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# On permutations containing no long arithmetic progressions

by

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**Introduction.** It has often been noted (e.g., see [1], [4], [5]) that it is possible to arrange $n$ consecutive integers into a sequence $a_1 a_2 \ldots a_n$ which contains no subsequence forming an increasing or decreasing 3-term arithmetic progression (A.P.). In other words, if $a_i = c$, $a_j = c+d$, $a_k = c+2d$ for some positive $d$, then either $j = \max\{i, j, k\}$ or $j = \min\{i, j, k\}$. In this note we investigate several questions related to this idea. For example, we show that any doubly-infinite permutation $\ldots a_{-2} a_{-1} a_0 a_1 a_2 \ldots$ of *all* the positive integers must contain an increasing or decreasing (i.e., monotone) 3-term A.P. as a subsequence. On the other hand, we construct a doubly-infinite permutation of the positive integers which contains no monotone 4-term A.P.

**Permutations of finite intervals.** Let us denote by $M(n)$ the number of permutations $a_1 a_2 \ldots a_n$ of $\{1, 2, \ldots, n\} \equiv [1, n]$ containing no monotone 3-term A.P. To see that $M(n) > 0$ for all $n$ simply note if $A = a_1 a_2 \ldots a_m$ has no monotone 3-term A.P. then

$$A' = (2A)(2A-1) \equiv (2a_1)(2a_2) \ldots (2a_m)(2a_1-1) \ldots (2a_m-1)$$

also has no monotone 3-term A.P. (since the first and last terms of a 3-term A. P. must have the same parity!) Of course, if $A$ is a permutation of $[1, m]$ then $A'$ is a permutation of $[1, 2m]$. Finally, since no monotone A.P.'s are created by *deleting* entries of $A$, the assertion $M(n) > 0$ for all $n$ follows immediately. In fact, much more is true.

FACT 1.

$$M(n) \geqslant 2^{n-1} \quad \textit{for} \quad n \geqslant 1. \tag{1}$$

Proof. As we have already noted, if $A$ has no monotone 3-term A.P., then neither do $2A$ and $2A-1$. Thus, if $A$ and $A'$ are 3-term A.P.-

free permutations of $[1, m]$, then $(2A)(2A'-1)$ and $(2A'-1)(2A)$ are 3-term A.P.-free permutations of $[1, 2m]$. Hence,

$$M(2n) \geqslant 2M(n)^2.$$

Similarly, we have

$$M(2n+1) \geqslant 2M(n+1)M(n).$$

Since $M(2) = 2$, $M(3) = 4$ then (1) follows. ∎

H. E. Thomas [6] has independently proved (1) by a somewhat more complicated construction.

In Table 1, we give a list of values of $M(n)$ for $n \leqslant 20$.

**Table 1**

| $n$ | $M(n)$ | $n$ | $M(n)$ |
|---|---|---|---|
| 1 | 1 | 11 | 2460 |
| 2 | 2 | 12 | 6128 |
| 3 | 4 | 13 | 12840 |
| 4 | 10 | 14 | 29380 |
| 5 | 20 | 15 | 74904 |
| 6 | 48 | 16 | 212728 |
| 7 | 104 | 17 | 368016 |
| 8 | 282 | 18 | 659296 |
| 9 | 496 | 19 | 1371056 |
| 10 | 1066 | 20 | 2937136 |

By using the fact that $M(16) = 212728$, it follows from the preceding argument that

$$M(2^t) > \tfrac{1}{2}(2.248)^{2^t}, \quad t \geqslant 4.$$

In the other direction, we have the following result:

FACT 2.

(2) $$M(2n-1) \leqslant (n!)^2, \qquad M(2n) \leqslant (n+1)(n!)^2.$$

Proof. Let $\mathscr{M}(t)$ denote the set of permutations of $[1, t]$ containing no monotone 3-term A.P.'s. Any permutation $X \in \mathscr{M}(n+1)$ generates a permutation $X' \in \mathscr{M}(n)$ by just deleting $n+1$. Consider an element $A = a_1 a_2 \ldots a_n \in \mathscr{M}(n)$ to which $n+1$ can be added *somewhere* to form an $A' \in \mathscr{M}(n+1)$. If $a_i$ satisfies

