some $x_i$, $N(x_i)$ may be 0.) Summing all the inequalities gives

$$m \cdot (\text{number of linear extensions of } P' \cup (b_1 < a_1)$$
$$\leqq n \cdot (\text{number of linear extensions of } P' \cup (a_1 < b_1),$$

which is what we wanted to show.

The preceding example suggests the following conjecture. Let $A = \{a_1, a_2, \cdots, a_m\}$, $B = \{b_1, b_2, \cdots, b_n\}$, $X = A \cup B$ and $(P, <)$ be a partial order on $X$ which contains no relation of the form $b_j < a_i$ or $a_i < b_j$.

Suppose $E = E_1 \cup \cdots \cup E_r$ and $E' = E'_1 \cup \cdots \cup E'_s$ where $E_i$ and $E'_j$ are events of the form $(a_{i_1} < b_{j_1}) \wedge (a_{i_2} < b_{j_2}) \wedge \cdots \wedge (a_{i_t} < b_{j_t})$.

*Conjecture.* Assuming all linear extensions of $P$ are equally likely, the events $E$ and $E'$ are mutually favorable, i.e.,

$$\Pr(E|P) \Pr(E'|P) \leqq \Pr(EE'|P).$$

In this paper, we shall prove several results related to this conjecture, which in particular imply the conjecture for the case when both $A$ and $B$ are linearly ordered under $P$ (see Corollary 2 to Theorem 1). The general conjecture, however, remains unresolved.

**2. A monotonicity theorem.** In this section, we shall prove a theorem which implies an important special case of the conjecture, namely, the case when $A$ and $B$ are each linearly ordered under $P$. In fact, in this case the conjecture is true even if $P$ includes relations of both of the types $a_i < b_j$ and $b_k < a_l$.

Let $A = \{a_1 < a_2 < \cdots < a_m\}$ and $B = \{b_1 < b_2 < \cdots < b_n\}$ be linear orders. Let $\Lambda$ denote the set of all linear extensions of $P = A \cup B$. A *cross-relation* between $A$ and $B$ is a set $Z \subseteq (A \times B) \cup (B \times A)$, interpreted as a set of comparisons $a_i < b_j$ and $b_k < a_l$. For a cross-relation $Z$, we define $\hat{Z} = \{\lambda \in \Lambda: \lambda(x) < \lambda(y) \text{ for all } (x, y) \in Z\}$.

It will be convenient to represent each $\lambda \in \hat{Z}$ as a lattice path $\bar{\lambda}$ in $\mathbb{Z}^2$ starting from the origin and terminating at the point $(n, m)$ (see Fig. 2). The interpretation is as follows: As we step along $\bar{\lambda}$ starting from (0, 0), if the $k$th step increases the $A$ (or $B$)

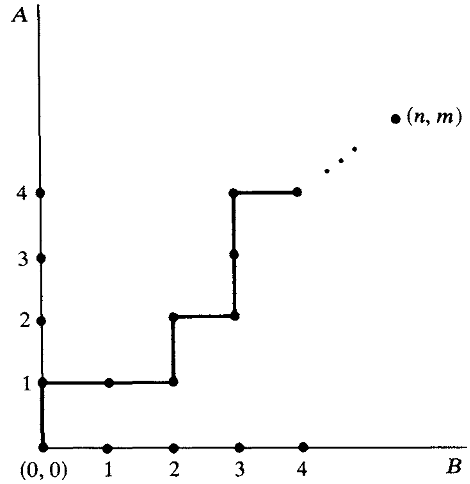


FIG. 1

coordinate from $i-1$ to $i$ then $\lambda$ maps $a_i$ (or $b_i$, respectively) to $k$. Thus, in Fig. 1, $\lambda(a_1)=1$, $\lambda(b_1)=2$, $\lambda(b_2)=3$, $\lambda(a_2)=4$, etc.

