RAMSEY THEORY

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INTRODUCTION

In 1930, F. P. Ramsey [27] proved a remarkable theorem as part of his investigations in 'formal logic'. The theorem is a profound generalization of the 'pigeon hole principle' or 'Dirichlet box principle'. As is the case with many beautiful ideas in mathematics, Ramsey's Theorem extends just the right aspect of an elementary observation and derives consequences which are extremely natural although far from obvious. Recently it has been recognized that many results in combinatorial theory and other areas have the same flavor as Ramsey's Theorem, and the attempt to capture this common flavor, and to develop some general ideas based on it, has led to a proliferation of results which consitute what we describe here as 'Ramsey Theory'. In many cases, including the original theorem itself, the existence of certain numbers is asserted. A large effort has gone into finding exact values and bounds for these 'Ramsey numbers'.

We shall try here to describe some interesting aspects of the subject. We will of necessity leave out many more specialized results as well as all proofs, with the exception of several simple ones which we include at the end of the chapter. Rather we will try to emphasize the general ideas of Ramsey Theory, hopefully sacrificing completeness for the sake of palatability.

We now describe a certain very general class of theorems which we shall call Ramsey theorems. In contrast to many subjects, here the most general formulation is by far the most elementary and immediate.

A *bipartite graph* G is a graph with its vertices divided into two classes A and B, such that each edge of G has one vertex in A and one vertex in B. If the vertices of A are partitioned into r classes (called *colors*) then a vertex b in B is called *monochromatic* if all vertices a in A which are adjacent to b lie in a single class (where possibly r is infinite). G is said to have the Ramsey property for r colors, or to be *r-Ramsey*, if for every partition of A into r or fewer parts (called *r-colorings*), there is a monochromatic vertex in B (see Figure 1). The entire field of Ramsey Theory basically can be thought of as the attempt to decide which graphs are r-Ramsey.

Although this general formulation is very simple, and all Ramsey theorems can be expressed in this common form, more intuitively convenient formulations are used for individual cases. Below we give a few of the more appealing examples of Ramsey theorems.

SOME EXAMPLES

One of the earliest of the Ramsey theorems is the following theorem of Schur [28]:

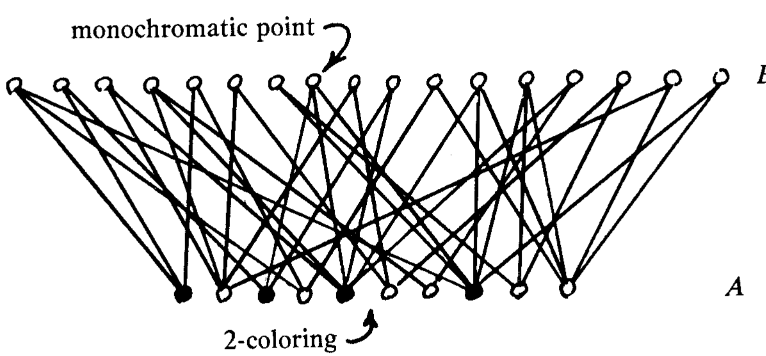

FIG. 1.

Example 1. For every positive integer r, there is an $N = N(r)$ such that if the set $[1, N]$ (integers x, with $1 \leq x \leq N$) is r-colored, then there must exist $x, y, z \in [1, N]$ all having the same color and satisfying $x + y = z$.

In terms of bipartite graphs, we define a graph G as follows. Let $A = [1, N]$, $B = \{\{x, y, z\}: x + y = z,\ x, y, z \in [1, N]\}$, and for $a \in A$, $b \in B$, let $\{a, b\}$ be an edge of G iff $a \in b$. That the bipartite graph G is r-Ramsey is Schur's theorem.

An example of a result not usually thought of as a Ramsey theorem is the Baire Category Theorem.

Example 2. Let A be the set of points of a complete metric space X. Let B be the set of subsets $S \subseteq X$ such that the closure $\overline{S}$ contains an open set of X. For $a \in A$, $b \in B$, let $\{a, b\}$ be an edge iff $a \in b$. Then this graph is $\aleph_0$-Ramsey.

This is just the statement that a complete metric space is not the countable union of nowhere dense sets. Of course, even though this theorem is of the Ramsey form, it is not really a combinatorial theorem.

The next example is Ramsey's original theorem.

Example 3. Let A be the set of all k element subsets of a countable set S. Let B be the set of all infinite subsets of S. For $a \in A$, $b \in B$, let $\{a, b\}$ be an edge iff $a \subseteq b$. Then this graph is r-Ramsey for all positive integers r.

