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Note

A Note on the Intersection Properties of Subsets of Integers

R. L. GRAHAM

Bell Laboratories, Murray Hill, New Jersey 07974

AND

M. SIMONOVITS AND V. T. SÓS

Eötvös Loránd University, Budapest, Hungary

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INTRODUCTION

Intersection properties of finite set systems have an extensive literature. Without going into details we mention just one of them, due to Erdös and de Bruijn [1]. According to the de Bruijn–Erdös theorem, if $A_1, ..., A_N$ are subsets of an n-element set S and $|A_i \cap A_j| = 1$ for $i \neq j$ (where $|X|$ denotes the cardinality of X), then $N \leqslant n$. This result is sharp, e.g., if $S = \{1, ..., n\} = [1, n]$ and $A_1 = \{1, n\}$, $A_2 = \{2, n\}, ..., A_{n-1} = \{n-1, n\}$, and $A_n = \{1, 2, ..., n-1\}$, then $A_i \cap A_j = 1$ for $1 \leqslant i < j \leqslant n$. Many similar theorems have been proved for sets. One could also ask what analogous results can be proved if the A_i have some extra structure and the condition on the intersection also refers to this structure (see [2, 3, 4]). For example, in [3] it is proved that if $A_1, ..., A_N$ are graphs on the same n vertices and the intersection of two graphs A_i and A_j is defined as the graph without isolated vertices whose edges are the common edges of A_i and A_j, then the condition "$A_i \cap A_j$ is a (nonempty) cycle for $1 \leqslant i < j \leqslant N$" implies that $N \leqslant \binom{n}{2} - 2$, which is again sharp. Here we shall investigate the case in which $A_1, ..., A_N$ is a system of subsets of $\{1, ..., n\}$ and the intersection condition is of a number-theoretic type.

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1. The Problem of Intervals

A subset A of $\{1, \ldots, n\}$ will be called an *interval* if for some integers a and b, $(b \geqslant a)$, $A = \{a, a+1, \ldots, b-1, b\}$. The first question considered here is the following:

Let $A_1, \ldots, A_N$ be subsets of $[1, n]$ such that $A_i \cap A_j$ is an interval for $1 \leqslant i < j \leqslant N$. How large N can be?

Proposition 1. *If $A_1, \ldots, A_N$ are subsets of $[1, n]$ and $A_i \cap A_j$ is an interval (possibly empty) whenever $i \neq j$, then $N \leqslant \binom{n}{2} + n + 1$.*

Remark 1. The bound in Proposition 1 is sharp: If $A_1, \ldots, A_N$ are the subsets of at most *two* elements, then $|A_i \cap A_j| \leqslant 1$, hence it is empty or an interval. There are also other extremal systems (that is, systems of maximum cardinality), e.g., the family of all intervals, together with the empty set forms an extremal system as well.

Proof. Let $A_1, \ldots, A_N$ be an arbitrary system of subsets for which $A_i \cap A_j$ is an interval if $i \neq j$, and let B_i consist of the smallest and the largest elements of A_i. Since $A_i \cap A_j$ is an interval, if $B_i = B_j$, then $A_i = A_j$. Hence the number of B's is the same as the number of A's and $N \leqslant \binom{n}{2} + n + 1$, as desired. ∎

If we exclude empty intersections, we have

Proposition 2. *Let $A_1, \ldots, A_N$ be a system of subsets of $[1, n]$ such that $A_i \cap A_j$ is a nonempty interval whenever $i \neq j$. Then $N \leqslant [(n+1)^2/4]$.*

Remark 2. The bound in Proposition 2 is also sharp. For consider all the intervals in $[1, n]$ containing $m = [n + 1/2]$. The number of these intervals is just $[(n+1)^2/4]$ and any two of them intersect in an interval.

Proof. Let $A_1, \ldots, A_N$ be a family of subsets of $[1, n]$ for which $A_i \cap A_j$ is a nonempty interval for $1 \leqslant i < j \leqslant N$. Let B_i be the smallest interval containing A_i. Clearly, if $B_i = B_j$, then $A_i = A_j$, since $A_i \cap A_j$ is an interval. Hence $B_1, \ldots, B_N$ is also a system of subsets of $[1, n]$ for which $B_i \cap B_j$ is a nonempty interval whenever $1 \leqslant i < j \leqslant N$. Trivially, there exists an $m \in B_i$ $(i = 1, \ldots, N)$ and therefore the upper endpoint of B_i can be chosen in $n - m + 1$ ways, the lower one in m ways. Thus, $N \leqslant m(n - m + 1) \leqslant [(n+1)^2/4]$ and we are done. ∎

(Observe that the method used to prove Proposition 2 yields another proof of Proposition 1.)

2. Generalizations of the Interval Problem

Let $\mathscr{A}$ be a family of $|\mathscr{A}|$ subsets of a finite set S which is closed under intersections (so that, in particular, $S \in \mathscr{A}$). Suppose A_i, $1 \leqslant i \leqslant n$, are subsets of S such that

$$A_i \cap A_j \in \mathscr{A}, \qquad 1 \leqslant i < j \leqslant n.$$

Under what conditions on $\mathscr{A}$ must we always have $n \leqslant |\mathscr{A}|$? One such condition is the following.

For $X \in S$, define $c_{\mathscr{A}}(X)$, the *convex hull of* X, by

$$c_{\mathscr{A}}(X) = \bigcap_{\substack{A \in \mathscr{A} \\ A \supseteq X}} A.$$

Thus, $c_{\mathscr{A}}(X)$ is the smallest set in $\mathscr{A}$ which contains X.