Let us consider the geometrical implications of a constraint of the form $\lambda(a_i)<\lambda(b_j)$. By definition, as we go along $\bar{\lambda}$ from (0, 0) to $(n, m)$, $\bar{\lambda}$ must achieve an $A$-value of $i$ before it achieves a $B$-value of $j$. But this means exactly that $\bar{\lambda}$ must not pass through the (closed) vertical line segment joining $(j, i)$ to $(j, 0)$. In general, a set $X \subseteq A \times B$ represents a set of vertical "barriers" of this type which, for any $\lambda \in \hat{X}$, the corresponding lattice path $\lambda$ is prohibited from crossing (Fig. 2). Of course, a set $Y \subseteq B \times A$ corresponds to a set of horizontal barriers in a similar way, with $(b_j, a_i) \in Y$ being represented by the line segment joining $(0, i)$ to $(j, i)$. We will also refer to such vertical and horizontal barriers as $x$-barriers and $y$-barriers. For a cross-relation $Z \subseteq (A \times B) \cup (B \times A)$, we define $Z_X = Z \cap (A \times B)$ and $Z_Y = Z \cap (B \times A)$. Thus $Z_X$ and $Z_Y$ are the vertical and the horizontal barriers determined by $Z$, respectively.

Let $Z$ and $W$ be two cross-relations between $A$ and $B$. We say $Z$ is more *A-selective* than $W$ if both $W_X \subseteq Z_X$ and $Z_Y \subseteq W_Y$. (For example, a set of $x$-barriers is

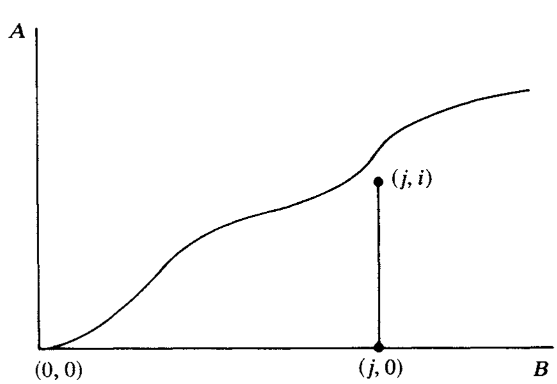


FIG. 2. *A vertical barrier corresponding to the condition* $\lambda(a_i)<\lambda(b_j)$.

always more $A$-selective than a set of $y$-barriers.) Intuitively, one would think that in this case linear extensions of $Z$ should have a greater probability for ranking $A$'s elements below $B$'s. Let $Z'$ and $W'$ be another pair of cross-relations with $Z'$ being more $A$-selective than $W'$. The basic result we prove is the following

THEOREM 1. $|\hat{Z} \cap \hat{Z}'| \cdot |\hat{W} \cap \hat{W}'| \geqq |\hat{Z}' \cap \hat{W}| \cdot |\hat{Z} \cap \hat{W}'|$.

COROLLARY 1. $\Pr(Z'|Z)/\Pr(W'|Z) \geqq \Pr(Z'|W)/\Pr(W'|W)$ *when the denominators are not zero.*

Corollary 1 follows immediately from Theorem 1. It asserts that the ratio $\Pr(Z')/\Pr(W')$ is larger when conditioned on $Z$ than when conditioned on $W$.

COROLLARY 2. $\Pr(V|Z) \geqq \Pr(V|W)$ *for any* $V$ *with* $V_Y \subseteq Z_Y$. *In particular,* $\Pr(X|Z) \geqq \Pr(X|W)$ *for any* $X \subseteq A \times B$.

This follows from Corollary 1 by letting $Z' = V$, and choosing $W'$ so that $W'_X = \emptyset$ and $W'_Y = V_Y$.

*Proof of Theorem* 1. We will construct a $1-1$ mapping of $(\hat{Z}' \cap \hat{W}) \times (\hat{Z} \cap \hat{W}')$ into $(\hat{Z} \cap \hat{Z}') \times (\hat{W} \cap \hat{W}')$. Suppose $\lambda \in \hat{Z}' \cap \hat{W}$ and $\lambda' \in \hat{Z} \cap \hat{W}'$. Let $\bar{\lambda}, \bar{\lambda}'$ be the corresponding lattice paths, and let $\{s_1, s_2, \cdots, s_r\}$ be the set of lattice points common to $\bar{\lambda}$ and $\bar{\lambda}'$.