Ramsey also proved a finite version of this theorem (where B is the set of all l element subsets of a sufficiently large finite set S') which we treat in the next section. We remark here that the reader is probably already familiar with the simplest case of this theorem, namely, that if all the edges of the complete graph on 6 vertices are 2-colored, then some monochromatic triangle must be formed. To see this, simply observe that for any vertex v there must be three vertices v_1, v_2, v_3 with all edges $\{v, v_i\}$ having the same color. If any edge $\{v_i, v_j\}$ also had this color, we are done. On the other hand, if all the edges $\{v_i, v_j\}$ have the opposite color, we are also done.

Example 4. Let A be the set of all subsets of cardinality 2 of a set S of cardinality $2^{\aleph_0 +}$. (If α is a cardinal, α^+ denotes the next largest cardinal.) Let B be the set of all subsets of S of cardinality $2^{\aleph_0}$. For $a \in A$, $b \in B$, let $\{a, b\}$ be an edge iff $a \subseteq b$. Then this graph is 2-Ramsey.

This is a theorem of Erdös. Replacing $2^{\aleph_0 +}$ by $2^{\aleph_0}$, Dushnik and Miller [7] show that the resulting graph is not 2-Ramsey. These results belong to a vast and still growing literature on Ramsey theorems for large cardinals, ordinals and order types. The assertions that certain graphs of this sort are Ramsey sometimes turn out to be independent of the usual axioms for set theory [10].

Example 5 (van der Waerden [30]). Let r and k be positive integers. Let $A = [1, W]$ and let B be the set of all k term arithmetic progressions in $[1, W]$. For $a \in A$, $b \in B$, let $\{a, b\}$ be an edge iff $a \in b$. Then there is a function $W(k, r)$ such that if $W \geq W(k, r)$ then the graph is r-Ramsey.

This theorem asserts that if we r-color a sufficiently large interval of integers, it must contain a monochromatic arithmetic progression of k terms. In fact, an awesome result has recently been proved by E. Szemerédi [29], settling a 40 year old conjecture of Erdös and Turán. Namely:

If R is a subset of the positive integers with positive upper density, i.e.,

$$\limsup_N \frac{|R \cap [1, N]|}{N} > 0,$$

then R contains arbitrarily long arithmetic progressions.

This theorem is equivalent to a deceptively simple strengthening of van der Waerden's Theorem. It is the statement that for Example 5, not only is there a monochromatic progression of length k, but in fact there must be such a progression having that color which occurs most frequently in $[1, W]$.

CATEGORIES

Until now we have considered individual graphs and their Ramsey properties. However, many of the most important theorems and their proofs, including Ramsey's Theorem itself, require the consideration of whole families of graphs. K. Leeb [21] pointed out that in such situations the use of category theory can be quite helpful both in the formulation and in the proofs of results. The introduction of categorical methods was initially inspired by the attempt to prove the insightful conjecture of G.-C. Rota, namely, that the analogue to Ramsey's Theorem for finite vector spaces holds. It is now known that certain categories are "Ramsey". These include the category of sets (Ramsey's Theorem) and the category of finite vector spaces (Rota's conjecture).

In their usual formulation these theorems are as stated below. The more formal statements in terms of categories will be given at the end of this section, where we state a general theorem for certain categories. These categories include the two above.

THEOREM [27]: *For all k, l, r, there exists a least integer $n(k, l, r)$ such that if $n \geq n(k, l, r)$ and the set of k-subsets of an n-set N is arbitrarily r-colored, all the k-subsets of some l-subset of N have a single color.*

THEOREM [14]: *For all k, l, r, there exists a least integer $n_q(n, l, r)$ such that if $n \geq n_q(n, l, r)$ and set of k-dimensional subspaces of an n-dimensional space N over $GF(q)$ is arbitrarily r-colored, then all the k-dimensional subspaces of some l-dimensional subspace have a single color.*

There are categories, however, for which the Ramsey property holds only in part. The proofs of the validity of the Ramsey property for the appropriate parts of these categories are among the nicest in the theory. For example, van der Waerden's theorem is one such case.

Consider the category P whose objects are the finite arithmetic progressions of the positive integers. Let the morphisms be the monomorphic affine maps from one progression to another (i.e., $x \to ax + b$). The Ramsey property for this category would assert: For every k, l, r, there is an $n = n(k, l, r)$ such that if the k term progressions in $[1, n]$ are r-colored, then all the k term subprogressions of some l term progression have the same color. For $k = 1$, this is just van der Waerden's theorem. However, for $k = 2$, it is false. To see this, consider the following 2-coloring of the two term arithmetic progressions:

$$c(\{x_1, x_2\}) = \alpha^* \text{ where } \alpha^* \equiv \alpha (\text{mod } 2),\ \alpha^* = 0 \text{ or } 1$$

and 2^α is the largest power of 2 dividing $x_2 - x_1$.

