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On Partitions of $\mathbb{E}^n$

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INTRODUCTION

Euclidean Ramsey theory [1–3] is that branch of combinatorics which deals with questions of the following type: Which finite subsets C of $\mathbb{E}^n$ have the property that for any partition of Euclidean n-space $\mathbb{E}^n$ into r classes, say $\mathbb{E}^n = C_1 \cup \cdots \cup C_r$, some C_i always contains a subset C' which is the image of C under some Euclidean motion? Such C are said to be *r-Ramsey for* $\mathbb{E}^n$. We usually use alternate "chromatic" terminology and say that C is r-Ramsey for $\mathbb{E}^n$ if any *r-coloring* (=partition into r classes) of $\mathbb{E}^n$ always contains a *monochromatic* copy of C (=image of Euclidean motion of C in one class). If for any r, a monochromatic copy of C always occurs in any r-coloring of $\mathbb{E}^n$ provided only that n is sufficiently large (as a function of r and C), we say that C is *Ramsey.*

This subject is relatively young and many fundamental questions are still unanswered. For example, is any three-point subset C of the plane consisting of the vertices of a nonequilateral triangle 2-Ramsey in the plane? It is known that every C which forms a *right* triangle is (see [10]). It is also known that (see [1]):

(i) If C is Ramsey then C must lie on the surface of some sphere;

(ii) Any subset of the vertices of a rectangular parallelepiped is Ramsey.

Whether the Ramsey sets are exactly characterized by either of these two conditions is still not known.

The subject of this note was suggested by the following question of Gurevich [7]: Must every $(r + 1)$-coloring of $\mathbb{E}^n$ contain in one color the vertices of a k-dimensional simplex with volume 1?

We settle this conjecture by proving a much stronger theorem of this type (see Theorem 4). The heart of the proof rests on a related discrete analog of the conjecture (Theorem 1) together with a general "product" theorem (Theorem 3) for Ramsey sets.

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TRIANGLES IN THE PLANE

We begin by proving the following discrete variation for the plane. Of course, when we say triangle, we mean the vertices of the triangle.

THEOREM 1. *For any r, there exists a positive integer $T(r)$ so that in any r-coloring of the lattice points $\mathbb{Z}^2$ of the plane, there is always a monochromatic right triangle with area exactly $T(r)$.*

Proof. We shall use the well-known theorem of van der Waerden (see [12, 5]) on arithmetic progressions. It asserts that for any k and m, there is an integer $W = W(k, m)$ such that in any m-coloring of the integers $\{1, 2,..., W\}$, there is a monochromatic arithmetic progression of k terms.

Let r be an arbitrary fixed integer. Define integers S_i, $1 \leqslant i \leqslant r$, as follows:

$$S_1 = 1, S_{i+1} = (S_i + 1)!\, W(2(S_i + 1)! + 1, i + 1)!, \qquad 1 \leqslant i < r. \tag{1}$$

Let $T = T(r)$ be defined to be

$$T = S_1 S_2 \cdots S_r .$$

Assume now that the lattice points $\mathbb{Z}^2$ have been r-colored so that no monochromatic right triangle of area T is formed.

Step 0. Let X_r denote the integer $S_{r-1}S_{r-2} \cdots S_2S_1$. Consider the sequence of $W_r = W(2(S_{r-1} + 1)! + 1, r)$ integer points $(iX_r, 0)$, $1 \leqslant i \leqslant W_r$, on the x axis. Since by hypothesis these points have been r-colored then by the definition of W, these points contain a monochromatic arithmetic progression of $2(S_{r-1} + 1)! + 1$ terms, say, $(a_r + ig_rX_r, 0)$, $0 \leqslant i \leqslant 2(S_{r-1} + 1)!$, where g_r is the *gap size* of the progression. Of course, $g_r \leqslant W_r$. We can assume without loss of generality that the points of the progression have color r. Since we have assumed there are no monochromatic right triangles of area T, then certain points of $\mathbb{Z}^2$ now cannot have color r. In particular, no point of the form