We assume that the $s_i$ are labeled so that $s_1 = (0, 0)$, $s_r = (n, m)$, and as we move along $\bar{\lambda}$ from $s_1$ to $s_r$, we reach $s_i$ before $s_{i+1}$. Consider the pair of path segments $\bar{\lambda}(s_i, s_{i+1})$ (defined to be the portion of $\bar{\lambda}$ between (and including) $s_i$ and $s_{i+1}$) and $\bar{\lambda}'(s_i, s_{i+1})$. We will call the closed region bounded by these two segments an *olive*, provided that the region is nondegenerate (i.e., $\bar{\lambda}(s_i, s_{i+1})$ and $\bar{\lambda}'(s_i, s_{i+1})$ do not coincide). Let $O_1, O_2, \cdots, O_t$ be the set of olives formed by $\bar{\lambda}$ and $\bar{\lambda}'$. The upper path segment bounding $O_k$ we denote by $O_k^+$; the lower we denote by $O_k^-$. Note that, given $\bar{\lambda} \cup \bar{\lambda}'$, the path $\bar{\lambda}$ can be determined by specifying which $O_i$ contribute $O_i^+$ to $\bar{\lambda}$ and consequently, which $O_i$ contribute $O_i^-$ to $\bar{\lambda}$.

We want to show that for each $\bar{\lambda} \in \hat{Z}' \cap \hat{W}$ with $\bar{\lambda}' \in \hat{Z} \cap \hat{W}'$, we can associate a unique $\bar{\mu} \in \hat{Z} \cap \hat{Z}'$ with $\bar{\mu}' \in \hat{W} \cap \hat{W}'$. In fact, $\bar{\mu}$ and $\bar{\mu}'$ will be constructed from the path segments of $\bar{\lambda}$ and $\bar{\lambda}'$ so that $\bar{\mu} \cup \bar{\mu}' = \bar{\lambda} \cup \bar{\lambda}'$. The rule for obtaining $\bar{\mu}$ (and consequently $\bar{\mu}'$) is as follows:

> Let $\bar{\mu}$ be the same as $\bar{\lambda}$ except that whenever an olive $O_k$ is intersected by a barrier of $Z$ or $W$, we let $O_k^+ \in \bar{\mu}$.

In the example illustrated in Fig. 3, $O_2$ is penetrated (from below) by an $x$-barrier in $Z-W$, and $O_4$ is penetrated (from the left) by a $y$-barrier in $W-Z$. Note that $\bar{\lambda}$ always contains the lower boundaries $O_k^-$ of the penetrated olives $O_k$. To obtain $\bar{\mu}$, we substitute $O_2^+, O_4^+$ for $O_2^-, O_4^-$ in the path $\bar{\lambda}$.

To show that $\bar{\mu} \in \hat{Z} \cap \hat{Z}'$ and that the complementary path $\bar{\mu}' \in \hat{W} \cap \hat{W}'$ we need only verify that $\bar{\mu}$ and $\bar{\mu}'$ clear their respective sets of barriers in $Z \cup Z'$ and $W \cup W'$.

Suppose $O_k$ is penetrated (from below) by an $x$-barrier in $Z-W$, such as $O_2$ in Fig. 3. Then $\bar{\lambda}$ contains $O_k^-$ and $\bar{\lambda}'$ contains $O_k^+$. We want to argue that $O_k^+$ must clear $Z$ and $Z'$, while $O_k^-$ must clear $W$ and $W'$. First of all, if $O_k^+$ clears $W'$ then it clears $W'_Y$ and hence $Z'_Y$. Secondly, $O_k^+$ clears $Z'_X$ since $O_k^-$ clears $Z'$. It follows that $O_k^+$ clears both $Z$ and $Z'$ as desired. The fact that $O_k^-$ clears $W$ and $W'$ can be shown in the same way.

Similarly, if $O_k$ is penetrated by a $y$-barrier in $W-Z$, such as $O_4$ in Fig. 3, then assigning $O_k^+$ to $\bar{\mu}$ and $O_k^-$ to $\bar{\mu}'$, will enable $\bar{\mu}, \bar{\mu}'$, to clear their respective barriers.

