JOURNAL OF COMBINATORIAL THEORY (B) 18, 164–169 (1975)

On Multicolor Ramsey Numbers for Complete Bipartite Graphs

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Communicated by W. T. Tutte

Received June 6, 1974

INTRODUCTION

It follows from a well-known theorem of Ramsey [7] that for any finite graph G and any positive integer k, there exists a least integer $r(G; k)$ which has the following property.

> Any k-coloring of the edges of the complete graph K_r on r edges always has a monochromatic subgraph isomorphic to G, provided only that $r \geqslant r(G; k)$.

Most work up to now has dealt with the case $k = 2$. The reader is invited to read Burr [2] for an excellent survey of the current state of affairs on this subject. In this paper, we examine the case in which G is the complete bipartite graph $K_{s,t}$ and k is arbitrary.

Without loss of generality we may assume $s \leqslant t$. For $s = 1$, the numbers $r(K_{1,t})$ are known exactly. They are given [3] by

$$r(K_{1,t}\,; k) = \begin{cases} k(t-1)+1 & \text{if } \ k \equiv t \equiv 0 \pmod 2, \\ k(t-1)+2 & \text{otherwise.} \end{cases}$$

SOME UPPER BOUNDS

THEOREM 1.

$$r(K_{s,t}\,; k) \leqslant (t-1)(k+k^{1/s})^s \qquad \textit{for} \quad k > 1, \quad t \geqslant s \geqslant 2.$$

Proof. We first obtain an upper bound on the number of edges e a graph G on n vertices may have if G contains no subgraph isomorphic to $K_{s,t}$. Let $M = (m_{ij})$ denote the adjacency matrix of G. Since $K_{s,t} \not\subseteq G$ then

$$\sum_{j=1}^{n} m_{i_1,j} \cdot m_{i_2,j} \cdots m_{i_s,j} \leqslant t - 1 \tag{1}$$

where $1 \leqslant i_1 < \cdots < i_s \leqslant n$. If c_j denotes $\sum_{i=1}^{n} m_{ij}$ then summing (1) over all choices of $i_1, \ldots, i_s$, we obtain

$$\sum_{j=1}^{n} c_j(c_j - 1) \cdots (c_j - s + 1) \leqslant (t-1)\, n(n-1) \cdots (n - s + 1). \tag{2}$$

Since $f(x) = x(x-1) \cdots (x - s + 1)$ is convex for $x > s - 1$, then (2) implies

$$nf\left(\frac{1}{n} \sum_{j=1}^{n} c_j\right) \leqslant (t-1)\, f(n),$$

provided the argument of f exceeds $s - 1$. Since $e = \frac{1}{2} \sum_{j=1}^{n} c_j$, we have in this case

$$n((2e/n) - (s-1))^s \leqslant (t-1)\, n^s. \tag{3}$$

Now, for an arbitrary fixed $n \geqslant (t-1)(k + k^{1/s})^s$, let the edges of K_n be k-colored. Thus, some color occurs on at least $(1/k)\binom{n}{2}$ edges. Let G denote the subgraph which has these edges. Since

$$k^{1/s}(k + k^{1/s})^{s-1} \geqslant k + 1$$

then

$$(t-1)(k + k^{1/s})^s(1 - k/(k + k^{1/s})) > k(s-1) + 1$$

for $t \geqslant s \geqslant 2$. But because of the assumption on n, we have

$$n(1 - k((t-1)/n)^{1/s}) > k(s-1) + 1$$

i.e.,

$$n - 1 > k(s - 1 + n((t-1)/n)^{1/s}).$$

Thus, G has more than

$$\frac{1}{k}\binom{n}{2} > \frac{n}{2}\left(s - 1 + n\left(\frac{t-1}{n}\right)^{1/s}\right)$$

edges. However, (3) can be rewritten as

$$e \leqslant (n/2)(s - 1 + n((t-1)/n)^{1/s}) \tag{3'}$$

and this is also clearly valid in the case that

$$\tfrac{1}{4} \sum_{j=1}^{n} c_j \leqslant s - 1.$$

Thus, by (3), G must contain a monochromatic copy of $K_{s,t}$. This proves the theorem. ∎

A more careful argument can be used to prove the following somewhat stronger theorem.

THEOREM 1′.

$$r(K_{s,t}\,; k) \leqslant (t-1)\, k^s(1 + e(k))^s \qquad \text{for} \quad k \geqslant 1, t \geqslant s \geqslant 2$$

where $e(k) = k^{1-s}(s - 1 + k^{-1})(t-1)^{-1}$.

For the special case $s = 2$, a closer analysis along the same lines can be used to establish the following result.

THEOREM 2.

$$r(K_{2,t}\,; k) \leqslant (t-1)\, k^2 + k + 2.$$

By a refinement of this argument for the case $t = 2$, one may obtain the following.

COROLLARY 1.

$$r(K_{2,2}\,; k) \leqslant k^2 + k + 1 \qquad \text{for} \quad k > 1.$$

As we shall see, this upper bound for the 4-cycle $K_{2,2}$ is fairly close to the known lower bound. The upper bound

$$r(K_{2,2}\,; k) < ck^2$$

for a suitable $c > 0$ had been previously obtained by Hajnal and Szemerédi (unpublished).