Proposition 3. *Suppose $\mathscr{A}$ satisfies*

$$c_{\mathscr{A}}(X) = c_{\mathscr{A}}(Y) \quad \text{and} \quad X \cap Y \in \mathscr{A} \Rightarrow X = Y. \tag{*}$$

Then $n \leqslant |\mathscr{A}|$.

Proof. Replace each A_i by $c_{\mathscr{A}}(A_i)$. By (*), if $i \neq j$, $c_{\mathscr{A}}(A_i) \neq c_{\mathscr{A}}(A_j)$. Since $c_{\mathscr{A}}(A_i) \in \mathscr{A}$, $1 \leqslant i \leqslant n$, then $n \leqslant |\mathscr{A}|$. ∎

Examples. (a) Let $\mathbb{Z}^k$ denote the set of integer points $(z_1, \ldots, z_k)$ in k-dimensional Euclidean space $\mathbb{E}^k$ and let S be a fixed finite subset of $\mathbb{Z}^k$. Let $\mathscr{A}$ denote the family of those subsets $C \subseteq S$ which contain all the lattice points in their ordinary (geometrical) convex hull. Then (*) is satisfied and hence Proposition 3 holds. (b) Similar to a notion arising in the theory of several complex variables, let a compact set $C \subseteq \mathbb{R}^k$ be called *polynomially convex* if for every $\mathbf{y} \notin C$ there exists a real polynomial P so that $P(\mathbf{y}) > \max_{\mathbf{x} \in C} P(\mathbf{x})$. Then again, in this case, (*) is easily verified and Proposition 3 holds.

3. The Problem of Arithmetic Progressions

For this variation, we would like to know how many subsets $A_1, \ldots, A_N$ of $[1, n]$ we can choose so that $A_i \cap A_j$, $i \neq j$, is an arithmetic progression (possibly empty). The answer is given by the following result.

Proposition 4. *If $A_1, \ldots, A_N$ are distinct subsets of $[1, n]$ and for $i \neq j$, and $A_i \cap A_j$ is an arithmetic progression (possibly empty), then*

$$N \leqslant \binom{n}{3} + \binom{n}{2} + n + 1.$$

The only extremal system is the family of all the subsets of $[1, n]$ with at most three elements.

Proof. Let $A_1,\ldots,A_N$ be some extremal system. Let $A_i = \{x_1 < x_2 < \cdots < x_r\}$ be a set with maximal cardinality and suppose $r \geqslant 4$. There are two cases.

(i) Suppose A_i is an arithmetic progression. Define $B = \{x_1, x_2, x_r\}$, $C = \{x_1, x_{r-1}, x_r\}$. If $B \subseteq A_j$ for some $j \neq i$ then $B \subseteq A_i \cap A_j$. But $A_i \cap A_j$ is an arithmetic progression so we must have $A_i \subseteq A_j$, and this contradicts the maximality of A_i (the same argument applies to C). Thus, $B \not\subseteq A_j$, $C \not\subseteq A_j$, $j \neq i$, and so, $|B \cap A_j| < |B| = 3$, $|C \cap A_j| < |C| = 3$. Therefore, $A_1,\ldots, A_{i-1}, B, C, A_{i+1},\ldots, A_N$ is a system of $N + 1$ distinct subsets satisfying the hypothesis of the theorem. This contradicts the maximality of N.

(ii) Suppose A_i is not an arithmetic progression. Then there exists $A' = \{x_k < x_{k+1} < x_{k+2} < x_{k+3}\} \subseteq A$ which is not an arithmetic progression. Form a new family by replacing A_i by A'. The new family still has N distinct sets since $A' = A_j$, $j \neq i$, implies $A_i \cap A_j = A'$ which is impossible. Furthermore, the intersection of any two sets of the new family forms an arithmetic progression (since A' consists of consecutive elements of A). It is easily seen that there must exist two distinct subsets B, C of A' which are not arithmetic progressions. If one of these, say B, is equal to some A_k then $A_i \cap A_k = B$ which as before is impossible. Thus, replacing A' by B and C we have a *larger* family satisfying the hypothesis of the theorem. This contradicts the maximality assumption on N.

Thus, any extremal family must have $|A_i| \leqslant 3$ for all i. Since taking all such sets forms a valid family, the proposition is proved.

Remark 3. The case when the intersection is required to be a *nonempty* arithmetic progression is more difficult and will be described elsewhere. The upper bound in this case is of the form cn^2.

4. An Open Problem

Returning to the problem of convexity, the following problem is of interest.

Suppose S is a "convex" subset of $\mathbb{Z}^k$ (in the sense of Section 2) and let A_i, $1 \leqslant i \leqslant n$, be subsets of S such that for $i \neq j$, $A_i \cap A_j$ is convex and nonempty. Is it true that if the A_i form a maximum such family (i.e., n is as large as possible) then $\bigcap_{1 \leqslant i \leqslant n} A_i \neq \varnothing$?

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REFERENCES

1. N. G. DEBRUIJN AND P. ERDÖS, On a combinatorial problem, *Nederl. Akad. Wetensch. Proc. Ser. A* **51** (1948), 1277–1279.
2. R. A. FISHER, On examination of the different possible solutions of a problem in incomplete blocks, *Ann. Eugenics* **10** (1940), 52–75.
3. M. SIMONOVITS AND V. T. SÓS, Intersection theorems for graphs, I, *in* "Problèmes Combinatoire et Théorie des Graphes," pp. 389–391, CNRS no. 260, 1978.
4. M. SIMONOVITS AND V. T. SÓS, Intersection theorem for graphs, II, "Proc. Int. Conf. on Combinatorics," pp. 1017–1030, Colloq. Math. Soc. János Bolyai, **18**, 1976.

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