The mapping $(\bar{\lambda}, \bar{\lambda}') \to (\bar{\mu}, \bar{\mu}')$ is $1-1$, since the path $\bar{\lambda}$ can be reconstructed from $\bar{\mu}$ by substituting $O_k^-$ for $O_k^+$ in those olives $O_k$ penetrated by a barrier of $Z$ or $W$. This completes the proof of Theorem 1. $\square$

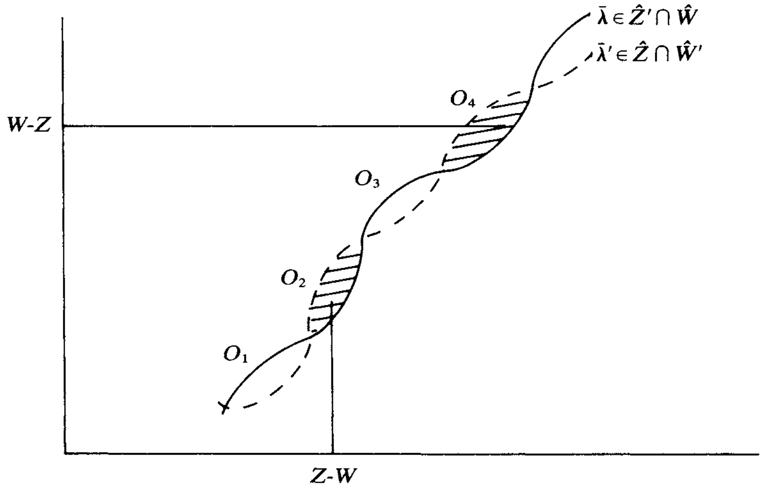


FIG. 3. *Olives which are penetrated by an x-barrier in Z–W and a y-barrier in W–Z.*

**3. Extension to disjunctions of partial orders.** In this section, we will consider pairs of cross relations $(Z, W)$ on $A=\{a_1<a_2<\cdots<a_m\}$ and $B=\{b_1<b_2<\cdots<b_n\}$, when $Z$ consists of just $x$-barriers and $W$ consists of just $y$-barriers. However, we now incorporate the concept of a disjunction of a set of cross-relations. For a disjunction $\mathcal{Z}=\bigcup_i Z_i$, where $Z_i\subseteq(A\times B)\cup(B\times A)$, we let $\hat{\mathcal{Z}}$ denote $\bigcap_i \hat{Z}_i$. Suppose $\mathcal{X}=\bigcup_i X_i$ and $\mathcal{Y}=\bigcup_j Y_j$ where $X_i\subseteq A\times B$ and $Y_j\subseteq B\times A$, with $\mathcal{X}'=\bigcup_i X_i'$ and $\mathcal{Y}'=\bigcup_j Y_j'$ defined similarly. The analogue[2] of Theorem 1 is the following

THEOREM 2. $|\hat{\mathcal{X}}\cap\hat{\mathcal{X}}|\,|\hat{\mathcal{Y}}\cap\hat{\mathcal{Y}}'|\geqq|\hat{\mathcal{X}}\cap\hat{\mathcal{Y}}|\,|\hat{\mathcal{X}}'\cap\hat{\mathcal{Y}}'|$.

As in the case of Theorem 1, here we can also derive as corollaries that $\Pr(\hat{\mathcal{X}}|\hat{\mathcal{X}}')/\Pr(\hat{\mathcal{Y}}|\hat{\mathcal{X}}')\geqq\Pr(\hat{\mathcal{X}}|\hat{\mathcal{Y}}')/\Pr(\hat{\mathcal{Y}}|\hat{\mathcal{Y}}')$, that is, the ratio $\Pr(\hat{\mathcal{X}}/\Pr\hat{\mathcal{Y}})$ is larger when conditioned on $\hat{\mathcal{X}}'$ than when conditioned on $\hat{\mathcal{Y}}$. For the special case that $\mathcal{Y}=\mathcal{Y}'=\varnothing$, we obtain

$$\Pr(\hat{\mathcal{X}}|\hat{\mathcal{X}}')\geqq\Pr(\hat{\mathcal{X}}). \tag{1}$$

*Proof of Theorem* 2. As in the proof of Theorem 1, we will show that for each $\bar\lambda\in\hat{\mathcal{X}}\cap\hat{\mathcal{Y}}$ with $\bar\lambda'\in\hat{\mathcal{X}}'\cap\hat{\mathcal{Y}}'$, we can associate a unique $\bar\mu\in\hat{\mathcal{X}}\cap\hat{\mathcal{X}}'$ with $\bar\mu\in\hat{\mathcal{Y}}\cap\hat{\mathcal{Y}}'$. Furthermore, $\bar\mu$ and $\bar\mu'$, will be constructed from $\bar\lambda$ and $\bar\lambda'$ by interchanging certain path segments. We may assume without loss of generality that no $X_i$, $X_i'$, $Y_i$, or $Y_j'$ have a barrier which penetrates both $\bar\lambda$ and $\bar\lambda'$.

