# ON PROPERTIES OF A WELL-KNOWN GRAPH OR WHAT IS YOUR RAMSEY NUMBER?

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## Introduction

During the past ten years various authors (see e.g., [1, 2, 3, 5]) have investigated properties of a certain graph $G$, often called the *collaboration graph of mathematicians* (see Figure 1). In this note we explore a number of questions concerning $G$ and suggest numerous new open problems, not only for $G$ but also for a variety of related graphs.

## Notation and Definitions

Strictly speaking, the collaboration graph $G$ is really a hypergraph. The *vertex set* $V(G)$ of $G$ is usually taken to be the set of mathematicians. A subset $X \subseteq V(G)$ is an *edge* of $G$ iff there exists a published paper (or book)* which has $X$ as its set of authors. However, for the purposes of this note, we consider $G$ as a graph by replacing each (hyper)edge $X$ by the complete graph $K(X)$ on $X$. In accordance with the usual conventions for this topic, at most one edge in $K(X)$ will be allowed in any solution of the various extremal problems for $G$ we will consider. We should note that occasionally the restricted graph $G^* \subseteq G$ is considered in which only papers having exactly two authors are allowed to generate edges [1].

We should say a few words about the notation used in the figures. A solid edge between $A$ and $B$ usually indicates that a paper having $A$ and $B$ as authors has appeared or is about to appear. A dotted line indicates that the paper in question will probably appear (perhaps at present is just a technical report). Other various notations will be described in the text.

## The Main Result

### *Some Comments*

1. It has long been known that $G$ is nonplanar. This seems to have first been noted by Schinzel (see [1]), who found a $K_{3,3}$ consisting of the two vertex sets {Chowla, Mahler, Schinzel} and {Davenport, Erdös, Lewis}. (In fact, this $K_{3,3} \subseteq G^*$.) However, a brief examination shows that $K_5$ is also a subgraph of $G$, the vertex set being {Erdös, Graham, Rothschild, Spencer, Straus}. In forming this $K_5$, it certainly did not hurt that these five authors are actually part of three 6-tuple papers (each of which however is allowed to contribute only one edge to the $K_5$).

* Various conventions are possible with regard to abstracts, technical reports, memoranda, lecture notes, correspondence, etc. However, we shall follow none of them.