Another category for which the partial validity of the Ramsey property holds is the category G of finite, undirected graphs for which the morphisms are induced subgraph embeddings. φ is such an embedding if it is an injective mapping taking vertices of a finite graph L into vertices of a finite graph N and edges of L into edges of N such that for any two vertices u, v, of L, $\{u, v\}$ is an edge of L iff $\{\varphi(u), \varphi(v)\}$ is an edge of N. The Ramsey property for this category then says that for any graphs K, L and any integer r, there is a graph N such that for every r-coloring of the K-subgraphs of N (i.e., induced subgraphs isomorphic to K), there is an L-subgraph with all of its K-subgraphs having the same color. For the case in which K is a single vertex (and L is arbitrary), the Ramsey property holds by a result of Folkman [13]. In the case where K consists of a single edge, the Ramsey property also holds. This is a powerful new result of Deuber [6]. If K is a complete graph on k vertices and L is a complete graph on l vertices, then the Ramsey property becomes just the statement of Ramsey's Theorem.

Finally, if we consider the category G_n of graphs with no complete subgraph on n vertices (and the same morphisms as before), then the Ramsey property holds if K is a single vertex or if K is a single edge. The first result here is due to Folkman [13]. The second result is a major new result of Nešetřil and Rödl [23].

However, just as in the case of arithmetic progressions, easy counterexamples show that the Ramsey property does not hold in general in these categories. For example, in the category G, let K

be P_3, the tree on three vertices and let L be the 4-cycle C_4. Let N be an arbitrary graph on n vertices labelled with the elements of $[1, n]$. Color the tree $\underset{x}{\bullet}\!-\!\underset{y}{\bullet}\!-\!\underset{z}{\bullet}$ red if $y > \max(x, z)$ and blue otherwise. Clearly, no C_4 in N can have all four of its subgraphs P_3 with the same color. Similar counterexamples work for the categories G_n.

Very recently, Ramsey properties for certain categories of hypergraphs have been established in work of Nešetřil and Rödl [24]. Deeper understanding of these categories will be required before the full role of the Ramsey property is apparent.

We conclude this section with the Ramsey theorem for categories. The later sections will not depend on ideas from category theory.

Formally, we can define the Ramsey property for a category C as follows: If N and K are objects of C, then we let $C\begin{bmatrix} N \\ K \end{bmatrix}$ denote the set of subobjects of N of type K, where a subobject of N is said to be of *type* K if it contains a monomorphism $K \to N$. If $\varphi: L \to N$ is a monomorphism then we let $\bar{\varphi}: C\begin{bmatrix} L \\ K \end{bmatrix} \to C\begin{bmatrix} N \\ K \end{bmatrix}$ denote the obvious induced map.

The category C is called *Ramsey* if for every positive integer r and every pair of objects K and L, there is an object N such that for every r-coloring $c: C\begin{bmatrix} N \\ K \end{bmatrix} \to [1, r]$ there exists a monomorphism $\varphi: L \to N$ and an $i \in [1, r]$ such that the following diagram commutes:

$$\begin{tikzcd}
C\begin{bmatrix} N \\ K \end{bmatrix} \arrow[r, "c"] & {[1, r]} \\
C\begin{bmatrix} L \\ K \end{bmatrix} \arrow[u, "\bar{\varphi}"] \arrow[r] & \{i\} \arrow[u, "\text{incl.}"']
\end{tikzcd}$$

For example, to state Ramsey's Theorem in this form, we consider the category S of finite sets with morphisms being injective mappings. We see that if N is an n-set, then its subobjects of type

K correspond to its subsets of $k = |K|$ elements. Note that $\left| S\begin{bmatrix} N \\ K \end{bmatrix} \right| = \binom{n}{k}$. Since all sets of the same cardinality are isomorphic, then the assertion that S is Ramsey is just Ramsey's original statement. For vector spaces, we consider the category $\mathcal{V}_q$ of finite-dimensional vector spaces over GF(q) with morphisms being injective linear mappings. Here the subobjects of type K of an n-dimensional space N correspond to the $|K|$-dimensional subspaces of N.