Let $O_1, O_2, \cdots, O_t$ be the set of olives formed by $\bar\lambda$ and $\bar\lambda'$. Thus $\bar\lambda$ corresponds to a subset $P\subseteq\{1,2,\cdots,t\}=T$ such that $\bar\lambda$ contains $O_k^+$ iff $k\in P$, and with this association $\bar\lambda'$ corresponds to the subset $Q=T-P=P^c$. For a given olive $O_k$, there may be various barriers which intersect it. For each $X_i$, let $G_i$ denote the set $\{k\in T\colon$ a barrier from $X_i$ intersects $O_k\}$. Similarly, define $G_i'$ for $X_i'$, $H_i$ for $Y_i$, and $H_i'$ for $Y_i'$. Observe that

$$\begin{aligned}\bar\lambda\in\hat{\mathcal{X}} &\text{ iff } \bar\lambda\in\hat{X}_i \text{ for some } i,\\ &\text{iff } P\supseteq G_i \text{ for some } i,\\ &\text{iff } P\in[\mathcal{G}]_U\equiv\text{upper ideal in } 2^T \text{ generated by } \mathcal{G}=\{G_1,G_2,\cdots\},\end{aligned}$$

where the meaning of the last statement is as follows.

[2] We could, of course, write this as $|\hat{\mathcal{X}}\cap\hat{\mathcal{X}}'|\,|\hat{\mathcal{Y}}\cap\hat{\mathcal{Y}}'|\geqq|\hat{\mathcal{X}}'\cap\hat{\mathcal{Y}}|\,|\hat{\mathcal{X}}\cap\hat{\mathcal{Y}}'|$ to make it resemble Theorem 1 more.

DEFINITION. For a finite set $T$, let $2^T$ denote the collection of all subsets of $T$ partially ordered by set inclusion (i.e., $C < D$ iff $C \supseteq D$). An *upper ideal* in $2^T$ is a subset $\mathscr{U} \subseteq 2^T$ such that if $S \in \mathscr{U}$ then any element $S'$ higher in the partial order (i.e., $S \subseteq S'$) must also be in $\mathscr{U}$. Similarly, a *lower ideal* $\mathscr{L} \subseteq 2^T$ has the property that if $S \in \mathscr{L}$ and $S' \subseteq S$, then $S' \in \mathscr{L}$.

As above, we have

$$\bar{\lambda} \in \hat{\mathscr{Y}} \text{ iff } \bar{\lambda} \in \hat{Y}_j \text{ for some } j,$$

$$\text{iff } P \subseteq H_j^c \text{ for some } j,$$

$$\text{iff } P \in [\mathscr{H}^c]_L \equiv \text{lower ideal in } 2^T \text{ generated by } \mathscr{H}^c = \{H_1^c, H_2^c, \cdots\}.$$

Now, what we are trying to show is that for each $\bar{\lambda} \in \hat{\mathscr{X}} \cap \hat{\mathscr{Y}}$ with $\bar{\lambda}' \in \hat{\mathscr{X}}' \cap \hat{\mathscr{Y}}'$, we can associate a unique $\bar{\mu} \in \hat{\mathscr{X}} \cap \hat{\mathscr{X}}'$ with $\bar{\mu}' \in \hat{\mathscr{Y}} \cap \hat{\mathscr{Y}}'$. Translating this into the language of ideals, we want:

For each $P \in [\mathscr{G}]_U \cap [\mathscr{H}^c]_L$ with $P^c \in [\mathscr{G}']_U \cap [\mathscr{H}'^c]_L$, there can be associated a unique $Q \in [\mathscr{G}]_U \cap [\mathscr{G}']_U$ with $Q^c \in [\mathscr{H}^c]_L \cap [\mathscr{H}'^c]_L$.

