ON THE SET OF DISTANCES DETERMINED BY THE UNION OF ARITHMETIC PROGRESSIONS

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1. *Introduction.*

Suppose the set of points $\{n\theta\}$, $0 \le n < N$, is placed on a circle of unit circumference, forming the increasing set

$$\{0 = s_1 < s_2 < \ldots < s_t = 1\},$$

where $\{x\}$, as usual, denotes the *fractional part* of x, i.e., $x - [x]$. Consider the set D of *distances* between consecutive s_k, i.e.,

$$D = \{s_{k+1} - s_k : 0 \le k < t\}.$$

In 1958, S. Świerczkowski [9] established the interesting result that D never has more than *three* elements, thereby confirming an assertion of H. Steinhaus (see [8]). (This result was also proved independently around the same time by P. Erdös and V.T. Sós [5], [6] and by P. Szüsz (unpublished)). Motivated by related work [2] on the distribution of $\{n\alpha\}$, one of the authors conjectured [1], [3] in 1969, that this result should be a special case of the following more general situation.

Suppose the k sets of points $\{n_i\theta + \alpha_i\}$, $0 \le n_i < N_i$, $1 \le i \le k$, are all placed on a circle of unit circumference, forming the increasing set

$$\{s_1 < s_2 < \ldots < s_{t^*} = s_1 + 1\}$$

Then the set D* of distances between consecutive s_k satisfies

$$|D^*| \le 3k \tag{1}$$

where $|D^*|$ denotes the cardinality of D*.

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In this paper, we prove (1). As a necessary preliminary result we also establish the interesting analog of (1) for the real line (as opposed to the circle), namely, that in this case we have

(1') $$|D^*| \le 3k - 3 \text{ for } k > 1.$$

2. *The Linear Case*

As mentioned in the introduction, before proving (1) it will be necessary to prove the analogous (and somewhat simpler) result (1'). We begin by making several definitions. For a fixed positive integer k, we assume we are given k real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ and k non-negative integers $n_1, n_2, \ldots, n_k$. Let A_i denote the set $\{\alpha_i + x;\ x = 0, 1, \ldots, n_i\}$ for $1 \le i \le k$ and let P_k be the ordered set formed from the union of the A_i, i.e.,

(2) $$P_k = \bigcup_{i=1}^{k} A_i = \{\pi_1 < \pi_2 < \ldots < \pi_n\}.$$

Let $D^*(P_k) = \{\pi_{i+1} - \pi_i : 1 \le i < n\}$ denote the set of distances between consecutive points of P_k. Our goal in this section will be to establish the following result.

THEOREM 1.

(3) $$|D^*(P_k)| \le 3k - 3 \text{ for } k \ge 2.$$

Of course, it is obvious that $|D^*(P_1)| \le 1$. The plan will actually be to prove a stronger pair of inequalities (see (5)), also depending on k, by induction on k.

Before doing this, we first "normalize" P_k a bit.

(a) We may assume $A_i \cap A_j = \emptyset$ for $i \ne j$. For if $A_i \cap A_j \ne \emptyset$, then $A_i \cup A_j$ forms an arithmetic progression and consequently P_k is a union of at most $k - 1$ arithmetic progressions to which the induction hypothesis will apply. The same argument shows that we may also assume the following.

(b) If $p_i \epsilon A_i$, $p_j \epsilon A_j$ with $i \neq j$ then $|p_i - p_j| \neq 1$.

(c) We may assume that $\pi_1 \epsilon A_1$, $\pi_2 \epsilon A_1$. For we can always extend the first term of the first progression one more step to the left and relabel as A_1 if necessary. Similarly, it will also be convenient to assume that $\pi_{n-1} \epsilon A_x$, $\pi_n \epsilon A_x$ for some x.

We next require several definitions. For $\pi_\ell \epsilon P_k$, define $\underline{F}: P_k \to \{1,2,\ldots,k\}$ by $F(\pi_\ell) = i$ where $\pi_\ell \epsilon A_i$. This is well-defined by (a). For $\pi_\ell, \pi_{\ell+1} \epsilon P_k$, define

(4) $\underline{d_p(\pi_\ell,\pi_{\ell+1})} = (\pi_{\ell+1} - \pi_\ell, F(\pi_\ell), F(\pi_{\ell+1}))$.

This we call the *P-length* of the interval $(\pi_\ell, \pi_{\ell+1})$. Thus, the P-length is a triple which indicates, in addition to the actual distance between π_ℓ and $\pi_{\ell+1}$, the corresponding A_i's to which they belong as well. We shall say that the intervals $(\pi_\ell, \pi_{\ell+1})$ and $(\pi_{\ell'}, \pi_{\ell'+1})$ have *equivalent* P-length provided that either

(i) $\pi_{\ell+1} - \pi_\ell = \pi_{\ell'+1} - \pi_{\ell'} = 1$,

or

(ii) $\pi_{\ell+1} - \pi_\ell = \pi_{\ell'+1} - \pi_{\ell'} \neq 1$ and

$F(\pi_\ell) = F(\pi_{\ell'})$, $F(\pi_{\ell+1}) = F(\pi_{\ell'+1})$.

