Maximum Antichains in the Partition Lattice

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Many questions in combinatorics deal with collections of subsets of a finite set. One of the most basic results of this type, first proved by E. Sperner [16] in 1928, asserts the following: If $S_1, S_2, \ldots, S_t$ are subsets of an n-element set S such that no S_i is a subset of any other S_j then

$$t \leqslant \binom{n}{[\frac{n}{2}]}$$

Furthermore, this bound on t can only be achieved by taking the S_k to be all the $[\frac{n}{2}]$ -element subsets of S, or, when n is odd, by taking all the $([\frac{n}{2}]+1)$-element subsets.

Sperner's theorem, like Schur's work [15] on the solutions of $x^m + y^m = z^m$ van der Waerden's theorem [17] on arithmetic progressions, Ramsey's fundamental result [12] on partitions of the subsets of a set, and Polya's approach [11] to the theory of enumeration, has been the seed from which a major branch of combinatorial theory has grown during the past 50 years. This branch, often called extremal set theory, has been especially active during the past 10 years. In particular, one of the outstanding open problems, which was first raised by G.–C. Rota nearly 15 years ago and which was responsible for much of this activity, has just been settled within the past year by E. Rodney Canfield of the University of Georgia. What is even more intriguing is that Canfield showed that the answer that everyone had expected (and was trying to prove) was wrong*.

In this note I would like to give a brief sketch of the background of Rota's problem and its resolution.

By a chain C in a (finite) partially ordered set P we mean a totally ordered subset of P; the length of C is just the number of elements in it. We say that P is *graded* if P has a unique minimal element 0 and for every $p \in P$, all maximal chains from 0 to p have the same length, called the *rank* of p. We denote the elements of P having rank k by R_k, also called the k^{th} level. By an *antichain* in P we mean a subset of mutually incomparable elements of P. For example, for any k the set R_k forms an antichain.

*This often explains the difficulty encountered in trying to prove a result.

What Sperner's theorem says is that for $P = \mathcal{S}_n$, the class of subsets of an n-element set S partially ordered by inclusion, the largest antichain must be of this form for the choice of k which maximizes $|R_k|$ (in this case $k = [\frac{n}{2}]$).

It is natural to ask to what extent this result holds for more general graded partially ordered sets. In particular, G.–C. Rota [14] raised the question of whether this is true for P_n, the lattice of partitions of an n-element set. The elements of P_n of rank k, denoted by R(n,k), are the partitions of a fixed n-element set, say $\{1,2,\ldots,n\}$, into k nonempty blocks. We say that the partition π *refines* the partition π' if each block of π is contained in a block of π'. P_n is partially ordered by refinement. In Fig. 1 we show P_4.

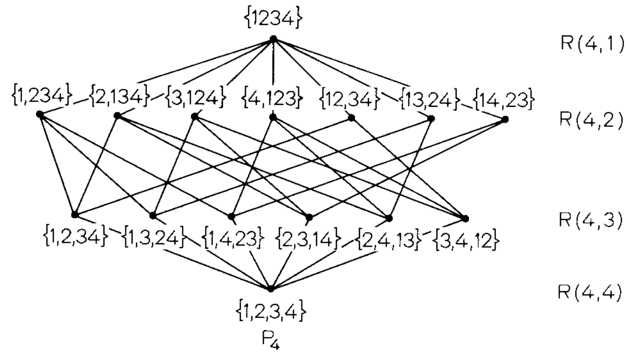

Figure 1

The number of elements of rank k is denoted by S(n, k). These numbers are known as the Stirling numbers of the second kind [13]. They satisfy the simple recurrence

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

and although they have been known for hundreds of years, there are still some surprising gaps in our knowledge about them. For example, it is not even known whether $S(n, k) = S(n, k+1)$ can ever hold for $n > 2$ (it can be shown that $S(n,k) = S(n,k+1) = S(n,k+2)$ is impossible). It is known however that the S(n, k) are *unimodal*, i.e., $S(n, k) - S(n, k+1)$ has only one change of sign as k goes from 1 to $n-1$. It follows from this that Rota's question is equivalent to showing that between any two consecutive levels R(n, k) and R(n, k+1) of P_n, one can always find what is called a *matching*. This is simply a one-to-one pairing of each partition π of the smaller of the two levels with a distinct *comparable* partition π' of the larger of the two levels. For example, one such matching between R(4,2) and R(4,3) in Fig. 1 is given by:

$$\{1,234\} - \{1,3,24\}, \quad \{2,134\} - \{1,2,34\},$$
$$\{3,124\} - \{2,3,14\}, \quad \{4,123\} - \{1,4,23\},$$
$$\{12,34\} - \{3,4,12\}, \quad \{13,24\} - \{2,4,13\}.$$

It is easy to see that the existence of these matchings implies that the maximum antichains of P_n have $\max_k S(n,k)$ elements. For if we have an alleged maximum antichain A having more than $\max_k S(n,k)$ elements, say A has elements below the largest level, simply replace each of A's lowest rank elements π by the (comparable) partition π' paired with it in the adjacent level above (a similar argument applies if A has elements above a maximum level). The new set A' is still an antichain, has as many elements as A and has its elements in one less level. Using the unimodality of the S(n, k), we can by induction find an antichain belonging entirely to a largest level, which is a contradiction.