THEOREM: *Let $\mathcal{C}$ be a class of categories such that for each category B in $\mathcal{C}$ there is a category A in $\mathcal{C}$ such that A and B satisfy the conditions below. Then all categories in $\mathcal{C}$ are Ramsey.*

The conditions are as follows, where we are assuming that all categories have as objects the set of nonnegative integers:

There is a function M from A to B with $M(l) = l + 1$, $l = 0, 1, 2, \ldots$, a functor P from B to A with $P(l) = l$, $l = 0, 1, 2, \ldots$, an integer $t \geq 0$, and for each $l \geq 0$, there are t monomorphisms $l \xrightarrow{\varphi_{lj}} l + 1, j \in [1, t]$, satisfying the following three conditions:

I. For each $k = 0, 1, 2, \ldots$, the diagonal d in the following diagram is epic, where $\amalg$ (together with the indicated injections) denotes coproduct, and d is the unique map determined by the coproduct which makes the diagram commute:

$$\begin{tikzcd}
B\begin{bmatrix} l \\ k+1 \end{bmatrix} \arrow[rrr, "i_1"] \arrow[d, "\bar{\varphi}_{l1}"'] & \cdots & B\begin{bmatrix} l \\ k+1 \end{bmatrix} \arrow[r, "i_t"] \arrow[dll, "\bar{\varphi}_{lt}"'] & \amalg \arrow[dlll, "d"'] \\
B\begin{bmatrix} l+1 \\ k+1 \end{bmatrix} & & & A\begin{bmatrix} l \\ k \end{bmatrix} \arrow[u, "i_0"'] \arrow[lll, "\bar{M}"]
\end{tikzcd}$$

Here, $\bar{M}$ is the mapping induced on subobjects by M.

II. For each $s \xrightarrow{g} l$ in B and each $j \in [1, t]$, the following diagram commutes:

$$\begin{tikzcd}
l \arrow[r, "\varphi_{lj}"] & l+1 \\
s \arrow[u, "g"] \arrow[r, "\varphi_{sj}"'] & s+1 \arrow[u, "M(P(g))"']
\end{tikzcd}$$

III. For some $l \xrightarrow{e} l + 1$ in A, the following diagram commutes for all $j \in [1, t]$:

$$\begin{tikzcd}
 & l+1 \arrow[dr, "\varphi_{l+1,j}"] & \\
l \arrow[ur, "\varphi_{lj}"] \arrow[dr, "\varphi_{lj}"'] & & l+2 \\
 & l+1 \arrow[ur, "M(e)"'] &
\end{tikzcd}$$

For $\mathcal{C}$ consisting of the single category $\mathcal{S}$ with morphisms $k \to l$ being the monomorphisms from $[1, k]$ to $[1, l]$, the theorem (with appropriate choices M, P and $t = 1$) becomes Ramsey's Theorem. In this case only condition I remains interesting, reflecting simply the basic relation for binomial coefficients, namely, $\binom{l+1}{k+1} = \binom{l}{k} + \binom{l}{k+1}$. In particular, the $(k + 1)$-subsets of an $(l + 1)$-set consist of those which contain a fixed element x together with those not containing x. The relation above is an example of a much more general "Pascal relation", as described by Leeb [22]. The study of such general theorems has sometimes been referred to as 'Pascal Theory'. Other categories, including those of finite binary trees (with appropriate inclusion morphisms) and finite

Boolean algebras (with sublattice monomorphisms) also have Pascal relations and the Ramsey property.

RAMSEY NUMBERS

Most of the finite Ramsey theorems we have considered up to this point involve the existence of "Ramsey numbers". In this section we will discuss some of the known values and bounds for these numbers.

We recall that Ramsey's Theorem asserts the existence of a least integer $n(k, l, r)$ such that any r-coloring of the k-subsets of an n-set S forces all the k-subsets of some l-subset of S to have a single color, provided $n \geq n(k, l, r)$. All the values of $n(k, l, r)$ currently known are given by (e.g., see [18]):

$$n(1, l, r) = r(l - 1) + 1; \; n(l, l, r) = l;$$

$$n(2, 3, 2) = 6, \; n(2, 3, 3) = 17, \; n(2, 4, 2) = 18.$$

The most thoroughly studied case has been $k = 2$, for which it is natural to phrase the results in terms of coloring the edges of a complete graph. We can define $R(l_1, \ldots, l_r)$ to be the least integer such that if $n \geq R(l_1, \ldots, l_r)$ and the edges of the complete graph K_n are arbitrarily r-colored, then there is a monochromatic K_{l_i} for some color i. The numbers known here [18] (which are not included above) are as follows:

$$R(3, 4) = 9, \; R(3, 5) = 14, \; R(3, 6) = 18, \; R(3, 7) = 23.$$

These are all the values known exactly. For other choices of the parameters, only estimates are available, some of which are:

$$27 \leq R(3, 8) \leq 30, \; 36 \leq R(3, 9) \leq 37,$$

$$c_1 k 2^{k/2} \leq R(k, k) \leq c_2 \frac{4^k}{\sqrt{k}} \frac{\log \log k}{\log k}, \qquad [9], [18]$$

$$R(x, y) \leq c_3 y^{x-1} \frac{\log \log y}{\log y}, \qquad [18]$$

(a proof for the lower bound on $R(k, k)$ may be found in the paper by Joel Spencer in this volume).