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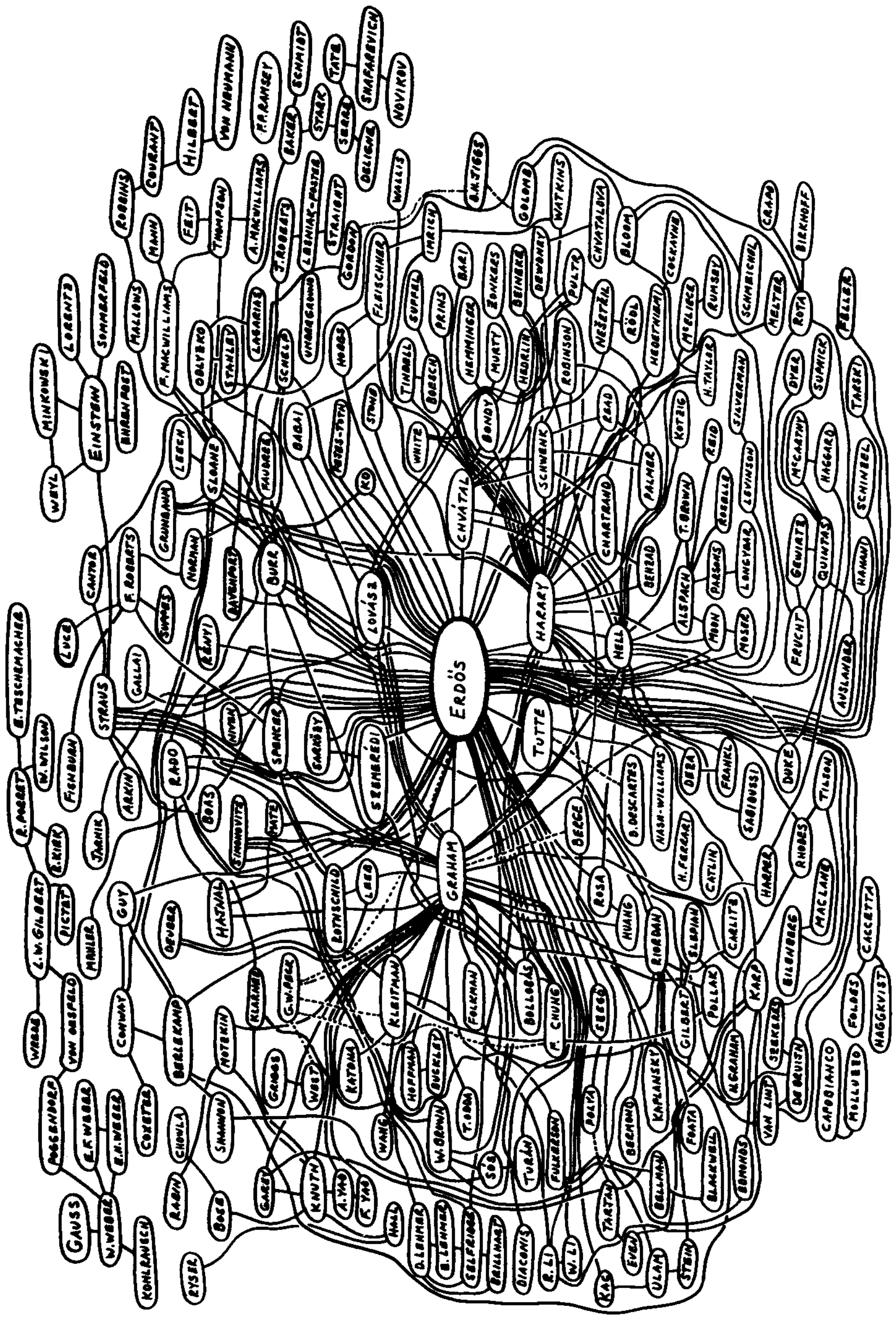
ERDÖS
GRAHAM
HARARY
TUTTE
LOVÁSZ
CHVÁTAL
BURR
SPENCER
SZEMERÉDI
HELL
EINSTEIN
GAUSS
HILBERT
VON NEUMANN
KNUTH
KLEITMAN
CONWAY
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ULAM
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CHARTRAND
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QUINTAS
F. ROBERTS
HAJNAL
BERLEKAMP
CHUNG
BOLLOBÁS
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WEYL
LORENTZ
SOMMERFELD
COURANT
COXETER
STEIN
DIACONIS
KAC

FIGURE 1.