$$p_{ij} = \left(a_r + ig_rX_r, \frac{2T}{g_rX_r(S_{r-1} + 1)!}\, j\right),$$

$0 \leqslant i \leqslant (S_{r-1} + 1)!$, $1 \leqslant j \leqslant S_{r-1} + 1$, can have color r. For if it did then using it together with the two color r points

$$p_i = (a_r + ig_rX_r, 0)$$

and

$$p_j = \left(a_r + ig_rX_r + \frac{g_rX_r(S_{r-1}+1)!}{j}, 0\right),$$

a color r right triangle of area T is formed (see Fig. 1). Note that the spacing between adjacent rows in this subarray is

$$\frac{2T}{g_rX_r(S_{r-1}+1)!} = \frac{2W_r!}{g_r}. \tag{2}$$

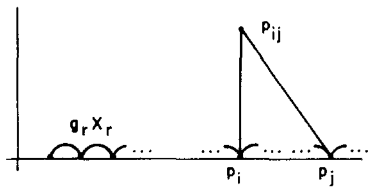

FIGURE 1

Step 1. We first transform the coordinate system slightly, making the bottom row of the array (p_{ij}) the new x axis and the left most column of (p_{ij}) the y axis. In this coordinate system, the points are now given (using (2)) by

$$\left(ig_rX_r, \frac{2W_r!}{g_r}j\right),$$

$$0 \leqslant i \leqslant (S_{r-1}+1)!, \qquad 0 \leqslant j \leqslant S_{r-1}.$$

Let X_{r-1} be defined by

$$X_{r-1} = (S_{r-1}+1)!\, g_rS_{r-2} \cdots S_2S_1.$$

Let us restrict our attention to those points in the base row of the form $(iX_{r-1}, 0)$. Since

$$\frac{X_{r-1}}{g_rX_r} = \frac{(S_{r-1}+1)!}{S_{r-1}},$$

then these are all points in our color r-free subarray for $0 \leqslant i \leqslant S_{r-1}$. But

$$S_{r-1} \geqslant W(2(S_{r-2}+1)!+1, r-1) \equiv W_{r-1}$$

by definition, so that in these points we must find some monochromatic arithmetic progression of $2(S_{r-2}+1)!+1$ terms, say $((a_{r-1}+ig_{r-1})X_{r-1}, 0)$, $0 \leqslant i \leqslant 2(S_{r-2}+1)!$, where the gap size g_{r-1} satisfies $g_{r-1} \leqslant W_{r-1}$ and we can assume without loss of generality that the progression has color $r-1$. As before, this arithmetic progression of color $r-1$ prevents certain points in our subarray from having color $r-1$. In particular, just as before, no point of the form

$$p'_{ij} = \left((a_{r-1} + ig_{r-1})\, X_{r-1}\,, \frac{2T}{g_{r-1}X_{r-1}(S_{r-2}+1)!}\, j\right),$$

where $0 \leqslant i \leqslant (S_{r-2}+1)!$, $1 \leqslant j \leqslant S_{r-2}+1$, can have color $r-1$. Note that since the row spacing in this subarray is

$$\frac{2T}{g_{r-1}X_{r-1}(S_{r-2}+1)!} = \frac{2W_r!}{g_r}\,\frac{W_{r-1}!}{g_{r-1}}$$

then these points p'_{ij} do form a subset of the p_{ij} and therefore they also do not have color r.

Step k. Suppose that at this point we have managed to isolate a subarray of points

$$q_{ij} = \left(ig_{r-k+1}X_{r-k+1}\,, \frac{2W_r!}{g_r}\,\frac{W_{r-1}!}{g_{r-1}} \cdots \frac{W_{r-k+1}!}{g_{r-k+1}}\, j\right),$$
$$0 \leqslant i \leqslant (S_{r-k}+1)!, \qquad 0 \leqslant j \leqslant S_{r-k} \tag{3}$$

which contains none of the colors $r, r-1,..., r-k+1$, where

$$X_{r-k+1} = (S_{r-1}+1)! \cdots (S_{r-k+1}+1)!\, g_r g_{r-1} \cdots g_{r-k+2} S_{r-k} \cdots S_2 S_1$$