For van der Waerden's theorem (Example 5), the estimates for the corresponding Ramsey numbers $W(k, r)$ are much less accurate. For small values, it is known that: $W(2, 2) = 3$, $W(3, 2) = 9$, $W(4, 2) = 35$ [4]. On the other hand, for larger values it is known that

$$k\, 2^k \leq W(k, 2) < A(k, 4),$$

where the first inequality holds for k prime [1] and $A(m, n)$ is defined by:

$$A(1, n) = 2^n,\ A(m, 2) = 4,\ m \geq 1,\ n \geq 2,$$

$$A(m, n) = A(m - 1, A(m, n - 1)),\ m \geq 2,\ n \geq 3.$$

The reader is invited to calculate a few values of A, e.g., $A(5, 5)$ or $A(12, 3)$, in order to get a feeling for the disparity between these upper and lower bounds.

In general, the best constructive upper bounds for Ramsey numbers are strongly correlated to the complexity of the arguments used to show their existence. Hence, it is not surprising that the estimates for some of the more subtle Ramsey numbers are extremely large. For example, consider the Ramsey number $N(l)$ defined to be the least integer such that if $n \geq N(l)$ and the line segments between all pairs of vertices of a given n-dimensional rectangular parallelepiped P_n are arbitrarily 2-colored, then all the line segments between the pairs of vertices of some l-dimensional rectangular subparallelepiped of P_n have a common color (see [15] for a proof of existence). Of course, $N(1) = 1$. The best available estimate for $N(2)$, however, is:

$$6 \leq N(2) \leq A(A(A(A(A(A(A(12, 3), 3), 3), 3), 3), 3), 3).$$

It is conjectured that $N(2) = 6$.

SOME OLD DIRECTIONS

An important area we have yet to discuss originated some 40 years ago with the fundamental work of R. Rado [25], [26]. It deals with the integer solutions to systems of linear equations and contains as special cases both the theorem of Schur (Example 1) and van der Waerden's theorem.

Let $\mathcal{L} = \mathcal{L}(x_1, \ldots, x_n)$ denote a system of homogeneous linear equations in the variables $x_1, \ldots, x_n$ with integer coefficients. We say that $\mathcal{L}$ is *r-Ramsey* (called *r-regular* by Rado) if for any partition of the positive integers **P** into r classes, $\mathcal{L}$ has a solution $(a_1, \ldots, a_n)$ with all the a_k in one class. If $\mathcal{L}$ is r-Ramsey for all r then $\mathcal{L}$ is called *Ramsey.*

The basic result here is the following theorem of Rado [25]:

THEOREM: *$\mathcal{L}$ is Ramsey if and only if there is a partition of* $[1, n] = S_1 \cup \cdots \cup S_l$ *so that for each* $k \in [1, l]$, *there is a solution* $u^{(k)} = (u_1{}^{(k)}, \ldots, u_n{}^{(k)})$ *satisfying*

$$u_i^{(k)} = \begin{cases} 0, \text{ if } i \in S_j \text{ for } j > k, \\ 1, \text{ if } i \in S_k, \\ \text{arbitrary, otherwise.} \end{cases}$$

For the special case in which $\mathcal{L}$ consists of a single equation, the result is particularly appealing:

COROLLARY: *The equation* $\sum_{k=1}^{m} a_k x_k = 0$ *is Ramsey if and only if some nonempty subset of the* a_k*'s sum to zero.*

Not only does the above theorem imply van der Waerden's theorem and the previously mentioned theorem of Schur, but it also proves the following interesting generalization (see also [16]):

THEOREM: *Given integers k and r, there exists an integer N(k, r) such that if* $n \geq N(k, r)$, *then for any r-coloring of* $[1, n]$ *there* *exists* $A \subseteq [1, n]$ *with* $|A| = k$ *for which all the sums* $\sum_{b \in B} b$, $\emptyset \neq B \subseteq A$, *have the same color.*

A striking extension of this result for the case of k infinite has recently been given by Hindman [20], [0].