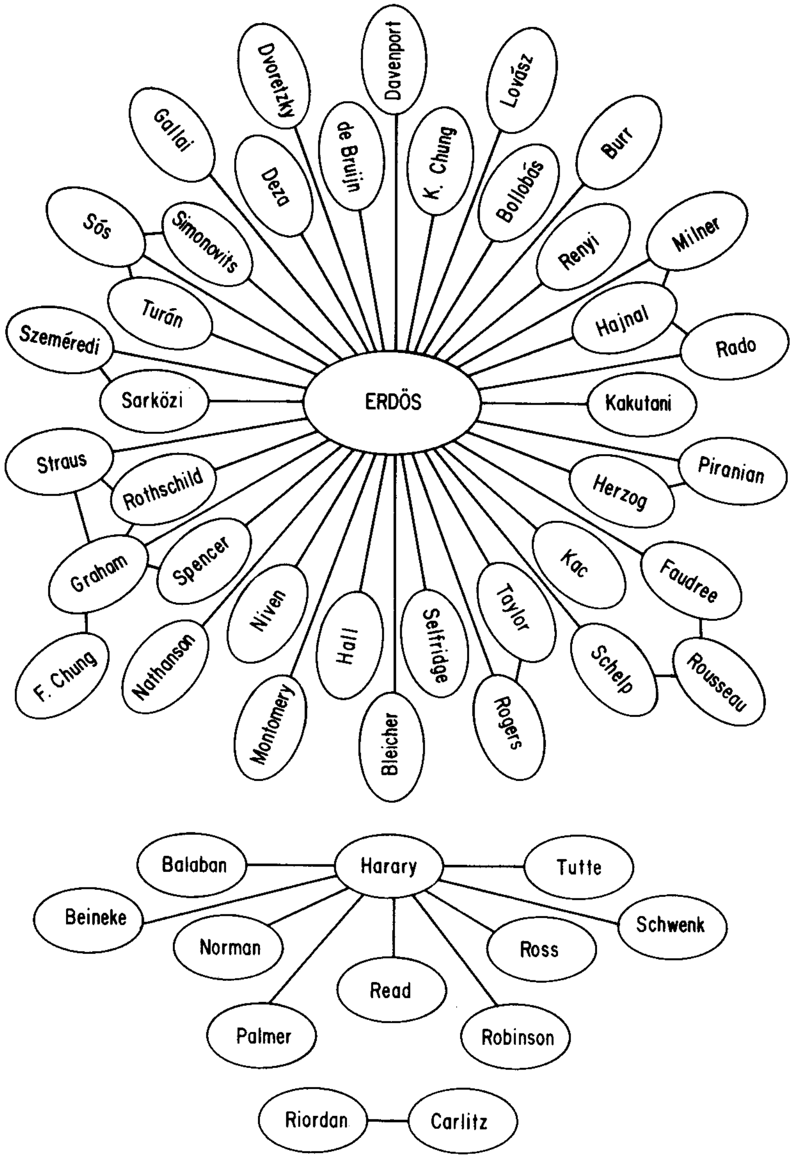


FIGURE 2. A portion of $G_5$.

2. D. West is the only known person to have a purely *imaginary* Erdös number,† where the Erdös number of a vertex $V$ is defined to be length of a shortest path from $V$ to Erdös (if its exists). This comes about because of the recent paper of G. W. Peck [6]. The actual authors are {Chung, Erdös, Graham, Kleitman, Purdy, West}. This has a precedent in the well-known paper of B. H. Jiggs [4], whose actual authors were {Baumert, Golomb, Gordon, Hales, Jewett, Selfridge}. The $i$ was imaginary (as it should be). As a consequence, West's coauthor, Griggs, has Erdös number $1 + i$.
3. It was pointed out by Erdös [1] that $G^*$ should actually be written as $G^{(t)*}$ since its current state depends on the time $t$ at which it is considered.

   Erdös went on to conjecture that, for any $r$, there is a time $t(r)$ so that $G^{(t(r))*}$ will contain a complete graph $K_r$ on $r$ vertices. However, there may be some reason for doubting the validity of this conjecture because of the following observations. If a person $P$ writes one (joint) paper per day for (say) eighty years, he/she will still only have written less than 30,000 papers. Of course, he/she may write a greater rate than one paper per day and may be productive for more than eighty years, but in any case, it seems reasonably safe to say that (at least in the foreseeable future) no one will write 100,000 papers, each with a different coauthor. If $G^{(t)*}$ were ever to contain $K_{100,000}$, a complete graph on 100,000 points, we would need 99,999 people doing this! Thus, it appears somewhat unlikely that $K_{100,000}$ will ever be a subgraph of $G^{(t)*}$, for any $t$.

   However, there is no obvious reason why $K_r$ could not be a subgraph of $G^{\omega} = \bigcup_{t \geq 0} G^{(t)}$ for any fixed value of $r$. All that is needed for $K_r \subseteq G^{\omega}$ is that we have an $r$-tuple-authored paper. For trends in this direction (in physics), see [7].
4. The vertex {Thompson} is the only known example of a person connected to both a woman (F. MacWilliams) and her daughter (A. MacWilliams). Curiously, no corresponding example for a father and son is known (to the author).
5. There is still no 4-set of vertices $X$ known in $G$ for which every nonempty subset of $X$ has a joint paper [1]. For the weak solution (every subset of $X$ occurs in the set of authors of a distinct paper), the closest example known is {Erdös, Graham, Rothschild, Straus} for which fourteen of the fifteen required papers exist.
6. An interesting question concerning $G$ which has never been investigated concerns its robustness. That is, suppose we consider the subgraph $G_k \subseteq G$ in which $A$ and $B$ have an edge between them only if they have at least $k$ joint papers. In FIGURE 2 we show a portion of $G_5$. In FIGURE 3 we show a portion of $G_{25}$.
7. A related graph is the *joint book graph* $B$ (whose name is self-explanatory). The strict interpretation does not allow jointly edited books or contributed chapters in a book to count. A portion of $B$ is shown in FIGURE 4.