and $g_t \leqslant W_t = W(2(S_{t-1}+1)!+1, t)$, $r-k+1 \leqslant t \leqslant r$. Define X_{r-k} by

$$\begin{aligned} X_{r-k} &= (S_{r-1}+1)! \cdots (S_{r-k}+1)!\, g_r \cdots g_{r-k+1} S_{r-k-1} \cdots S_2 S_1 \\ &= X_{r-k+1}(S_{r-k+2}+1)!\, g_{r-k}/S_{r-k+1} \\ &= X_{r-k+1}(S_{r-k}+1)!\, g_{r-k+1}/S_{r-k}\,. \end{aligned}$$

Since

$$\frac{X_{r-k}}{g_{r-k+1}X_{r-k+1}} = \frac{(S_{r-k}+1)!}{S_{r-k}}$$

then the points $(iX_{r-k}\,, 0)$, $0 \leqslant i \leqslant S_{r-k}$, belong to our subarray (q_{ij}). Since

$$S_{r-k} \geqslant W(2(S_{r-k-1}+1)!+1, r-k) \equiv W_{r-k}$$

then the points $(iX_{r-k}, 0)$ contain a monochromatic arithmetic progression of $2(S_{r-k-1}+1)!+1$ terms, say $((a_{r-k}+ig_{r-k})X_{r-k}, 0)$, $0 \leqslant i \leqslant 2(S_{r-k-1}+1)!$, where the gap size g_{r-k} satisfies $g_{r-k} \leqslant W_{r-k}$ and we can assume without loss of generality that the progression has color $r-k$. The locations of these points now imply that none of the points of the subarray of (q_{ij}) given by

$$q'_{ij} = \left((a_{r-k}+ig_{r-k})\,X_{r-k}, \frac{2T}{g_{r-k}X_{r-k}(S_{r-k-1}+1)!}\,j\right),$$

$0 \leqslant i \leqslant (S_{r-k-1}+1)!$, $1 \leqslant j \leqslant S_{r-k-1}+1$, can have color $r-k$. We also see that the spacing between adjacent rows of (q'_{ij}) is

$$\frac{2T}{g_{r-k}X_{r-k}(S_{r-k-1}+1)!} = \frac{2W_r!}{g_r}\,\frac{W_{r-1}!}{g_{r-1}}\cdots\frac{W_{r-k}!}{g_{r-k}}.$$

A simple change of coordinates now results in an array in the form (3) with $r-k$ replacing $r-k+1$.

After completing Step $r-2$, we will finally have formed a rectangular array of points p^*_{ij}, *all of which have color* 1, and which can be written (in an appropriate coordinate system) as

$$p^*_{ij} = \left(ig_2X_2, \frac{2T}{g_2X_2(S_1+1)!}\,j\right),$$

$$0 \leqslant i \leqslant (S_1+1)!, \qquad 0 \leqslant j \leqslant S_1+1,$$

where

$$g_2 \leqslant W_2 \equiv W(2(S_1+1)!+1, 2)$$

and

$$X_2 = (S_{r-1}+1)!\cdots(S_2+1)!\,g_r\cdots g_3S_1.$$

The vertical spacing of the rows of (p^*_{ij}) is

$$\frac{2T}{g_2X_2(S_1+1)!} = \frac{2W_r!}{g_r}\,\frac{W_{r-1}!}{g_{r-1}}\cdots\frac{W_2!}{g_2}.$$

However, the three points

$$p^*_{0,0}, \; p^*_{2,0}, \text{ and } p^*_{0,1}$$

form a color 1 right triangle of area

$$\frac{1}{2}(2g_2X_2)\cdot\left(\frac{2T}{2g_2X_2}\right) = T$$

since $S_1 = 1$, which is a *contradiction*. This proves the theorem. ∎

It is easy to see that the preceding arguments apply to nonrectangular arrays as well. Thus, by suitable scaling we obtain the following result as an immediate corollary of Theorem 1.