(3) $$\left[\frac{n+3}{2}\right] \leqslant a_i \leqslant n,$$

then the three values

$$n+1,\ a_i,\ 2a_i - n - 1$$

form an arithmetic progression which is not allowed to occur monotonely in $A'$. Hence, for each $a_i$ satisfying (3), $n+1$ is prohibited from being placed just to the right (left) of $a_i$ if $2a_i-n-1$ occurs to the left (right) of $a_i$. Also, if $n+1$ were prohibited from going to the right of $a_i$ *and* to the left of $a_{i+1}$ then $A$ could not be extended to an element of $\mathcal{M}(n+1)$. Hence, each of the $n-\left[\frac{n+3}{2}\right]+1$ values $a_i$ satisfying (3) rules out at least one of the $n+1$ possible locations in $A$ for $n+1$, leaving at most $\left[\frac{n+3}{2}\right]$ places where $n+1$ might go. This implies

$$M(n+1) \leqslant \left[\frac{n+3}{2}\right] M(n)$$

which, in turn, implies (2). ∎

**Permutations of the positive integers.** Let $A = a_1a_2a_3\ldots$ be a permutation of the set $\mathbf{Z}^+$ of positive integers. Denote by $\mathcal{S}_k$ the set of those $A$ which contain no monotone $k$-term A.P.

FACT 3.

$$\mathcal{S}_3 = \emptyset.$$

Proof. Let $A = a_1a_2a_3\ldots$ be a permutation of $\mathbf{Z}^+$. If $i$ denotes the least index for which $a_i > a_1$ then for some $j > i$,

$$a_j = 2a_i - a_1$$

and so we always have, in fact, an increasing 3-term A.P. in $A$. ∎

FACT 4.

$$\mathcal{S}_5 \neq \emptyset.$$

Proof. For $k \geqslant 0$, define the intervals $A_k$ and $B_k$ as follows:

$$A_k = [a_k+1, a_k+10^k], \qquad B_k = [b_k+1, b_k+10^k]$$

where $a_0 = 0$, $b_0 = 1$, and in general,

$$a_k = 2\sum_{i=0}^{k-1} 10^i, \qquad b_k = a_k+10^k.$$

Thus, $\mathbf{Z}^+$ is partitioned into disjoint intervals $A_k, B_k, k \geqslant 0$. Note that $A_0 = \{1\}$ and

$$|A_k| = |B_k| = 10^k.$$

Let $A_k^*$ and $B_k^*$ denote arbitrary fixed permutations of $A_k$ and $B_k$, respectively, which contain no monotone 3-term A.P.'s. Finally, let $P$ be the permutation of $\mathbf{Z}^+$ given by

$$P = B_0^*A_0^*B_1^*A_1^*B_2^*A_2^*\ldots B_k^*A_k^*\ldots$$

We claim that $P$ contains no monotone 5-term A.P. Suppose the contrary, i.e., suppose $X = \{x_1, x_2, x_3, x_4, x_5\}$ with $x_{k+1} - x_k = d > 0$ is a 5-term A.P. occurring monotonely in $P$. There are several possibilities:

(i) $X$ is a *decreasing* subsequence of $P$. Thus, for some $k$, $X \subseteq A_k \cup B_k$. But this implies that either $x_5, x_4, x_3$ is a decreasing A.P. in $B_k^*$ or $x_3, x_2, x_1$ is a decreasing A.P. in $A_k^*$. Since neither of these possibilities can occur, this case is impossible.

(ii) $X$ is an *increasing* subsequence of $P$.

(a) Suppose $|X \cap (A_k \cup B_k)| \leqslant 1$ for all $k$. Let $x_k \in A_{i_k} \cup B_{i_k}$, $1 \leqslant k \leqslant 5$. Thus, $i_1 < i_2 < i_3 < i_4 < i_5$. Since

$$x_5 - x_3 > a_{i_5} - a_{i_4} \geqslant 2 \cdot 10^{i_5}$$

then

$$d = \tfrac{1}{2}(x_5 - x_3) > 10^{i_5}.$$

Thus,

$$x_2 = x_3 - d < a_{i_4} - 10^{i_5} \leqslant 2(1 + 10 + \ldots + 10^{i_4}) - 10^{i_4+1} < 0$$

which is impossible. Hence, in this case we cannot even have a 4-term A.P.