Note that by (b), (i) implies $F(\pi_\ell) = F(\pi_{\ell+1})$, $F(\pi_{\ell'}) = F(\pi_{\ell'+1})$.

Let
$$S = \{p \in P_k : p - 1 \notin P_k\},$$
$$T = \{p \in P_k : p + 1 \notin P_k\}.$$

S is called the set of *starting points* and T is called the set of *terminal points*. The elements of $S \cup T$ are called *critical points*; the elements of $P \setminus (S \cup T)$ are called *regular points*. Finally, define $\underline{f(P_k)}$ to be the number of equivalence classes of the $d_p(\pi_\ell, \pi_{\ell+1})$, $1 \le \ell < n$, and define $f^*(P_k)$ to be the number of equivalence classes of the $d_p(\pi_\ell, \pi_{\ell+1})$ with $\pi_{\ell+1} - \pi_\ell \neq 1$. Equation (3) will follow from the following stronger inequalities:

(5) $f(P_k) \le 3k - 3$, $f^*(P_k) \le 3k - 4$ for $k \ge 2$.

A brief calculation shows that (5) holds for $k = 2$. We shall assume k is a fixed integer greater than 2 and that (5) holds for all unions of fewer than k arithmetic progressions A_i.

Fact 1. *Let* π_ℓ, $\pi_{\ell+1} \in P_k$ *with* $F(\pi_\ell) = i$, $F(\pi_{\ell+1}) = j \neq i$. *Suppose for some integer* $t > 1$, $\pi_\ell + t = \pi_{\ell'}$, $\pi_{\ell+1} + t = \pi_{\ell'+1}$ *but that* $\pi_\ell + t'$ *and* $\pi_{\ell+1} + t'$ *are* not *adjacent for any* t', $0 < t' < t$. *Then*

$$f(P_k) \le 3k - 4.$$

Proof. Let $X = \{\pi_m : \pi_\ell + t' < \pi_m < \pi_{\ell+1} + t' \text{ for some } t',\ 0 < t' < t\}$ and let $X' = P \setminus X$. Thus, $X = P_{k-k'}$ and $X' = P_{k'}$ for some k', $2 \le k' \le k-1$. Consider an interval (π_u, π_{u+1}) in P_k. If π_u and π_{u+1} are both in X' then the P-length $d_p(\pi_u, \pi_{u+1})$ also occurs in X'. If π_u and π_{u+1} are both in X then in fact $d_p(\pi_u, \pi_{u+1})$ occurs in $X \cup A_i$. If $\pi_u \in A_i$ and $\pi_{u+1} \in X$ then $d_p(\pi_u, \pi_{u+1})$ also occurs in $X \cup A_i$. Each P-length

$d_p(\pi_u, \pi_{u+1})$ with $\pi_u \in X \cup A_i$ and $\pi_{u+1} \in X'$ must have $\pi_{u+1} \in A_j$, and so corresponds to a unique P-length $d_p(\pi_u, \pi_v)$ where $\pi_v \in A_i$ (π_v is the element in $X \cup A_i$ which follows π_u). Furthermore, $d_p(\pi_u, \pi_v)$ does not occur in P_k. Finally, we note that since the P-length (1,1,1) occurs in P_k then it also occurs in X'. Thus, both $X \cup A_i$ and X' are the unions of at least 2 and at most k-1 arithmetic progressions so that the induction hypotheses applies, yielding

$$f(P_k) \le f^*(X \cup A_i) + f(X')$$
$$\le 3(k-k'+1) - 4 + 3k' - 3 = 3k - 4$$

and the fact is proved. □

The following result is immediate.

Fact 2. *Let* t *denote the number of* n_i, $1 \le i \le k$, *for which* $n_i = 0$. *Then for* $t \ge 1$ *we have*

$$\text{(i)} \quad f(P_k) \le 3k - 3 - t \quad \text{for} \quad k > 1;$$
$$\text{(ii)} \quad f(P_k) = k - 1 \quad \text{if} \quad t = k > 1.$$

Fact 3. *Suppose there exist* $p < \pi_\ell < \pi_{\ell+1} < p'$ *in* P_k *so that* $\pi_{\ell+1} - \pi_\ell > 1$ *and*

$$F(p) \ne F(\pi_\ell), \; F(p') \ne F(\pi_{\ell+1}).$$

Then

$$f(P_k) \le 3k - 6.$$

Proof. By hypotheses, we have the situation illustrated in Fig. 1.