For the case $s = t$, Chvátal [6] has obtained the bound

$$r(K_{t,t}\,; k) \leqslant 2tk^t$$

which differs asymptotically from our bound for this case by a factor of 2.

SOME LOWER BOUNDS

We begin with a bound on $r(K_{2,2}\,; k)$.

THEOREM 3. *For $k - 1$ a prime power,*

$$r(K_{2,2}\,; k) > k^2 - k + 1.$$

Proof. Since $k - 1$ is a prime power, then it is well known that there exists a simple difference set $D = \{d_1, ..., d_k\}$ modulo $(k^2 - k + 1)$. For each t, $1 \leqslant t \leqslant k$, form a cyclic (symmetric) matrix $B_t = (b_t(i, j))$ as follows:

$$b_t(i,j) = \begin{cases} 1 & \text{if } \ i + j + d_t \equiv d_s \ (\text{mod } k^2 - k + 1) \ \text{ for some } \ d_s \in D, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Since D is a difference set, then it follows that for $i, j \in \mathbb{Z}_{k^2-k+1}$ (the integers modulo $(k^2 - k + 1)$), there exists a t such that $b_t(i, j) = 1$. Furthermore, for each t, no two rows of B_t have a common pair of 1's.

We now form a k-colored K_{k^2-k+1} as follows. The vertices of K_{k^2-k+1} will be the elements of $\mathbb{Z}_{k^2-k+1}$. The color of the edge $\{i, j\}$ for $i, j \in \mathbb{Z}_{k^2-k+1}$ is defined to be the least integer t such that $b_t(i, j) = 1$. By the preceding remark, no two rows of any B_t have a common pair of 1's and so, no monochromatic 4-cycle $K_{2,2}$ occurs in K_{k^2-k+1}. This shows that $r(K_{2,2}; k) > k^2 - k + 1$ and the theorem is proved. ▮

A somewhat similar technique, based on n-dimensional projective geometries over finite fields, can be used to prove the following result:

$$r(K_{2,k}n; k) = k^{n+2} + o(k^{n+2}). \tag{5}$$

The details of the proof of (5) are a bit complicated and will not be given here (cf. [5]).

We remark that for two colors, it has been shown [4], [5] that

$$r(K_{2,t}; 2) \geqslant 4t - 2, \qquad 4t - 3 \text{ a prime power.}$$

The best lower bound we know for the general case is given by a simple counting argument.

THEOREM 4.

$$r(K_{s,t}; k) > (2\pi\sqrt{st})^{1/(s+t)}((s + t)/e^2)\, k^{(st-1)/(s+t)}.$$

Proof. Call a k-coloring of K_n *bad* if it contains a monochromatic $K_{s,t}$. It is easy to see that there are at most

$$\binom{n}{s+t}\binom{s+t}{s} k \cdot k^{\binom{n}{2}-st}$$

bad colorings. Hence, if this expression is less than the total number of k-colorings $k^{\binom{n}{2}}$ then we can deduce the inequality

$$r(K_{s,t}; k) > n.$$

Elementary calculations now show that if

$$n \leqslant (2\pi \sqrt{st})^{1/(s+t)}((s+t)/e^2)\, k^{(st-1)/(s+t)}$$

where e denotes the base for natural logarithms, then

$$\binom{n}{s+t}\binom{s+t}{s} k^{\binom{n}{2}-st+1} < k^{\binom{n}{2}}$$

as required. This proves the theorem. ∎

Note that for $t >> s$, (6) becomes essentially

$$r(K_{s,t}\,; k) > (t/e^2)\, k^s \tag{6'}$$

which is fairly close to the upper bound in Theorem 1.

Concluding Remarks

For a given graph G and integer $n \geqslant |G|$, define $T(G; n)$ to be the least integer m such that if H is any graph on n vertices with m edges then H must contain a subgraph isomorphic to G. These numbers are known as the *Turán* numbers for G. Clearly, if $R(G;n)$ denotes the minimum number of colors necessary to color K_n without forming a monochromatic G, then

$$R(G; n) > \binom{n}{2}\Big/T(G; n). \tag{7}$$

Since

$$r(G; R(G; n) - 1) \leqslant n < r(G; R(G; n)) \tag{8}$$

then knowledge of $T(G; n)$ can be used to deduce bounds on $r(G; k)$. It was pointed out by Spencer [8] that in certain cases a simple probabilistic argument can be given which establishes *upper* bounds on $R(G; n)$. In particular, if $T(G; n) = o(n^2)$, then we have

$$R(G; n) = O((n^2 \log n)/T(G; n)). \tag{9}$$

For example, since it has been shown by Brown [1] that $T(K_{3,3}\,; n) = (n^{5/3}/2)(1 + o(1))$, then we can conclude

$$r(K_{3,3}\,; k) > ck^3/\log^3 k \tag{10}$$

for some $c > 0$. Unfortunately, no very good bounds are currently known for $T(K_{r,s}\,; n)$.

It can also be shown using results from the theory of cyclotomy that

$$\lim_{t\to\infty} (1/t)\, r(K_{2,t}\,;\, k) = k^2.$$

The details may be found in [5].

It does not seem unreasonable to conjecture that in general, for $t \geqslant s \geqslant 2$,

$$r(K_{s,t}\,;\, k) \sim (t-1)\, k^s + o(k^s). \tag{11}$$

The authors take pleasure in acknowledging the valuable suggestions of H. S. Wilf.

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