(b) Suppose for some $k$, $|X \cap (A_k \cup B_k)| \geqslant 2$. Of course, since $X$ is increasing and $B_k$ precedes $A_k$ in $P$, then $X$ cannot intersect both $A_k$ and $B_k$. Therefore, by the construction of $P$ (which uses $A_k^*$ and $B_k^*$), we must have $|X \cap (A_k \cup B_k)| = 2$. There are two possibilities.

(α) Suppose $|X \cap B_k| = 2$. If $x_2, x_3 \in B_k$ then $d = x_3 - x_2 < 10^k$ and

$$x_1 = x_2 - d > b_k - 10^k = a_k,$$

i.e., $x_1 \in A_k$ which, as we have just noted, is impossible. A similar argument applies if $x_3, x_4 \in B_k$ or $x_4, x_5 \in B_k$. Thus,

$$x_5 = x_2 + 3d < a_{k+1} + 3 \cdot 10^k$$

which implies $x_5 \in A_{k+1}$ and consequently, $x_3, x_4 \in A_{k+1}$ as well, which is impossible.

(β) Suppose $|X \cap A_k| = 2$. If $x_3, x_4 \in A_k$ then $d = x_4 - x_3 < 10^k$ and

$$x_5 = x_4 + d < a_k + 10^k + 10^k = a_{k+1},$$

i.e., $x_5 \in B_k$ which is impossible. The same argument applies if $x_1, x_2 \in A_k$ or $x_2, x_3 \in A_k$. Thus, the only possibility remaining is $x_4, x_5 \in A_k$.

Now, if $x_2 \in B_{k-1}$ then we also must have $x_3 \in B_{k-1}$ and this is impossible from Case (i). On the other hand, if $x_2 \in A_{k-1}$ then $x_3 \in A_{k-1}$ and $d = x_3 - x_2 < 10^{k-1}$ which implies

$$x_4 = x_3 + d < x_3 + 10^{k-1},$$

i.e., $x_4 \in B_{k-1}$, a contradiction. Thus, $x_2 \leqslant a_{k-1}$ and so,

$$d = \tfrac{1}{2}(x_4 - x_2) > \tfrac{1}{2}(a_k - a_{k-1}) = 10^{k-1}.$$

Therefore,

$$x_1 = x_2 - d < a_{k-1} - 10^{k-1} < 0$$

which is a contradiction.

This completes the proof that $P$ contains no monotone 5-term A.P. and Fact 4 is proved. ∎

One of the most tantalizing questions still open is whether or not $\mathscr{S}_4$ is empty; i.e., whether every permutation of $\boldsymbol{Z}^+$ must contain monotone 4-term A.P.'s. Current opinions are about evenly divided.

**Doubly-infinite permutations of the positive integers.** If we are allowed to arrange the positive integers into a *doubly-infinite* sequence $A = \ldots a_{-2} a_{-1} a_0 a_1 a_2 \ldots$ then, in principle, we have more opportunity to prevent the occurrence of monotone A.P.'s. Denote by $\mathscr{D}_k$ the set of those $A$ which contain no monotone $k$-term A.P. As in the case of $\mathscr{S}_3$, $\mathscr{D}_3$ is also empty. This time however, a little more work is required to prove it.

FACT 5.

$$\mathscr{D}_3 = \emptyset.$$

Proof #1 (J. H. Folkman [2]). Let $A = \ldots a_{-2} a_{-1} a_0 a_1 a_2 \ldots$ be a doubly-infinite permutation of $\boldsymbol{Z}^+$. For $n \in \boldsymbol{Z}^+$, let $A(n)$ denote the *index* of $n$ in $A$, i.e., $A(n)$ is defined by

$$a_{A(n)} = n.$$

Suppose $A$ contains no monotone 3-term A. P. Thus, for all $a, d > 0$,

$$A(a) < A(a+d) \quad \text{iff} \quad A(a+d) > A(a+2d)$$

and

$$A(a) > A(a+d) \quad \text{iff} \quad A(a+d) < A(a+2d).$$

Iterating these relations we obtain

$$\text{(4)} \quad A(a) < A(a+d) \quad \text{iff} \quad \begin{cases} A(a+2md) < A(a+d+2md) \text{ and} \\ A\big(a+(2m+1)d\big) > A\big(a+d+(2m+1)d\big), \\ \qquad m = 0, 1, 2, \ldots \end{cases}$$

$$\text{(4')} \quad A(a) > A(a+d) \quad \text{iff} \quad \begin{cases} A(a+2md) > A(a+d+2md) \text{ and} \\ A\big(a+(2m+1)d\big) < A\big(a+d+(2m+1)d\big), \\ \qquad m = 0, 1, 2, \ldots \end{cases}$$