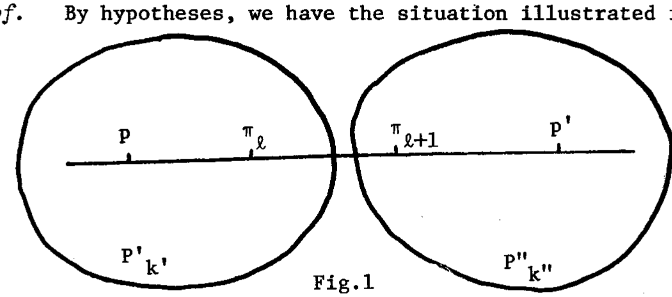

Fig.1

That is, P_k can be decomposed into two sets $P'_{k'}$ and $P''_{k''}$ with $k' \geq 2$, $k'' \geq 2$ and $k' + k'' = k$. Thus, by induction

$$f(P_k) \leq f^*(P'_{k'}) + f(P''_{k''}) + 1$$
$$\leq (3k' - 3 + 3k'' - 3 - 1) + 1 = 3k - 6$$

where the +1 term comes from the P-length $d_p(\pi_\ell, \pi_{\ell+1})$. □

Fact 4. *Suppose there exist* $p < \pi_\ell < \pi_{\ell+1} < p'$ *in* P_k *so that*

$$F(p) \neq F(\pi_\ell) = F(\pi_{\ell+1}) \neq F(p').$$

Then

$$f(P_k) \leq 3k - 4.$$

Proof. Let $F(\pi_\ell) = i$ and let $P'_{k'}$ denote the set of all points $\pi \epsilon P_k$ with $\pi \leq \pi_\ell$ together with all points of A_i. Similarly, let $P''_{k''}$ denote the set of all points $\pi \epsilon P_k$ with $\pi \geq \pi_{\ell+1}$ together with all points of A_i. Then $P'_{k'}$ and $P''_{k''}$ are unions of arithmetic progressions and $k' \geq 2$, $k'' \geq 2$ and $k' + k'' = k + 1$. Thus, by induction

$$f(P_k) = f(P'_{k'}) + f^*(P''_{k''}) + 1$$
$$\leq 3k' - 3 + 3k'' - 4 = 3k - 4$$

where the +1 term accounts for the P-length $d_p(\pi_\ell, \pi_{\ell+1})$ which by hypotheses is equivalent to (1,1,1). □

Fact 5. *Suppose there exist* $\pi_\ell < \pi_{\ell+1} < \pi_{\ell+2} < p$ *in* P_k *so that* $\pi_{\ell+1} \in T$ *and*

$$F(\pi_{\ell+1}) \neq F(\pi_\ell) = F(\pi_{\ell+2}) \neq F(p).$$

Then

$$f(P_k) \leq 3k - 3.$$

Proof. As before, let $F(\pi_\ell) = i$ and let $P'_{k'}$ denote $\{\pi \in P_k : \pi \leq \pi_{\ell+1}\} \cup A_i$, and let $P''_{k''}$ denote $\{\pi \in P_k : \pi \geq \pi_{\ell+2}\} \cup A_i$. Thus, we have the situation shown in Fig. 2, where $k' \geq 2$, $k'' \geq 2$ and $k' + k'' = k + 1$.

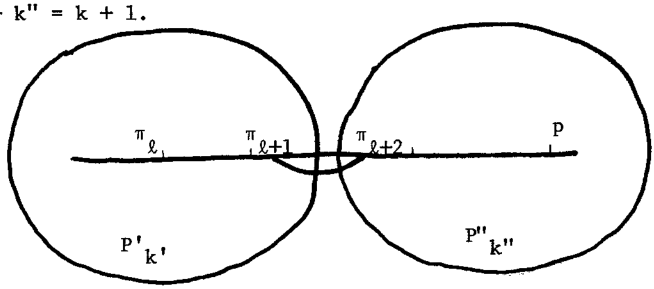

Fig. 2.

Therefore, by induction,

$$f(P_k) = f(P'_{k'}) + f^*(P''_{k''}) + 1$$
$$\leq 3k' - 3 + 3k'' - 4 + 1 = 3k - 3$$

where the +1 term comes from $d_p(\pi_{\ell+1}, \pi_{\ell+2})$. By reflecting the above picture, a similar argument proves the following result.

Fact 6. *Suppose there exist* $p < \pi_\ell < \pi_{\ell+1} < \pi_{\ell+2}$ *in* P_k *with* $\pi_{\ell+1} \in S$ *and*

$$F(p) \neq F(\pi_\ell) = F(\pi_{\ell+2}) \neq F(\pi_{\ell+1}).$$

Then

$$f(P_k) \leq 3k - 3.$$

Fact 7. *Suppose there exist* $\pi_\ell < \pi_{\ell+1} < \pi_{\ell'} < \pi_{\ell'+1}$ *in* P_k *with* $\pi_{\ell'} = \pi_\ell + 1$ *and so that both* $\pi_{\ell+1}$ *and* $\pi_{\ell'+1}$ *are regular points. Then* $\pi_{\ell'+1} = \pi_{\ell+1} + 1$.

Proof. Since $\pi_{\ell+1}$ is regular then $\pi_{\ell+1} + 1 = \pi_{\ell''} \in P_k$. Since $\pi_{\ell''} > \pi_\ell + 1 = \pi_{\ell'}$ then $\pi_{\ell''} \geq \pi_{\ell'+1}$. But $\pi_{\ell'+1}$ is not a starting point (by hypothesis) so that $\pi_{\ell'+1} - 1$ is a point of P_k which satisfies

$$\pi_\ell = \pi_{\ell'}-1 < \pi_{\ell'+1}-1 \leq \quad \pi_{\ell''}-1 = \pi_{\ell+1}.$$

However, this forces $\pi_{\ell'+1}-1 = \pi_{\ell+1}$ as asserted. □

In the same way the following fact is proved.