COROLLARY. *For any $\alpha > 0$ and any pair of nonparallel lines L_1 and L_2, in any partition of the plane into finitely many classes, some class contains the vertices of a triangle which has area α and two sides parallel to the L_i.*

It is relatively straightforward to extend the previous ideas to prove the corresponding results in $\mathbb{E}^n$, using the Gallai–Witt extension of van der Waerden's theorem to $\mathbb{E}^n$ (see [8, 6]). The reader should have no difficulty in filling in the necessary details for the proof of the following extension.

THEOREM 2. *For any $\alpha > 0$ and any set of n lines $L_1, \ldots, L_n$ which span E^n, $n \geqslant 2$, in any partition of $\mathbb{E}^n$ into finitely many classes, some class contains the vertices of an n-dimensional simplex which has (n-dimensional) volume α and edges through one vertex parallel to the L_i.*

A GENERAL PRODUCT THEOREM

By the corollary we know that for any fixed r-coloring of the plane, there is a monochromatic triangle of area 1 and also a monochromatic triangle of area 2. It is natural to ask whether we can guarantee that the two triangles actually have the *same* color. It turns out that we can and, in fact, much more is true. In this section we prove a general result of this type. To state the theorem we need some terminology.

Let (G, t) be an Abelian group. We say that a family $\mathscr{F}$ of finite subsets of G is a *Ramsey* family if for any partition of G into finitely many classes, there exist $F \in \mathscr{F}$ and $g \in G$ such that

$$F + g \equiv \{f + g : f \in F\}$$

is contained entirely in one of the classes. In other words, in any r-coloring of G, there is a monochromatic *translate* of some $F \in \mathscr{F}$.

THEOREM 3. *Let (G, t) be an Abelian group and suppose $\{\mathscr{F}_\alpha\}_{\alpha \in A}$ is a collection of Ramsey families of G. Then for any partition of G into finitely many classes, say $G = C_1 \cup \cdots \cup C_r$, there are elements $F_\alpha \in \mathscr{F}_\alpha$ and $g_\alpha \in G$ such that for some k, $F_\alpha + g_\alpha \subseteq C_k$ for all $\alpha \in A$.*

Proof. We first show by induction that for any m and any finite collection $\mathscr{F}_{\alpha_1}, \ldots, \mathscr{F}_{\alpha_t}$ of elements of $\{\mathscr{F}_\alpha\}_{\alpha \in A}$, there is a finite set $[\mathscr{F}_{\alpha_1}, \ldots, \mathscr{F}_{\alpha_t}]_m \subseteq G$ such that for any partition of it into m classes, say $[\mathscr{F}_{\alpha_1},..., \mathscr{F}_{\alpha_t}]_m = C_1 \cup \cdots \cup C_m$, some C_k contains a translate of some element of each $\mathscr{F}_{\alpha_i}$.

For $t = 1$ this assertion follows at once from a well known compactness principle (see [9]). Let $\bar{t} > 1$ be fixed and suppose the assertion holds for all $t < \bar{t}$. Of course, the assertion also holds if $m = 1$. Thus, let $\bar{m} > 1$ be fixed and suppose the assertion also holds for $t = \bar{t}$ and all $m < \bar{m}$.

Let $\mathscr{F}_{\alpha_1},..., \mathscr{F}_{\alpha_{\bar{t}}} \in \{\mathscr{F}_\alpha\}_{\alpha\in A}$ be arbitrary and consider the set

$$F^* = [\mathscr{F}_{\alpha_1},..., \mathscr{F}_{\alpha_{\bar{t}-1}}]_{\bar{m}} \cup ([\mathscr{F}_{\alpha_1},..., \mathscr{F}_{\alpha_{\bar{t}}}]_{\bar{m}-1} + [\mathscr{F}_{\alpha_{\bar{t}}}]_{m^*})$$