We may assume without loss of generality that $A(1) < A(2)$ (otherwise, reverse the sequence). By (4), we have

$$\text{(5)} \qquad A(2m-1) < A(2m), \quad m = 1, 2, \ldots$$

We claim that for *any* odd $a$ and $d$,

$$A(a) < A(a+d). \tag{6}$$

For $d = 1$, this is just (5). Assume (6) holds for a fixed odd $d \geqslant 1$. Let $a$ be odd and let $b = a+2d+4$. By assumption

$$A(b) < A(b+d).$$

(i) Suppose $A(b+d) < A(b+d+2)$. Then $A(b) < A(b+d+2)$ and so

$$A(a) = A\big(b-2(d+2)\big) < A\big(b+d+2-2(d+2)\big) = A(a+d+2)$$

by (4).

(ii) Suppose $A(b+d) > A(b+d+2)$. Then by (5)

$$A(a+d) = A\big(b+d-(d+2)\cdot 2\big) < A\big(b+d+2-(d+2)\cdot 2\big) = A(a+d+2).$$

Since $A(a) < A(a+d)$ then $A(a) < A(a+d+2)$.

Thus, in either case, we have $A(a) < A(a+d+2)$. This completes the induction step and (6) is proved. We are now finished, since by (6)

$$A(1) < A(2m) \quad \text{for all } m > 0.$$

Thus, as in the argument that $\mathscr{S}_3 = \emptyset$, if $2r$ is the first even number to the right of 1 and $2r+2d$ is the first even number to the right of $2r$ which is larger than $2r$, then $2r+4d$ is to the right of $2r+2d$ and $2r$, $2r+2d$, $2r+4d$ forms an increasing 3-term A.P. in $A$. This completes Proof #1 of Fact 5.

We sketch another proof of Fact 5 which is conceptually somewhat simpler although it involves some computation.

Proof #2. We form a directed tree $T$ as follows. The vertices of $T$ will be certain permutations $A \in \mathscr{M}(n)$ for various $n$. $T$ will have 4 root vertices 132, 213, 231 and 312. Suppose $A$ is a vertex of $T$ in which the subblock $B = a_i a_{i+1} \ldots a_{i+r}$ spanned by $\{1, 2, 3\}$ contains some *other* 3-term A.P. (necessarily non-monotone). We call such a vertex *special*. If $A \in \mathscr{M}(n)$ is a non-special vertex of $T$ and $A$ is a subsequence of $A' \in \mathscr{M}(n+1)$ then $A'$ is also a vertex of $T$ and $(A, A')$ is a directed edge of $T$. If no such $A'$ exists for $A$ then $A$ is called a *terminal* vertex of $T$. We show a portion of $T$ in Fig. 1. The basic fact concerning $T$ is that it is *finite*. In fact, straightforward computation shows that $T$ contains no vertices $A \in \mathscr{M}(n)$ with $n > 17$.

To complete the proof, we make the following observation. As we adjoin consecutive integers, starting with $A^* \in \mathscr{M}(3)$, to form a permutation $P$ of $\boldsymbol{Z}^+$, we move in the obvious way along a directed path in the tree. Suppose we reach a special vertex $A = a_1 \ldots a_n$. By definition, the block of $A$ spanned by $\{1, 2, 3\}$ contains a subsequence $a_{i_1} a_{i_2} a_{i_3}$ which