THEOREM: *For any* r *and any* r*-coloring of the positive integers* $\mathbf{P}$ *there is an infinite set* $A \subseteq \mathbf{P}$ *such that all sums* $\sum_{b \in B} b$, $\emptyset \neq B \subseteq A$, *have the same color.*

An intriguing problem is to determine those infinite systems of homogeneous linear equations which are Ramsey.

We call a set $A \subseteq \mathbf{P}$ *regular* if any Ramsey system $\mathcal{L}$ of linear equations has a solution entirely in A. More than 35 years ago, Rado put forth the following conjecture: If a regular set of integers is partitioned into a finite number of classes then at least one of the classes is regular. This conjecture has very recently been proved in a study of Deuber [5] in which he characterizes regular sets of integers in terms of certain high-dimensional array-like subsets for which he is able to establish a Ramsey property.

SOME NEW DIRECTIONS

Certain questions have only recently been pursued. We discuss here two of the most active areas. The first of these involves the determination of numbers somewhat more general than those arising from Ramsey's Theorem, an immediate corollary of which is the following statement: For any choice of finite graphs $G_1, \ldots, G_r$, there is an $N = N(G_1, \ldots, G_r)$ such that if $n \geq N$ and the edges of K_n are r-colored, then there is an $i \in [1, r]$ and a subgraph (not necessarily induced) isomorphic to G_i with all its edges having color i. If $l = \max(|G_1|, \ldots, |G_r|)$ where $|G_i|$ denotes the number of vertices of G_i then clearly $N(G_1, \ldots, G_r) \leq n(2, l, r)$, the number from Ramsey's Theorem. The estimation of the "graph Ramsey numbers" provides some information on the size of the original

Ramsey numbers. However, in most of the cases considered up to now, the graphs G_i have been too simple for the resulting bounds on $N(G_1, \ldots, G_r)$ to be of much use in estimating Ramsey numbers. Some of these cases do turn out to have nice, exact answers and are themselves fascinating results in graph theory. For example,

$$N(P_m, P_n) = m + \left[\frac{n}{2}\right] - 1,$$

$$N(mK_3, nK_3) = 3m + 2n \text{ for } m \geq n,\ m \geq 2,$$

$$N(\underbrace{C_4, \ldots, C_4}_{r}) = r^2 + O(r),$$

where P_m denotes a path with m vertices, mK_3 denotes the union of m disjoint triangles and C_4 denotes a 4-cycle. For a survey of results in this direction, the reader is referred to Burr [2].

We have already seen one area of Ramsey theory where geometric considerations arise, namely, the Ramsey theorem for finite vector spaces. The theory is rather complete here, except for the calculation or estimation of the corresponding Ramsey numbers. When we consider Euclidean geometry, however, the corresponding theorem is clearly false. That is, there are 2-colorings of the points of Euclidean n-space $\mathbf{E}^n$ so that no Euclidean line is monochromatic. (For example, consider concentric spherical shells of alternating colors). The question of which monochromatic configurations must occur is the subject of "Euclidean Ramsey Theory" [11], [12]. In some sense, the obstruction to finding a monochromatic line is the fact that the underlying field is infinite, and consequently there are infinitely many points on each line. (Cates and Hindman [3] have investigated Ramsey properties for other infinite fields and for spaces of infinitely many dimensions, mostly along lines similar to the studies of the Ramsey properties of large cardinals.)

Thus, it is natural to consider finite configurations $C \subseteq \mathbf{E}^n$. Let G be a permutation group acting on $\mathbf{E}^n$. The problem is to determine those configurations C such that for any r-coloring of $\mathbf{E}^n$, for some $g \in G$, all the points of $g(C)$ have a single color.

For the case when G is the affine group on $\mathbf{E}^n$, it was shown by Gallai [26] that all finite configurations are Ramsey, i.e., are r-Ramsey for all r, thus generalizing van der Waerden's theorem. For the case when G is the identity group, then, of course, only one-point configurations are Ramsey.