where $m^* = \bar{m}^{|[\mathscr{F}_{\alpha_1},...,\mathscr{F}_{\alpha_{\bar{t}}}]_{\bar{m}-1}|}$ and for subsets $X, Y \subseteq G$, $X + Y$ is defined as usual to be $\{x + y : x \in X, y \in Y\}$. Consider an $\bar{m}$-coloring of F^*. By the definition of $[\mathscr{F}_{\alpha_1},..., \mathscr{F}_{\alpha_{\bar{t}-1}}]_{\bar{m}}$, for some color, say color k, there are translates of some $F_i \in \mathscr{F}_{\alpha_i}$ with color k for $1 \leqslant i \leqslant \bar{t} - 1$. Now consider the $\bar{m}$-coloring of the other part of F^*, i.e., $[\mathscr{F}_{\alpha_1},..., \mathscr{F}_{\alpha_{\bar{t}}}]_{\bar{m}-1} + [\mathscr{F}_{\alpha_{\bar{t}}}]_{m^*}$. This induces a new m^*-coloring of $[\mathscr{F}_{\alpha_{\bar{t}}}]_{m^*}$ as follows: Two elements $x, x' \in [\mathscr{F}_{\alpha_{\bar{t}}}]_{m^*}$ will have the same "new" color if corresponding points in $[\mathscr{F}_{\alpha_1},..., \mathscr{F}_{\alpha_{\bar{t}}}]_{\bar{m}-1} + x$ and $[\mathscr{F}_{\alpha_1},..., \mathscr{F}_{\alpha_{\bar{t}}}]_{\bar{m}-1} + x'$ have the same original color, i.e., for all $y \in [\mathscr{F}_{\alpha_1},..., \mathscr{F}_{\alpha_{\bar{t}}}]_{\bar{m}-1}$, $y + x$ and $y + x'$ have the same original color. By the definition of m^*, this gives an m^*-coloring of $[\mathscr{F}_{\alpha_{\bar{t}}}]_{m^*}$. Thus, there is a monochromatic translate of some $F \in \mathscr{F}_{\alpha_{\bar{t}}}$, say $F + g$, in the induced coloring. This means that

$$y + f + g \quad \text{and} \quad y + f' + g$$

have the same original color for all $y \in [\mathscr{F}_{\alpha_1},..., \mathscr{F}_{\alpha_{\bar{t}}}]_{\bar{m}-1}$ and all $f, f' \in F$. There are two possibilities:

(i) Suppose some color, say color j, is missing in $[\mathscr{F}_{\alpha_1},..., \mathscr{F}_{\alpha_{\bar{t}}}]_{\bar{m}-1} + F + g$. This induces an obvious j-free coloring of $[\mathscr{F}_{\alpha_1},..., \mathscr{F}_{\alpha_{\bar{t}}}]_{\bar{m}-1}$ and so by definition, each $\mathscr{F}_{\alpha_i}$ has an element F_i with an appropriate monochromatic translate, which is exactly what we wanted to show.

(ii) Suppose all colors occur in $[\mathscr{F}_{\alpha_1},..., \mathscr{F}_{\alpha_{\bar{t}}}]_{\bar{m}-1} + F + g$.

Thus, there is a $y \in [\mathscr{F}_{\alpha_1},..., \mathscr{F}_{\alpha_{\bar{t}}}]_{\bar{m}-1}$, $f \in F$, so that $y + f + g$ has color k. But for all $f' \in F$, $y + f + g$ and $y + f' + g$ have the same color. Thus, all points of

$$y + F + g = F + (y + g)$$

have the color k, that is, this is a translate of $F \in \mathscr{F}_{\alpha_{\bar{t}}}$ with color k. But in the $\bar{m}$-coloring of $[\mathscr{F}_{\alpha_1},..., \mathscr{F}_{\alpha_{\bar{t}-1}}]_{\bar{m}}$ we already have color k translates of $F_i \in \mathscr{F}_{\alpha_i}$ for $1 \leqslant i \leqslant \bar{t} - 1$. Hence, in this case, all the $\mathscr{F}_{\alpha_i}$, $1 \leqslant i \leqslant \bar{t}$, have elements with translates of color k. This shows that the required finite set $[\mathscr{F}_{\alpha_1},..., \mathscr{F}_{\alpha_{\bar{t}}}]_{\bar{m}} = F^* \subseteq G$ exists. The general assertion now follows by induction on t and m.