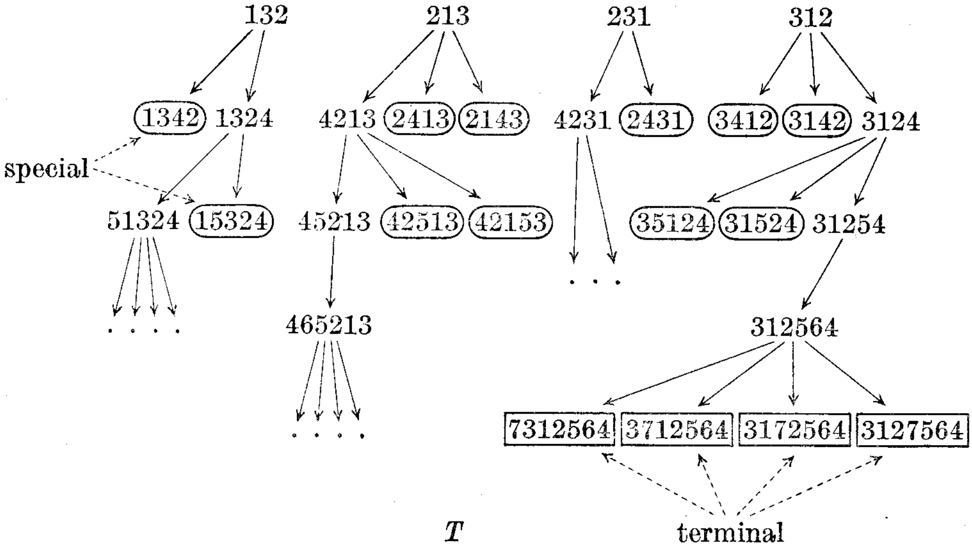


Fig. 1

is a permutation of $\{a, a+d, a+2d\} \neq \{1, 2, 3\}$. If we restrict our attention from now on to just those integers of the form $a+md$, $m \geq 0$, then we can move back to the appropriate *root* of $T$, i.e., the permutation of $\{1, 2, 3\}$ having the same relative order as $a_{i_1} a_{i_2} a_{i_3}$. Since $T$ is finite then as we form $P$, we must pass through the roots of $T$ an unbounded number of times. However, this implies that in $P$ some pair of integers in $\{1, 2, 3\}$ must have an unbounded number of integers separating them. This, however, contradicts the definition of a permutation of $\boldsymbol{Z}^+$, and the proof is completed. ■

The additional freedom allowed by doubly-infinite permutations can be used to prevent the occurrence of monotone 4-term A.P.'s.

FACT 6. $\mathscr{D}_4 \neq \emptyset$.

Proof. Define the blocks $B_i$, $i > 0$, as follows:

$$B_0 = 1, \quad B_{2i+1} = (2B_{2i})'(2B_{2i}+1)', \quad B_{2i+2} = (2B_{2i+1}+1)'(2B_{2i+1})', \quad i \geq 0,$$

where $B'$ denotes the block $B$ written in reverse order. Define the doubly-infinite permutation $P$ of $\boldsymbol{Z}^+$ by

$$P = \ldots B_4 B_2 B_0 B_1 B_3 \ldots$$
$$= \ldots 28, 20, 24, 16, 7, 5, 6, 4, 1, 2, 3, 8, 12, 10, 14, 9, 13, 11, 5, \ldots$$

We claim that $P \in \mathscr{D}_4$.

We first note that for all $i \geq 0$, $B_i$ is a permutation of $[2^i, 2^{i+1}-1]$ containing no monotone 3-term A.P. Suppose now that $P$ contains a monotone 4-term A.P. $X = \{x, y, z, w\}$ with either $x > y > z > w$ or

$x < y < z < w$, where we have chosen $X$ so that $d = |x-y|$ is minimal. There are several possibilities:

(i) The smallest two elements of $X$ belong to the same block $B_i$. Then $d < 2^i$ so that the largest two elements of $X$ are in $B_{i+1}$. Consequently, $x, y, z$ and $w$ all have the same parity. If $2j+1$ and $2k+1$ are in $B_i$ then $2j$ and $2k$ are also in $B_i$ with the same relative order. Hence, we may assume $x$, $y$, $z$, and $w$ are all even. But then

$$\frac{1}{2}X = \left\{\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right\}$$

is a monotone 4-term A.P. in $P$ since the smallest two elements of $\frac{1}{2}X$ appear in $B_{i-1}$ in reverse order of their appearance in $B_i$, the largest two appear in $B_i$ in reverse order of the appearance in $B_{i+1}$, and the order of $B_i$ and $B_{i-1}$ in $P$ is the reverse of that of $B_{i+1}$ and $B_i$. However, this contradicts the minimality of $d$.