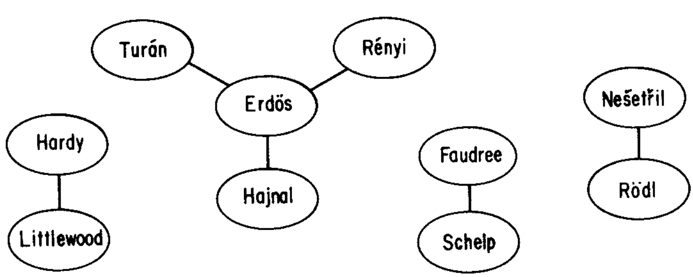


FIGURE 3. A portion of $G_{25}$.

† This was first noted by Kleitman.

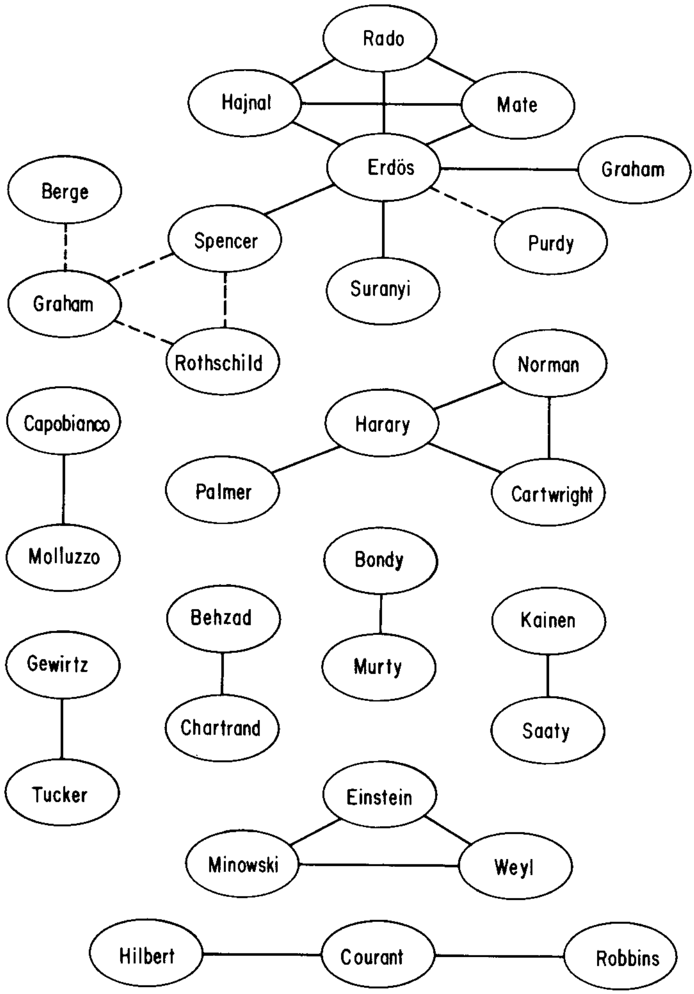


FIGURE 4. A portion of $B$.