Fact 8. *Suppose there exist* $\pi_\ell < \pi_{\ell+1} < \pi_{\ell'} < \pi_{\ell'+1}$ *in* P_k *with* $\pi_{\ell'+1} = \pi_{\ell+1} + 1$ *and so that both* π_ℓ *and* $\pi_{\ell'}$ *are regular points. Then* $\pi_{\ell'} = \pi_\ell + 1$.

We next show that by a suitable modification of P_k, *we may form another set* P_k^* *satisfying:*

(i) P_k^* *is the union of* k *arithmetic progressions* A_i^*;

(ii) $f(P_k^*) \geq f(P_k)$;

(iii) *If* a *and* b *are distinct critical points of* P_k^* *then* $|a-b| \geq 10$;

(iv) *If* a *is a starting point of* P_k^* *and* b *is a terminal point of* P_k^* *then* $a < b$.

To achieve this, we make a sequence of minor transformations. To begin with, suppose $a \in A_i$, and $b \in A_j$ are both terminal points of P_k with $a < b$. (We can call this a T-T pair). For each m such that $a_m + n_m$, the largest element of A_m, satisfies

$$a_m + n_m > a,$$

replace A_m by $A'_m = \{a_m + x\colon 0 \leq x \leq n_m + 1\}$. Otherwise, let $A'_r = A_r$. Then in $P'_k = \bigcup_{t=1}^{k} A'_t$, no pairs of critical points are closer than the corresponding pairs in P_k were, and the distance between terminal points of A'_i and A'_j is strictly greater than that between a and b. By continuing this process, we can transform P_k to $\bar{P}_k$ in which all pairs of terminal points differ by at least 10.

Exactly the same techniques can be applied to pairs of starting points (S-S pairs) as well as to all pairs $\{a,b\}$ where a is a starting point, b is a terminal point and $a < b$ (i.e., S-T pairs). Thus, we may assume that we now have a set $\bar{P}_k$ in which the only pairs of critical points $\{a,b\}$ with $|a-b| < 10$ are of the form: a is a terminal point, b is a starting point and $a < b$. Let us consider such a pair $\{a,b\}$ with b-a minimal. By Fact 2, we may assume that $b-a \neq 0$, i.e., no starting points of $\bar{P}_k$ are also terminal points of $\bar{P}_k$. Let $a \epsilon \bar{A}_i$ and $b \epsilon \bar{A}_j$. By hypothesis, there is some largest element $\pi \epsilon \bar{P}_k$ with $\pi < a$. By Fact 4, we may assume $F(\pi) \neq F(a)$, i.e., $a - \pi < 1$. There are two possibilities:

(a) There exists $\pi' \epsilon \bar{P}_k$ with $a < \pi' < b$. We may assume without loss of generality that π' is the least such point. By hypothesis, π and π' are regular points. Furthermore, all the translates $\pi + x$ and $\pi' + x$ which fall in between a and b + 10 are regular points of $\bar{P}_k$. Hence, if we extend $\bar{A}_i$ to $\bar{A}_i'$ by letting $\bar{A}_i' = \{\alpha_i + x : 0 \leq x \leq \bar{n}_i + c\}$ where $b < a + c < b + 1$, keeping all other $\bar{A}_t$ the same, then the resulting set $\bar{P}_k'$ has $f(\bar{P}_k') \geq f(\bar{P}_k)$ and also has one less occurrence of a terminal point being smaller than a starting point. Now, we apply the previous transformations to separate all the S-S , S-T and T-T pairs to have mutual distances at least 10 again.

(b) a and b are adjacent points of $\bar{P}_k$. Thus, $b - a < 1$. In this case, we extend $\bar{A}_i$ by one more term, i.e., $\bar{A}_i' = \{\alpha_i + x : 0 \leq x \leq \bar{n}_i + 1\}$. But $\pi + 2$ and $b + 2$ are adjacent points of $\bar{P}_k$ (by Fact 1). Hence, the only P-length a + 1 might have destroyed, namely $d_p(\pi + 1, b + 1)$, is in fact equivalent to $d_p(\pi + 2, b + 2)$. Thus, as in (a), the new $\bar{P}_k'$ has

$f(\overline{P}'_k) \geq f(\overline{P}_k)$ and one less occurrence of a terminal point preceding a starting point. Again, the previous transformations may be applied to separate all the S-S, S-T and T-T pairs which are too close together.

It now follows by repeated application of the preceding process, we can reach the desired set

$$P_k^* = \bigcup_{i=1}^{k} A_i^* = \{\pi_1^* < \pi_2^* < \dots < \pi_n^*\}$$

satisfying (i) - (iv). □

In addition, by the preceding remarks we may also assume P_k^* satisfies the following conditions:

(v) If $i \neq j$ and $p_i \in A_i$, $p_j \in A_j$ then $|p_i - p_j| \neq 0,1$.