An extremely interesting case is that of G being the group of Euclidean motions in $\mathbf{E}^n$. Thus, we must find a monochromatic set C' which is *congruent* to C. For the configuration C_1 consisting of two points separated by a fixed distance d, it is easily seen that any 2-coloring of $\mathbf{E}^2$ contains a monochromatic set congruent to C_1, e.g., by just considering the three vertices of an equilateral triangle of side d. A similar argument using higher dimensional simplexes shows that some regular simplex of side d will occur monochromatically in any r-coloring of a sufficiently high dimensional space. The minimum sufficient dimension $n(k, r)$ is a function of k, the number of vertices of the simplex, and r. At present, even the value of $n(r, 2)$ is unknown. It is known that $n(2, 2) = 2$, $n(7, 2) > 2$ and $n(2, 3) = 3$. In general, if for each r there is a monochromatic set congruent to C in any r-coloring of $\mathbf{E}^n$, provided only that n is sufficiently large depending upon r and C, then we say that C is Ramsey. At present the only configurations known [11] to be Ramsey are subsets of the vertices of a rectangular parallelepiped. In the other direction, it has been shown [11] that any Ramsey configuration must lie on some (perhaps very high dimensional) sphere. For configurations not in either of these classes, the simplest being the set of vertices of an obtuse triangle, it is not known whether any of them are Ramsey. The arguments for the spherical sets involve an extension of certain negative results of Rado (mentioned in the previous section) to a larger class of underlying fields than he considered.

One of the most appealing questions in this area is the conjecture that any 2-coloring of the Euclidean plane must contain a monochromatic set congruent to any given 3-set, with the possible exception of the set of vertices of a single equilateral triangle. Many families of triangles (i.e., 3-sets) are known to occur in this case, but for most triangles it is undecided [12]. Since this area is so new, very few of the obvious questions have been studied, e.g., other configurations, other groups G, and other geometries, to mention a few.

A FEW PROOFS

This section contains short proofs of the earliest Ramsey theorems, namely, Ramsey's Theorem, van der Waerden's theorem, Schur's theorem and a proof that the Ramsey number $r(K_3, K_3, K_3) = 17$ (due to Greenwood and Gleason [19]).

THEOREM (Ramsey): *For every k, l, r there is a least integer $n(k, l, r)$ such that if $n \geq n(k, l, r)$ and all the k-subsets of an n-set S are r-colored, then all the k-subsets of some l-subset of S have the same color.*

Proof: We use induction on k. For $k = 1$, this is just the box principle and $n(1, l, r) = (l - 1)r + 1$. Assume that the theorem holds for some $k \geq 1$ and all l and r. We prove the following: For each $j \leq k$, there is a number $f(j, k)$ such that if $n \geq f(j, k)$ and the $(k + 1)$-subsets of $[1, n]$ are r-colored, then there exists a k-subset $H = X \cup Y$ with $|X| = j$ such that if $x \in X$, $y \in Y$, then $x < y$ and, if K is any $(k + 1)$-subset of H with $K \cap X \neq \emptyset$, then the color of K is determined only by $\min\limits_{x \in K} x$.

We prove this by induction on j. For $j = 0$ it is trivial and we can choose $f(0, k) = k$. Assume that it holds for some $j \geq 0$. Let $f(j + 1, k) = f(j, n(k, k - j - 1, r) + j + 1)$, let $n \geq f(j + 1, k)$ and suppose the $(k + 1)$-subsets of $[1, n]$ are arbitrarily r-colored. By the induction hypothesis, there is a subset $T = X' \cup Y'$ with $|X'| = j$ which satisfies the required conditions. Let $x_{j+1} = \min\limits_{y \in Y'} y$. We now r-color the k-subsets of $Y' - \{x_{j+1}\}$ by assigning to the k-set $Z \subseteq Y' - \{x_{j+1}\}$ the same color that the $(k + 1)$-set $Z \cup \{x_{j+1}\}$ has. By the definition of $n(k, k - j - 1, r)$, there is a $(k - j - 1)$-subset Y of $Y' - \{x_{j+1}\}$ with all its k-subsets having the same color. Hence, if $X = X' \cup \{x_{j+1}\}$, then $X \cup Y$ satisfies the required conditions for the value $j + 1$ and the induction step is complete.

Now, consider an $n \geq f((l - 1)r + 1, (l - 1)r + 1)$ and let the $(k + 1)$-subsets of $[1, n]$ be r-colored. By the definition of f, there is an $((l - 1)r + 1)$-subset X of $[1, n]$ such that the color of any $(k + 1)$-subset of X is determined only by its least element. Hence, each element determines a color and since $|X| = (l-1)r + 1$, there is an l-subset L of X with all $(k+1)$-subsets of L having the same color. This completes the induction step and the theorem is proved. ■

THEOREM (van der Waerden): *Given k and r, there exists an integer $W(k, r)$ such that any r-coloring of* $[1, W(k, r)]$ *must contain an arithmetic progression of k terms all having the same color.*

Proof [17]: For a positive integer l, let us call two m-tuples $(x_1, \ldots, x_m)$, $(x_1', \ldots, x_m') \in [0, l]^m$ *l-equivalent* if they agree up through their last occurrences of l. For any l, $m \geq 1$, consider the statement

$S(l, m)$: For any r, there exists $W(l, m, r)$ so that for any function $C:[1, W(l, m, r)] \rightarrow [1, r]$, there exist positive $a, d_1, \ldots, d_m$ such that $C(a + \sum_{i=1}^{m} x_i d_i)$ is constant on each l-equivalence class of $[0, l]^m$.