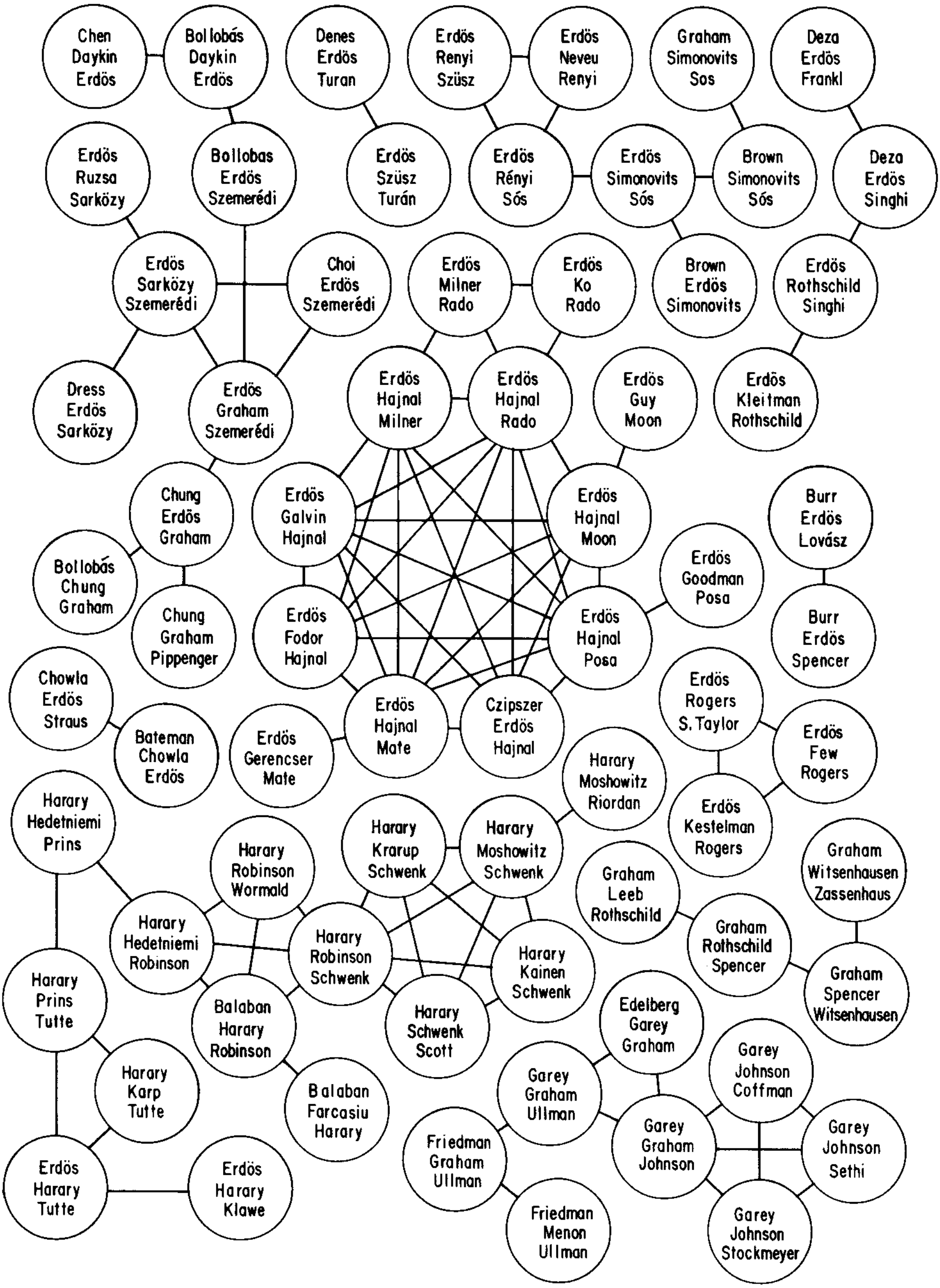


FIGURE 5. A portion of $G_{STP}$.

8. An interesting variation is to consider the strict triple paper graph $G_{STP}$. The vertices of $G_{STP}$ are sets of three people which exactly form the set of authors of a published paper. Two such vertices are joined by an edge if they have two people in common. We show a portion of $G_{STP}$ in FIGURE 5. It is actually somewhat surprising that $G_{STP}$ has such large components.
9. Just as in the case of the Erdös number of a vertex $V$, one can just as well ask about the Hilbert number, the Einstein number, the Gauss number, or even the Ramsey number‡ of $V$. However, it is not yet known whether Gauss or Ramsey are actually connected to Erdös in $G$. It may be reassuring to know however that everyone in the largest component of $G$, shown in FIGURE 1, at least has a Rumsey number. We also note, in passing, that Tarjan has an odd Even number, and an even Odda number.

## CONCLUDING REMARKS

We hope we have given the reader a feeling for the main ideas of this subject. Of course, the variety and depth of future results in this area are limited only by the ingenuity and persistence of the investigators.§ We look forward with anticipation to the great strides which will (unfortunately) inevitably occur in the field and hope that this modest effort has made some small contribution to filling a much needed gap in the literature.

## ACKNOWLEDGMENTS

The author wishes to acknowledge here the various suggestions of S. A. Burr, F. R. K. Chung, P. Erdös, A. Gewirtz, R. L. Graham, L. V. Quintas, E. G. Straus, and M. Van Ness in spite of whose comments this note was written.

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‡ This should not be confused with the generalized Ramsey numbers referred to by Burr in his paper [0] in this Volume. On the other hand, perhaps it should.

§ No doubt other qualities are also helpful.