(vi) If π_ℓ^*, $\pi_{\ell+1}^* \in P_k^*$ and $\pi_{\ell+1}^*$ is not adjacent to $\pi_{\ell+1}^* + 1$ then $\pi_\ell^* + t$ is not adjacent to $\pi_{\ell+1}^* + t$ for any $t \geq 1$.

(vii) All $A_i^* = \{a_i + x\colon 0 \leq x \leq n_i^*\}$ have $n_i^* \geq 1$.

(viii) $\pi_2 - \pi_1 = \pi_N - \pi_{N-1} = 1$.

(ix) If $\pi_{\ell+1} - \pi_\ell = 1$ then either $p > \pi_{\ell+1}$ for all p with $F(p) \neq F(\pi_\ell)$, or $p < \pi_\ell$ for all p with $F(p) \neq F(\pi_\ell)$.

It remains to show

$$f(P_k^*) \leq 3k - 3. \tag{6}$$

Proof of (6). Suppose (6) does not hold (so that $k \geq 3$) Let $P_{k-1} = \bigcup_{i=1}^{k-1} A_i^*$. By the induction hypothesis.

$$f(P_{k-1}) \leq 3k - 6.$$

We may assume $a_k \neq \pi_1$ in P_k^* where $A_k^* = \{a_k + x\colon 0 \leq x \leq n_k^*\}$ defines a_k.

Let $Q_i = P_{k-1} \cup \{a_k + x: 0 \le x \le i\}$ for $i = 0,1,\ldots,n_k^*$.

By Fact 2,

$$f(Q_0) \le 3k - 4.$$

Suppose a is the least integer so that $f(Q_a) < f(Q_{a+1}) = 3k - 2$, where $0 \le a < n_k^*$. Let $\pi_\ell = a_k + a$, $\pi_{\ell'} = a_k + a + 1$ in P_k^*. We first note that we may assume $a \ge 1$. For suppose $f(Q_0) < f(Q_1)$. By (iii), since $\pi_\ell = a_k \in S$ then $\pi_{\ell-1}$ and $\pi_{\ell+1}$ are regular points. If $\ell' = \ell + 1$ then by (viii), $f(Q_0) = f(Q_1)$. If $\ell' > \ell + 1$ then by (iii) no new P-length is created and consequently $f(Q_0) = f(Q_1)$ which again is impossible. In what follows, the reader may find it helpful to construct linear diagrams representing the various cases under consideration. There are several possibilities.

Case 1. $\ell' = \ell + 1$. If $p < \pi_\ell$ for all $p \in P_k^*$ with $F(p) \ne F(\pi_\ell) = k$ then $f(Q_{a+1}) = f(Q_a)$ which contradicts our assumption. On the other hand if there exists $p \in P_k^*$ with $p > \pi_{\ell'}$ and $F(p) \ne k$ then (ix) is contradicted.

Case 2. $\ell' > \ell + 1$ and $\pi_{\ell'} > p$ for all $p \in Q_{a+1}$. Since every point between π_ℓ and $\pi_{\ell'}$ must be a terminal point of P_k^* then by (iii) there can be just one such point, which is $\pi_{\ell+1}$. Thus, by (iii) $\pi_\ell \notin S$ and so $\pi_{\ell-1}$ exists and $\pi_{\ell-1} \notin S$. Hence, $\pi_{\ell-1} + 1 = \pi_{\ell+1}$ and $f(Q_{a+1}) = f(Q_a)$ which contradicts our assumption.

Case 3. $\ell' = \ell + 2$ and there exists $p \in Q_{a+1}$ with $p > \pi_{\ell'}$. If $\pi_{\ell-1}$, $\pi_{\ell+1}$ and $\pi_{\ell+3}$ are regular points then by Facts 7 and 8, $F(\pi_{\ell-1}) = F(\pi_{\ell+1}) = F(\pi_{\ell+3})$ and $f(Q_a) = f(Q_{a+1})$. Hence, we may assume exactly one of them is critical. If $\pi_{\ell+1}$ is critical then at least one of

$\pi_{\ell-1}$, $\pi_{\ell+3}$ must be critical, which is impossible by (iii). Thus, we must have $\pi_{\ell+1}$ regular.

(a) $\pi_{\ell-1} \epsilon S$. In this case, we still must have $F(\pi_{\ell-1}) = F(\pi_{\ell+1}) = F(\pi_{\ell+3})$ and so, $f(Q_a) = f(Q_{a+1})$ which is impossible.

(b) $\pi_{\ell-1} \epsilon T$. Let us consider $d_p(\pi_{\ell-1}, \pi_{\ell+1})$ in Q_{a-1} (where Q_{-1} denotes P_{k-1}). Suppose the P-length $d_p(\pi_{\ell-1}, \pi_{\ell+1})$ of Q_{a-1} also occurs somewhere in Q_a. By Fact 1 we then have

$$f(Q_{a-1}) \leq 3k - 4.$$

Since $\pi_{\ell+1} - 1$ is a regular point then $f(Q_a) = f(Q_{a-1})$. Finally, since $\pi_{\ell+1}$ is also regular,

$$f(Q_{a+1}) \leq f(Q_a) + 1 \leq 3k - 3$$

which contradicts our assumption on $f(Q_{a+1})$. Suppose the P-length $d_p(\pi_{\ell-1}, \pi_{\ell+1})$ does not occur in Q_a. Then