Thus, Rota's problem is reduced to deciding whether there is always a matching between R(n,k) and R(n, k+1). The most well-known tool for proving the existence of matchings is the so-called Marriage Theorem of P. Hall [8]. This result applies to the following situation. Suppose A and B are sets of men and women, respectively, and each man $a \epsilon A$ is acquainted with a set of women $T(a) \subseteq B$. We can conveniently represent this by a bipartite graph where an edge between a and b indicates acquaintanceship (and we assume without loss of generality that $|A| \leqslant |B|$). Then a necessary and sufficient condition that each man be able to marry a women he knows is that for all k, each set of k men know altogether at least k women. In other words, there is a matching between A and B iff for all $X \subseteq A$,

$$\left| \bigcup_{x \in X} T(x) \right| \geqslant |X| .$$

Unfortunately, unless there is a fair amount of regularity in the bipartite graphs in question, the use of Hall's theorem can be impractical in proving the existence of matchings (since it requires verifying $2^{|A|}$ conditions).

Other approaches to Rota's problem during the past 10 years resulted in the introduction of a variety of fruitful concepts and significant results into matching theory, such as the idea of a normalized matching [5], Harper's product theorem [9], and the far-reaching work of Greene and Kleitman [6], [7] and Katona [10], but the original problem itself still stood relatively untouched. It was shown that any antichain in P_n had at most $\max_k S(n, k)$ elements for $n \leqslant 20$ (and even this was not completely trivial because of the size of the S(n, k)'s, e.g., S(20,8) = 15170932662679) and it was generally believed this would continue to hold for all n.

Thus, it was quite unexpected when Canfield showed that there are antichains in P_n having many *more* than $\max_k S(n, k)$ elements when n becomes very large.

Basically, his technique for proving this involved:

(i) determining to within one the value k_n for which $S(n, k_n) = \max_k S(n, k)$;

(ii) defining a special class C of partitions in the level $R(n,k_n\text{-}1)$; more specifically, C consists of all $\pi \epsilon R(n,k_n\text{-}1)$ having exactly t blocks of size $\leqslant$ m and the remaining $k_n - t - 1$ blocks of size $>$ m and $\leqslant$ 2m, where t and m are appropriately chosen;

(iii) obtaining precise estimates of the size of C and the size of span (C), defined to be the set of all $\pi' \epsilon R(n, k_n)$ which are refinements of some $\pi \epsilon C$; this is done by using local limit theorems and the Berry-Esséen inequality for the estimation of coefficients of certain polynomials;

(iv) using these estimates to show that $|\text{Span}(C)| < |C|$.

Thus

$$R(n,k_n) - \text{Span}(C) \cup C$$

is an antichain in P_n with more than $S(n, k_n)$ elements (see Fig. 2). The details can be found in [1], [2], [3], [4].

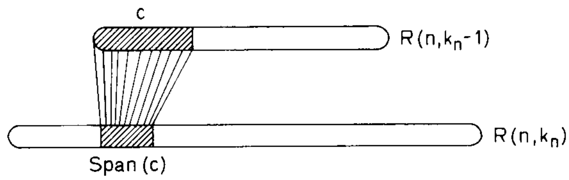

Figure 2

One might well ask just where these large antichains are located in P_n and, in particular, what the least n is for which P_n has antichains with more than $S(n, k_n)$ elements.

As to the first question, Canfield has shown that the maximum antichains A_n in P_n for large n must contain elements in levels quite far from the largest level $R(n,k_n)$. In fact, for any $\epsilon > 0$, A_n must have elements in both $R(n, k_n+t_1)$ and $R(n,k_n-t_2)$ for some $t_1, t_2 > \sqrt{n/(\log n)}^{\,1+\epsilon}$ when n is sufficiently large.

As to the second question, Canfield estimates that his techniques will start working at about $n = 6.526 \times 10^{24}$. At this value of n, $S(n, k_n)$ can be enormous, e.g., exceeding $10^{10^{20}}$.

Thus, it is conceivable that we will never know the first time P_n has an antichain with more than $\max_k S(n, k)$ elements!

REFERENCES

[1] E. Rodney Canfield, On a problem of Rota, Bull. Amer. Math. Soc., 84 (1978), p. 164.

[2] , Application of the Berry-Esséen inequality to combinatorial estimates, (1977) (preprint).

[3] , On the location of the maximum Stirling number(s) of the second kind, (1977) (preprint).

[4] , On a problem of Rota, (1977) (preprint).

[5] R. L. Graham and L. H. Harper, Some results on matching in bipartite graphs, SIAM Jour. on Appl. Math., 17 (1969), 1017–1022.

[6] C. Greene, Sperner families and partitions of a partially ordered set, Combinatorics, Math. Centre Tract 56, (1974) Amsterdam, 91–106.

[7] C. Greene and D. J. Kleitman, On the structure of Sperner k-families, Jour. Comb. Th. (A), 20 (1976), 41–68.

[8] P. Hall, On representatives of subsets, Jour. London Math. Soc., 10 (1935), 26–30.

[9] L. H. Harper, The morphology of partially ordered sets, Jour. Comb. Th. (A), 17 (1974), 44–58.

[10] G. O. H. Katona, Extremal problems for hypergraphs, Combinatorics, Math. Centre Tract 56, (1974) Amsterdam 13–42.

[11] G. Pólya, Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen, Acta Math. 68 (1937) 145–254.

[12] F. P. Ramsey, On a problem of formal logic, Proc. London Math. Soc., 2nd ser., 30 (1930) 264–286.

[13] J. Riordan, An introduction to combinatorial analysis, Wiley, New York, 1958.

[14] G. –C. Rota, Research problem 2-1, Jour. Comb. Th. 2 (1967), p. 104.

[15] I. Schur, Über die Kongruenz $x^m + y^m = z^m$ (mod p) Jahrb. Deutsch Math. – Verein. 25 (1916) 114 – 116

[16] E. S. Sperner, Ein Satz über Untermengen einer endlichen Menge, Math. Z. 27 (1928), 544–548.

[17] B. L. van der Waerden, Beweis einer Baudetschen Vermutung, Nieuw Archief voor Wiskunde, 15 (1927) 212–216.

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