We claim that, in fact, we will be able to find such a mapping for *arbitrary* upper ideals $\mathscr{U}$, $\mathscr{U}'$ and lower ideals $\mathscr{L}$, $\mathscr{L}'$ in $2^T$. In other words, there is a $1-1$ mapping $(P, P^c) \to (Q, Q^c)$ such that if $P \in \mathscr{U} \cap \mathscr{L}$ and $P^c \in \mathscr{U}' \cap \mathscr{L}'$ then $Q \in \mathscr{U} \cap \mathscr{U}'$ and $Q^c \in \mathscr{L} \cap \mathscr{L}'$. Further, we will restrict the mapping so that

$$P \subseteq Q. \tag{2}$$

If (2) holds then

$$P \in \mathscr{U} \Rightarrow Q \in \mathscr{U} \quad \text{since } \mathscr{U} \text{ is an upper ideal,}$$

$$P^c \in \mathscr{L}' \Rightarrow Q^c \in \mathscr{L}' \quad \text{since } \mathscr{L}' \text{ is a lower ideal.}$$

Thus, we want

$$\begin{matrix} P \in \mathscr{U} \cap \mathscr{L} \\ P^c \in \mathscr{U}' \cap \mathscr{L}' \end{matrix} \Rightarrow \begin{matrix} Q \in \mathscr{U}' \\ Q^c \in \mathscr{L} \end{matrix} \quad \text{with } P \subseteq Q.$$

We claim even further that we can find the required mapping for the more general domain

$$\begin{matrix} P \in \mathscr{L} \\ P^c \in \mathscr{U}' \end{matrix} \Rightarrow \begin{matrix} Q \in \mathscr{U}' \\ Q^c \in \mathscr{L} \end{matrix} \quad \text{with } P \subseteq Q.$$

But notice that if $\mathscr{U}'$ is an upper ideal then $\mathscr{U}'^c$ is a lower ideal. Thus, the condition

$$\begin{matrix} P \in \mathscr{L} \\ P^c \in \mathscr{U}' \end{matrix} \Rightarrow \begin{matrix} Q \in \mathscr{U}' \\ Q^c \in \mathscr{L} \end{matrix} \quad \text{with } P \subseteq Q$$

becomes

$$P \in \mathscr{L} \cap \mathscr{U}'^c \equiv \mathscr{W} \Rightarrow Q^c \in \mathscr{W} \quad \text{with } P \subseteq Q,$$

where $\mathscr{W}$, being the intersection of two lower ideals, is also a lower ideal. Of course,

$$P \subseteq Q \quad \text{iff} \quad P \cap Q^c = \varnothing.$$

Thus, the theorem will be proved if we show the following result, which is actually of independent interest:

For an *arbitrary* lower ideal $\mathcal{W}$ in $2^T$, there is always a permutation $\pi: \mathcal{W} \to \mathcal{W}$ such that for all $w \in \mathcal{W}$, $w \cap \pi(w) = \emptyset$.

For each $x \in \mathcal{W}$, let $d(x)$ denote the set $\{w \in \mathcal{W} : x \cap w = \emptyset\}$. By Hall's theorem [1], it is enough to show that

$$\left|\bigcup_{x \in \mathcal{S}} d(x)\right| \geqq |\mathcal{S}|$$

for all $\mathcal{S} \subseteq \mathcal{W}$. In fact, for $\mathcal{S} \subseteq \mathcal{W}$, let $d_{\mathcal{S}}(x)$ denote $d(x) \cap [\mathcal{S}]_L$. What we will actually show is the stronger assertion

$$\left|\bigcup_{x \in \mathcal{S}} d_{\mathcal{S}}(x)\right| \geqq |\mathcal{S}| \tag{3}$$

for any $\mathcal{S} \subseteq 2^T$. So, suppose $\mathcal{S} = \{S_1, \cdots, S_k\}$ with $S_i \subseteq T$. Thus,

$$\begin{aligned} y \in \bigcup_{x \in \mathcal{S}} d_{\mathcal{S}}(x) \text{ iff } & y \in [\mathcal{S}]_L \text{ and } y \cap x = \emptyset \text{ for some } x \in \mathcal{S}, \\ \text{iff } & y \subseteq S_i \text{ for some } i \text{ and } y \cap S_j = \emptyset \text{ for some } j, \\ \text{iff } & y \subseteq S_i - S_j \text{ for some } i, j. \end{aligned}$$