$$f(Q_a) = f(Q_{a-1}) - 1$$

and

$$f(Q_{a+1}) \leq f(Q_a) + 1 = f(Q_{a-1}) \leq 3k - 3.$$

Case 4. $\ell' > \ell + 2$ and there exists $p \epsilon Q_{a+1}$ with $p > \pi_{\ell'}$. Thus, $F(p) \neq k$ and $\pi_{\ell+1} < \pi_{\ell'-1}$. By (iii), at most one of $\pi_{\ell-1}$, $\pi_{\ell+1}$, $\pi_{\ell'-1}$, $\pi_{\ell'+1}$ is a critical point. But since $f(Q_a) < f(Q_{a+1})$ at least one of them must be a critical point. In fact we must have exactly one of $\pi_{\ell-1} \epsilon T$, $\pi_{\ell+1} \epsilon T$, $\pi_{\ell'-1} \epsilon S$ or $\pi_{\ell'+1} \epsilon S$, since otherwise $f(Q_{a+1}) \leq f(Q_a)$ follows at once.

(a) Suppose $\pi_{\ell-1} \epsilon T$. By Fact 7, $\pi_{\ell'+1} = \pi_{\ell+1} + 1$. Thus, $F(\pi_{\ell-1}) \neq F(\pi_{\ell'-1})$, $F(\pi_{\ell+1}) = F(\pi_{\ell'+1})$. There are three possibilities.

(i) $\pi_{\ell'+1} - \pi_{\ell'-1} = 1$. But this implies

$$(\pi_{\ell'+1} - 1) - (\pi_{\ell'-1} - 1) = 1 = \pi_{\ell+1} - (\pi_{\ell'-1} - 1)$$

i.e.,

$$\pi_{\ell+1} = \pi_{\ell'-1}$$

which is a contradiction.

(ii) The P-length $d_p(\pi_{\ell-1},\pi_{\ell+1})$ of Q_{a-1} does not occur in Q_a. Then

$$f(Q_a) \le f(Q_{a-1}) - 1 \le 3k - 4$$

and

$$f(Q_{a+1}) \le f(Q_a) + 1 \le 3k - 3$$

since $\pi_{\ell+1}$ is regular.

(iii) The P-length $d_p(\pi_{\ell-1},\pi_{\ell+1})$ of Q_{a-1} occurs in Q_a. Then by Fact 1

$$f(Q_{a-1}) \le 3k - 4$$

and we also note that $f(Q_a) \le f(Q_{a-1})$, so

$$f(Q_{a+1}) \le f(Q_a) + 1 \le f(Q_{a-1}) + 1 \le 3k - 3.$$

The cases in which $\pi_{\ell+1}$, $\pi_{\ell'-1}$ and $\pi_{\ell'+1}$ are critical follow in a similar way and the arguments will be omitted. Hence in all cases $f(Q_{a+1}) \le 3k - 3$ which contradicts our hypothesis on Q_{a+1}. This completes the proof of (b) and Theorem 1 is proved. □

To see that (3) is best possible, we consider the following partition $P_k = \{\pi_1 < \pi_2 < \dots < \pi_N\}$, $k \ge 2$, defined as follows:

$$A_0 = \{0,1,\dots,k\},$$

$$A_i = \{x + \frac{1}{3^i} : x = i,i+1,\dots,i+k\},\ 1 \le i < k,$$

$$P_k = \bigcup_{i=0}^{k-1} A_i.$$

Then $D^*(P_k)$ consists exactly of the 3k - 3 distances

$$\bigcup_{i=0}^{k-1} \frac{1}{3^i} \quad \cup \quad \bigcup_{i=1}^{k-1} \frac{2}{3^i} \quad \cup \quad \bigcup_{i=1}^{k-2} \left(1 + \frac{1}{3^{k-1}} - \frac{1}{3^i}\right)$$

which is just the bound of (3).

3. The Circular Case

It is perhaps not surprising that the arguments needed to prove (1) are very similar to those used in the proof of Theorem 1. In addition, the inequalities (5) are themselves also of considerable assistance in the proof. Rather than give the step-by-step verification of the corresponding Facts for the circular case, we shall just state the required results with various additional comments from which the interested reader should have little difficulty in reconstructing a complete proof.

We first assume we are given a fixed θ with $0 < \theta < 1$, real numbers a_i and nonnegative integers n_i, $1 \le i \le k$. Since (1) is known to hold for $k = 1$, we shall assume $k > 1$. We let B_i denote the set $\{\{a_i + x\theta\}: 0 \le x \le n_i\}$ for $1 \le i \le k$, where $\{y\}$ denotes the *fractional part* of y. We may assume without loss of generality that $|B_i| = n_i + 1$ and $a_i = 0$. We denote the union of the B_i by

$$Q_k = \bigcup_{i=1}^{k} B_i = \{0 = \pi_1 < \pi_2 < \ldots < \pi_n < \pi_{n+1} = 1\} .$$