To complete the proof of the theorem, assume that it fails for some value r. Thus, there is an r-coloring of G, say, $G = C_1 \cup \cdots \cup C_r$ such that for each i there is an $\mathscr{F}_{\beta_i} \in \{\mathscr{F}_\alpha\}_{\alpha\in A}$ so that for all $F_i \in \mathscr{F}_{\beta_i}$, $g_i \in G$,

$$F_i + g_i \not\subseteq C_i, \qquad 1 \leqslant i \leqslant r.$$

But by the preceding paragraph G contains a (finite) subset $[\mathscr{F}_{\beta_1},..., \mathscr{F}_{\beta_r}]_r$. Hence, in the given r-coloring of G, some C_k contains translates of some element $F_i' \in \mathscr{F}_{\beta_i}$ for each i, in particular, for $i = k$. However, this is a contradiction and the theorem is proved. ∎

To apply this result to our Euclidean situation, we take $(G, +)$ to be the additive group of $\mathbb{E}^n$ viewed as a vector space over $\mathbb{R}$. By Theorem 2, we can take the $\mathscr{F}_\alpha$ to be $\mathscr{F}_{\alpha;L_1,...,L_n}$, the family of all sets of $(n+1)$-tuples in $\mathbb{E}^n$ which form a simplex having (n-dimensional) volume $\alpha > 0$ and edges through one vertex parallel to the L_i. Applying Theorem 3, we obtain the following.

THEOREM 4. *For all partitions of E^n into finitely many classes, some class contains, for all $\alpha > 0$ and sets of lines L_1 ,..., L_n which span $\mathbb{E}^n$, a simplex having volume α and edges through one vertex parallel to the L_i.*

CONCLUDING REMARKS

It is not known to what extent the preceding results can be generalized to configurations which are not simplexes. For example, does Theorem 1 apply to parallelograms (i.e., the four vertices of a parallelogram)? Rhombuses? Rectangles? of course, it is easy to see that it cannot hold for squares.

The same questions can be raised about other constrained configurations rather than those with a given area. Not too much is known here. It has been noted by E. G. Straus that Theorem 1 cannot hold for triangles with a prescribed *perimeter*. For if it did, then the corresponding corollary would have to hold for perimeter 1, which, however, is clearly false (as can be seen by coloring the plane in a multicolored cyclic "checkerboard" pattern with squares of side $\frac{1}{10}$).

There is a *density* version of Theorem 1 which can be proved, based on using Szemerédi's theorem [11] in place of van der Waerden's theorem. This density version asserts that for any $\epsilon > 0$, there is an integer $T(\epsilon)$ so that if $n \geqslant n(\epsilon)$ and $R \subseteq \{(i,j) : 1 \leqslant i, j \leqslant n\}$ with $|R| > \epsilon n^2$, then R contains the vertices of a triangle of area $T(\epsilon)$. We will not discuss those directions in this note, however. The approximate continuous analog is an old theorem of Erdös [4]. It asserts that if X is an unbounded set with positive measure in the plane then X contains the vertices of triangles of all areas.

Finally, it would be interesting to have better estimates for the function $T(r)$ of Theorem 1. Does it really have the hyperexponential growth associated with the Ackermann function type upper bounds on $W(k, r)$ or is it actually more modest in its behavior? The best lower bound on $T(r)$ currently known (based on cyclic colorings of $\mathbb{Z}^2$) is

$$T(r) \geqslant \tfrac{1}{2} \text{ l.c.m. } (2,..., r) = e^{(1+o(1))r}.$$

Note added in proof. Since submitting this note, a much more general version of Theorem 3 has been found and has just appeared in [13].

Also added in proof. K. Kunen has just shown that under the Continuum Hypothesis it is possible to color the plane with $\aleph_0$ colors so that every non-degenerate triangle determined by three monochromatic points has an irrational area.

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