(ii) Suppose $y$ and $z$ occur in the same block $B_i$. Then the largest element of $X$ occurs in $B_{i+1}$ and the smallest occurs in $B_j$ for some $j < i$. But this requires $B_i$ to appear between $B_{i+1}$ and $B_j$ in $P$ which is impossible.

(iii) Suppose the largest two elements of $X$ occur in the same block $B_i$. The third largest element of $X$ must be at least as large as $2^{i-1}$ since otherwise, we would have $d < 2^{i-1}$ and consequently, the second largest element of $X$ would be less than $2^i$ and therefore, not in $B_i$. Thus, the third largest element of $X$ is in $B_{i-1}$. Hence, by (i), the smallest element of $X$ is in $B_j$ for some $j < i-1$. As before, this requires $B_{i-1}$ to appear between $B_j$ and $B_i$ in $P$ which is impossible.

(iv) Suppose each element of $X$ belongs to a different block $B_i$ of $P$. Let $B_i$ denote the block containing the largest element of $X$. Then we may argue as in (ii) and (iii) that the second largest element of $X$ is not contained in $B_{i-1}$. Consequently $d > 2^{i-1}$ so that the third largest element of $X$ must be negative, a contradiction.

Since the construction of the $B_i$ prohibits the occurrence of 3 elements of $X$ in a single block then we have proved that $P$ has no monotone 4-term A.P. ∎

**Concluding remarks.** There are a number of questions which we were either unable to resolve or did not have a chance to look at. We mention a few of these.

1. The most natural question remaining is whether or not $\mathscr{S}_4 = \emptyset$, i.e., whether or not every singly-infinite permutation of $\mathbf{Z}^+$ contains a monotone 4-term A.P. It is not clear at present in which direction the truth lies.

2. The following modular analogue to the finite problem has been studied by M. Nathanson [3]. A subsequence $a_{i_0}, \ldots, a_{i_{t-1}}$ of a permutation $a_1 a_2 \ldots a_n$ of $[1, n]$ is called a *monotone A. P. modulo* $n$ if for some $a$ and $d \neq 0$,

$$a_{i_k} \equiv a + kd \pmod{n}, \qquad 0 \leqslant k < t.$$

Nathanson has shown (see [3]) that:

(i) If $n \neq 2^r$ then any permutation of $[1, n]$ contains a monotone 3-term A.P. modulo $n$.

(ii) If $n = 2^r$ then there is a permutation of $[1, n]$ which contains no monotone 3-term A.P.

On the other hand, it is easily seen that a permutation of $[1, n]$ which contains no monotone 3-term A. P. also contains no monotone 5-term A. P. modulo $n$. As in the preceding question, the situation for 4-term A. P.'s modulo $n$ is unclear.

3. It is possible to partition $\boldsymbol{Z}^+$ into three sets, each of which can be permuted so as to have no monotone 3-term A. P. For example, define the partition of $\boldsymbol{Z}^+$ into consecutive intervals $A_k$ by:

$$A_1 = [1, 100], \qquad |A_{k+1}| = [\tfrac{3}{2}|A_k|], \qquad k \geqslant 1.$$

Now, rearrange each $A_k$ into $A_k^*$ containing no monotone 3-term A.P. and define

$$\mathscr{A} = A_1^* A_4^* A_7^* A_{10}^* \ldots,$$
$$\mathscr{B} = A_2^* A_5^* A_8^* A_{11}^* \ldots,$$
$$\mathscr{C} = A_3^* A_6^* A_9^* A_{12}^* \ldots$$

It is easily checked that $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ form the desired partition. Whether this can be done for some partition of $\boldsymbol{Z}^+$ into *two* sets is not known.

4. Let $\mathscr{A}$ denote the set of all infinite subsets $A$ of $\boldsymbol{Z}^+$ for which there exists a (singly-infinite) permutation of $A$ having no monotone 3-term A. P. What is

$$\sup_{A \in \mathscr{A}} \liminf_{n} \frac{|A \cap [1, n]|}{n} ?$$

What is

$$\sup_{A \in \mathscr{A}} \limsup_{n} \frac{|A \cap [1, n]|}{n} ?$$

5. The preceeding questions could also be asked for $\boldsymbol{Z}$, the set of all the integers, as well. Only preliminary results are known for this case. For example, using Fact 4, it is easy to construct permutations of $\boldsymbol{Z}$ which have no monotone 7-term A. P.

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