As before, we may also assume that for $p_i \in B_i$, $p_j \in B_j$, $i \neq j$, we have

$$|p_i - p_j| \neq 0, \theta .$$

For $\pi_\ell \in Q_k$, define $G: Q_k \to \{1,2,\ldots,k\}$ and $G': Q_k \to \{0,\infty\}$ by

$$G(\pi_\ell) = i \quad \text{where } \pi_\ell \in B_i,$$
$$G'(\pi_\ell) = m \quad \text{where } \pi_\ell = a_i + m\theta.$$

For $\ell \in \{1,2,\ldots,n\}$, define

$$d_Q(\pi_\ell,\pi_{\ell+1}) = (|\pi_{\ell+1} - \pi_\ell|,\ G(\pi_\ell),\ G(\pi_{\ell+1}),\ G'(\pi_\ell) - G'(\pi_{\ell+1})).$$

We call this the *Q-length* of the interval $(\pi_\ell,\pi_{\ell+1})$.

As before, we say that $(\pi_\ell,\pi_{\ell+1})$ and $(\pi_{\ell'},\pi_{\ell'+1})$ have *equivalent* Q-lengths provided either

(i) $\quad |\pi_{\ell+1} - \pi_\ell| = |\pi_{\ell'+1} - \pi_{\ell'}| = \theta$

or

(ii) $\quad |\pi_{\ell+1} - \pi_\ell| = |\pi_{\ell'+1} - \pi_{\ell'}| \neq \theta,\ G(\pi_\ell) = G(\pi_{\ell'}),$
$G(\pi_{\ell+1}) = G(\pi_{\ell'+1})$ and $G'(\pi_\ell) - G'(\pi_{\ell+1}) = G'(\pi_{\ell'}) - G(\pi_{\ell+1})$.

The definitions of *starting* point, *terminal* point, *critical* point and *regular* point are similar to those for the linear case. Finally, we let $g(Q_k)$ denote the number of inequivalent Q-lengths $d_Q(\pi_\ell,\pi_{\ell+1})$, $1 \le \ell \le n$, and we let $g^*(Q_k)$ denote the number of inequivalent Q-lengths $d_q(\pi_\ell,\pi_{\ell+1})$, $1 \le \ell \le n$, for which $|\pi_{\ell+1} - \pi_\ell| \neq \theta$. What we prove, which implies (1), is

THEOREM 2.

$$g(Q_k) \le 3k \quad \text{for } k \ge 1. \tag{7}$$

We shall also give examples to show that the bound of 3k in (1) can be achieved, so that (1) is best possible.

As before, the strategy will be to perform a sequence of normalizations on Q_k, eventually obtaining another set Q_k^* for which

$g(Q_k) \le g(Q_k^*)$ and so that the interactions between the various arithmetic progressions of Q_k^* have been "isolated". This will then allow $g(Q_k^*) \le 3k$ to be proved rather quickly. We assume that (7) holds for all values less than some fixed value of $k > 1$. (It is not difficult to show that it holds for $k=1$).

Fact 1'. *Let* $\pi_\ell, \pi_{\ell+1} \in Q_k$ *with* $G(\pi_\ell) \neq G(\pi_{\ell+1})$. *Suppose for some integer* $t > 1$, $\pi_\ell + t\theta = \pi_{\ell'}$, $\pi_{\ell+1} + t\theta = \pi_{\ell'+1}$ *but that* $\pi_\ell + t'\theta$ *and* $\pi_{\ell+1} + t'\theta$ *are* not *adjacent for any* t', $0 < t' < t$. *Then*

$$g(Q_k) \le 3k - 1.$$

The proof of this result is similar to that of Fact 1; one considers the set $\{\pi_m : \pi_\ell + t'\theta < \pi_m < \pi_{\ell+1} + t'\theta,\ 0 < t' < t\}$ corresponding to the set X in the proof of Fact 1.

Fact 2'. *Let* t *denote the number of* n_i, $1 \le i \le k$, *for which* $n_i = 0$. *Then*

$$g(Q_k) \le 3k - t,\ k > 1.$$

Also,

$$g(Q_k) = k \quad for \quad k = t.$$

Fact 3'. *Suppose there exist* $\pi_\ell, \pi_{\ell+1} \in Q_k$ *with* $\pi_{\ell+1} - \pi_\ell > \theta$.
Then

$$g(Q_k) \le 3(k-1) + 1.$$

To prove this, one simply breaks the circle between π_ℓ and $\pi_{\ell+1}$, unfolds it into a straight line and applies Theorem 1.

Fact 4'. *Suppose there exist* π_ℓ, $\pi_{\ell+1} \in Q_k$ *with* $\pi_{\ell+1} - \pi_\ell = \theta$.
Then

$$g(Q_k) \le 3k.$$

If there is only *one* such pair $\pi_\ell, \pi_{\ell+1}$ with $\pi_{\ell+1} - \pi_\ell = \theta$, then an argument similar to that used in the proof of Fact 3' applies. If there is more than one such pair then we apply induction using Fact 3.

In a similar way the following result can be established.