Fact 1. $S(l, m)$ for some $m \geq 1 \Rightarrow S(l, m+1)$.

Proof: For a fixed r, let $M = W(l, m, r)$, $M' = W(l, 1, r^M)$ and suppose $C:[MM'] \rightarrow [1, r]$ is given. Define $C':[1, M'] \rightarrow [1, r^M]$ so that $C'(k) = C'(k')$ iff $C(kM - j) = C(k'M - j)$ for all $0 \leq j < M$. By the inductive hypothesis, there exist a' and d' such that $C'(a' + xd')$ is constant for $x \in [0, l-1]$. Since $S(l, m)$ can apply to the interval $[a'M + 1, (a'+1)M]$, then by the choice of M, there exist $a, d_1, \ldots, d_m$ with all sums $a + \sum_{i=1}^{m} x_i d_i$, $x_i \in [0, l]$, in $[a'M + 1, (a'+1)M]$ and with $C(a + \sum_{i=1}^{m} x_i d_i)$ constant on l-equivalence classes. Set $d_i' = d_i$ for $i \in [1, m]$ and $d_{m+1}' = d'M$; then $S(l, m+1)$ holds.

Fact 2. $S(l, m)$ for all $m \geq 1 \Rightarrow S(l+1, 1)$.

Proof: For a fixed r, let C:$[1, 2N(l, r, r)] \to [1, r]$ be given. Then there exist $a, d_1, \ldots, d_r$ such that for $x_i \in [0, l]$, $a + \sum_{i=1}^{r} x_i d_i \leq W(l, r, r)$ and $C(a + \sum_{i=1}^{r} x_i d_i)$ is constant on l-equivalence classes. By the box principle there exist $u < v$ in $[0, r]$ such that $C(a + \sum_{i=1}^{u} ld_i) = C(a + \sum_{i=1}^{v} ld_i)$. Therefore $C((a + \sum_{i=1}^{u} ld_i) + x(\sum_{i=u+1}^{v} d_i))$ is constant for $x \in [0, l]$. This proves $S(l + 1, 1)$.

Since $S(1, 1)$ holds trivially, then by induction $S(l, m)$ is valid for all $l, m \geq 1$. Van der Waerden's theorem is $S(l, 1)$.■

THEOREM (Schur): *For all r there is an integer $N(r)$ such that any r-coloring of $[1, N(r)]$ contains three elements x, y, z having the same color and which satisfy $x + y = z$.*

Proof: Choose $N(r) = n(2, 3, r)$, the Ramsey number from Ramsey's Theorem. Any r-coloring of $[1, N(r)]$ induces an r-coloring of the edges of $K_{N(r)}$ by assigning to the edge $\{i, j\}$ the color that $|i - j|$ has. By the definition of $N(r)$, there exists a monochromatic triangle in $K_{N(r)}$, i.e., $x < y < z$ such that $z - x$, $z - y$, $y - x$ all have the same color. But $(z - y) + (y - x) = z - x$ so we are done.■

THEOREM (Greenwood and Gleason): $r(K_3, K_3, K_3) = 17$.

Proof: Form a 3-coloring of K_{16} by labelling the vertices with the elements of $GF(16)$ and coloring the edge $\{x, y\}$, $x, y \in GF(16)$, according to the coset of the group of cubic residues in which the difference $x - y$ lies. This coloring is well defined since $-1 \equiv 1 \pmod 2$. It is not hard to check that no monochromatic triangle is formed and so $r(K_3, K_3, K_3) > 16$.

To show that $r(K_3, K_3, K_3) \leq 17$, let K_{17} be arbitrarily 3-colored. For a fixed vertex x, some color, say blue, occurs in at least six edges incident to x. If the vertices at the other ends of these six blue edges span a blue edge then we have a blue triangle. On the other hand, if no blue edge is so spanned, then we have a 2-colored K_6. Since $r(K_3, K_3) = 6$, then we have a monochromatic K_3 in this case as well. Thus $r(K_3, K_3, K_3) \leq 17$ and the proof is completed.■

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