Therefore, if we can in fact show that there are always at least $k$ different sets of the form $S_i$–$S_j$ then (3) will follow. However, this is exactly the result of Marica and Schönheim [3]. Hence (3) holds and the theorem follows. □

Theorem 2 can be generalized slightly by allowing the partial order $(P, <)$ underlying $\hat{\mathcal{X}}, \hat{\mathcal{Y}}, \hat{\mathcal{X}}', \hat{\mathcal{Y}}'$ to be more than just $A \cup B$; i.e., $P$ may itself include relations of the form $a_i < b_j$ and $b_k < a_l$. In this case, all such relations can also be interpreted as barriers which cannot be crossed by a linear extension $\bar{\sigma}$ of $P$. Since both paths $\bar{\lambda}$ and $\bar{\lambda}'$ avoid all these barriers, then so will any path $\bar{\mu}, \bar{\mu}'$ constructed from their path segments.

**4. Concluding remarks.** We should point out that if we weaken the hypotheses on the structure of $(P, <)$ even slightly, then (1) can fail. To see this, consider the following partial order $(P, <)$ on the set $\{a_1, a_2, b_1, b_2, c\}$ as shown in Fig. 4.

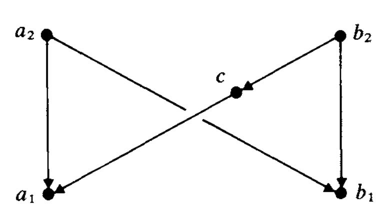


FIG. 4. *An example violating* (1).

Choose $X = X_1 = \{(1, 1)\}$, $X' = X'_1 = \{(2, 2)\}$, and all other $X_i, X'_i, Y_j, Y'_j$ to be $\emptyset$. An easy enumeration yields

$$|\Lambda| = 8, \quad |\hat{X}| = 3 = |\hat{X}'|, \quad |\hat{X} \cap \hat{X}'| = 1.$$

Thus,

$$\Pr(\hat{X}|\hat{X}') = \tfrac{1}{3} < \tfrac{3}{8} = \Pr(\hat{X})$$

which violates (1).

We should also note (as pointed out by D. Kleitman and J. Shearer) that the conjecture is not true if we allow $P$ to have even one relation of the form $a_i < b_j$ as the following example shows.

Let

$$A = \{a_1, a_2\}, \quad B = \{b_1, b_2\}, \quad P = \{a_1 < b_1\},$$
$$E = \{a_1 < b_2\} \quad \text{and} \quad E' = \{a_2 < b_1\}.$$

A simple calculation shows that

$$\Pr(E|P)\Pr(E'|P) = \tfrac{2}{3} \cdot \tfrac{2}{3} = \tfrac{4}{9} > \tfrac{5}{12} = \Pr(EE'|P).$$

*Note added in proof.* V. Chvátal has pointed out that the fact we use concerning mappings of lower ideals seems to be due to Erdös, Herzog and Schönheim in *An extremal problem on the set of noncoprime divisors of a number*, Israel J. Math., (1970), pp. 408–412. Very recently, L. A. Shepp has managed to settle the conjecture in the affirmative by an ingenious application of the FKG inequalities.

## REFERENCES

[1] P. HALL, *On representations of subsets*, J. London Math. Soc. 10 (1935), pp. 26–30.

[2] D. E. KNUTH, *The Art of Computer Programming*, Vol. 3, Sorting and Searching, Addison-Wesley, Reading, MA., 1973.

[3] J. MARICA AND J. SCHÖNHEIM, *Differences of sets and a problem of Graham*, Canad. Math. Bull., 12 (1969), pp. 635–637.

[4] A. C. YAO AND F. F. YAO, *On the average-case complexity of selecting the k-th best*, Proc. 19th Annual IEEE Symp. on Foundations of Computer Science, Ann Arbor, Michigan 1978, pp. 280–289.