Fact 5'. *Suppose there exist* $\pi_\ell, \pi_{\ell+1}, \pi_{\ell+2} \in Q_k$ *with* $\pi_{\ell+2} - \pi_\ell = \theta$ *and* $\pi_{\ell+1} \in S \cup T$. *Then*

$$g(Q_k) \le 3k.$$

Fact 6'. *Suppose there exist* π_ℓ, $\pi_{\ell+1}$, $\pi_{\ell'}$, $\pi_{\ell'+1} \in Q_k$ *with* $\pi_{\ell'} = \pi_\ell + \theta$ *and suppose both* $\pi_{\ell+1}$ *and* $\pi_{\ell'+1}$ *are regular points. Then*

$$\pi_{\ell'+1} = \pi_{\ell+1} + \theta.$$

The proof is similar to that of Fact 7.

Fact 7'. *Suppose there exist* π_ℓ, $\pi_{\ell+1}$, $\pi_{\ell'}$, $\pi_{\ell'+1} \in Q$ *with* $\pi_{\ell'+1} = \pi_{\ell+1} + \theta$ *and suppose* π_ℓ *and* $\pi_{\ell'}$ *are regular points. Then*

$$\pi_{\ell'} = \pi_\ell + \theta .$$

The following result will now be basic.

Fact 8'. *For any given set* Q_k *we may form another set* $Q_k^* = \{0 = \pi_1^* < \ldots < \pi_N^* < \pi_{N+1}^* = 1\}$ *satisfying*

(i) Q_k^* *is the union of* k *arithmetic progressions* B_i^* *on the circle,*

(ii) $g(Q_k^*) \ge g(Q_k)$,

(iii) *If* a *and* b *are distinct critical points of* Q_k^*, *then* $|a-b| \ge 10\ \theta$,

(iv) *If* $p_i \in B_i^*$, $p_j \in B_j^*$, $i \ne j$ *then* $|p_i - p_j| \ne 0, \theta$,

(v) *If* π_ℓ, $\pi_{\ell+1} \in Q_k$ *and* $\pi_\ell + \theta$ *is not adjacent to* $\pi_{\ell+1} + \theta$

then for any $t \geq 1$, $\pi_\ell + t\theta$ *is not adjacent to* $\pi_{\ell+1} + t\theta$,

(vi) *Each* B_i^* *consists of at least two points,*

(vii) *For all* ℓ, $\pi_{\ell+1}^* - \pi_\ell^* < \theta$.

This result follows from the preceding facts, much in the same way as in the construction of P_k^*, except that θ must be replaced with a smaller value θ^*. To illustrate the type of reduction involved, suppose a and b are distinct starting points with $a < b < a + 10\theta$. To separate them we simply increase the length of the circle by 10θ. by adjoining an appropriate segment of arc A at the point a (as shown in Fig.3).

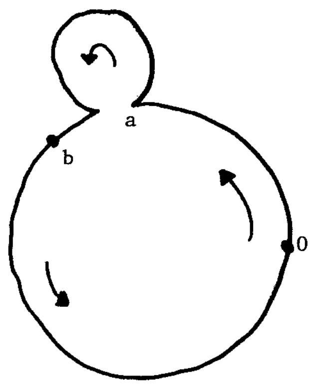

Fig. 3.

It is not difficult to see how to extend the various progressions B_i so that all the old Q-lengths still occur in the new circle (e.g., each time B_i hits $(a, a+\theta)$ we must take 10 extra steps to account for A). Then we scale the new circle down to a circle of circumference 1 by replacing by $\frac{\theta}{1+10\cdot\theta}$. Iterating this procedure eventually results in a partition which satisfies (iii).

Finally, the proof of (7) is now relatively straightforward, following roughly the same lines as the proof of (6). □

To see that (1) is best possible, we consider the following partition Q_k (where we may assume $k \geq 3$ since the construction for $k = 1$ and 2 are immediate).

Define
$$A_0 = \{-1,0,1,\ldots,k\},$$
$$A_i = \{x + \frac{1}{3^i} : x=i,\ i+1,\ldots,i+k\},\ 1 \leq i \leq k-3,$$
$$A_j = \{x + \frac{1}{3^j} : x=j,\ j+1,\ldots,j+k+1\},\ j = k-2,k-1,$$

and assume that these progressions denote taking steps of arc length 1 on a circle of circumference $2k + \frac{1}{\sqrt{2}}$. (This is just a slight modification of the example for the linear case which is wrapped around an appropriate circle). It is easily checked that for $Q_k = \bigcup_{i=0}^{k-1} A_i$, in addition to the $3k-3$ distances generated as in the linear case, we get 3 new distances as well, so that

$$|D^*(Q_k)| = 3k$$

as required, showing that (1) is tight.

Concluding Remarks.

It is not clear in which directions interesting generalizations of Theorems 1 and 2 lie. If we allow two different step sizes in the arithmetic progressions then it is possible to have an arbitrarily large number of distances between consecutive points. One could look at questions of this type in the plane or on a torus but these have not yet been investigated. It seems likely that in order to achieve $|D(P_k^*)| = 3k-3$ (or $|D(Q_k^*)| = 3k$ on a circle), one must have a fairly large total number of points. In our constructions, we used $O(k^2)$ points. Perhaps this is the correct order of magnitude for the minimum number required.

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