Therefore it follows from (15) and (16) that $\{h_{4n}(x)\}_1^\infty$ and $\{h_{4n-2}(x)\}_1^\infty$ are monotonic and consequently convergent. According to (16) their limits must be equal to the solution of $H_2(u) = 0$ so that

$$\lim_{n\to\infty} h_{2n}(x) = u_0 = -\tfrac{1}{2} + \sqrt{x - \tfrac{3}{4}} \quad \text{if } x > 8. \tag{17}$$

On account of (8) we can apply our method in any simply-connected domain excluding the circle $|z| < 2$ and including the points $z = x > 2$, and obtain (17) even for $x > 2$.

For odd indices we get from (17) and definition (13)

$$\lim_{n\to\infty} h_{2n-1}(x) = -(\tfrac{1}{2} + \sqrt{x - \tfrac{3}{4}}) \quad (x > 2).$$

Hence, $\lim_{n\to\infty} h_{2n}(x) \neq \lim_{n\to\infty} h_{2n-1}(x)$, i.e., $\{h_n(x)\}_1^\infty$ diverges in $x > 2$.

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ARE THERE $n+2$ POINTS IN E^n WITH ODD INTEGRAL DISTANCES?

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In this note we answer the question posed in the title.

THEOREM 1. *For the existence of $n+2$ points in E^n so that the distance between any two of them is an odd integer, it is necessary and sufficient that $n+2 \equiv 0 \pmod{16}$.*

There are analogous results concerning integral distances relatively prime to 3 or 6 which we mention at the end of this work.

The main tool in the proof of the necessity part of Theorem 1 is a theorem of Cayley (see, e.g. [1], p. 122).

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THEOREM 2. *Let the set of* $\binom{n+2}{2}$ *nonnegative numbers* d_{ij}; $1 \leqq i < j \leqq n+2$ *be a set of distances* $d_{ij} = d(p_i, p_j)$ *of points* $p_1, p_2, \cdots, p_{n+2}$ *in* E^n. *Then*

$$\Delta = \begin{vmatrix} 0 & d_{12}^2 & d_{13}^2 & \cdots & d_{1\,n+2}^2 & 1 \\ d_{21}^2 & 0 & d_{23}^2 & \cdots & d_{2\,n+2}^2 & 1 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ d_{n+2\,1}^2 & d_{n+2\,2}^2 & \cdot & \cdots & 0 & 1 \\ 1 & 1 & \cdot & \cdots & 1 & 0 \end{vmatrix} = 0,$$

where $d_{ij} = d_{ji}$.

Proof. We consider the points p_i as vectors in R^n and assume without loss of generality that $p_{n+2} = 0$, the origin. Then

$$d_{ij}^2 = |p_i - p_j|^2 = |p_i|^2 + |p_j|^2 - 2(p_i, p_j)$$

and

$$\Delta = \begin{vmatrix} (|p_i|^2 + |p_j|^2 - 2(p_i, p_j)) & & & & & 1 \\ & & & & & \vdots \\ & & & & & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & 1 & 0 \end{vmatrix}.$$

Subtracting $|p_i|^2$ times the last row from the ith row and $|p_j|^2$ times the last column from the jth column we get

$$\Delta = \begin{vmatrix} & & & 1 \\ & -2(p_i, p_j) & & \vdots \\ & & & 1 \\ 1 & \cdots & 0 & 1 \end{vmatrix} = \begin{vmatrix} -2(p_1, p_1) & \cdots & -2(p_1, p_{n+1}) & 0 & 1 \\ \vdots & & & & \\ -2(p_{n+1}, p_1) & \cdots & -2(p_{n+1}, p_{n+1}) & 0 & 1 \\ 0 & \cdots & 0 & 0 & 1 \\ 1 & \cdots & 1 & 1 & 0 \end{vmatrix}$$

$$= (-1)^n 2^{n+1} \begin{vmatrix} (p_1, p_1) & \cdots & (p_1, p_{n+1}) \\ \vdots & & \vdots \\ (p_{n+1}, p_1) & \cdots & (p_{n+1}, p_{n+1}) \end{vmatrix}$$

$$= (-1)^n 2^{n+1} \det(P \cdot P^{tr}),$$

where the $n \times (n+1)$ matrix

$$P = \begin{bmatrix} p_1 \\ \vdots \\ p_{n+1} \end{bmatrix}$$

has rank $P \leqq n$, and hence rank $(P \cdot P^{tr}) \leqq n$, so that $\det(P \cdot P^{tr}) = 0$.

REMARK. For an alternate proof, consider the linear mapping $(a_1, \cdots, a_{n+2}) \to (\Sigma a_j p_j, \Sigma a_j)$ on R^{n+2} into the $(n+1)$-dimensional space $E^n \oplus R$. It has non-zero kernel, so there is a vector $(a_1, \cdots, a_{n+2}) \neq \mathbf{0}$ such that $\Sigma a_j p_j = \mathbf{0}$ and $\Sigma a_j = 0$. Set $c = -\Sigma a_j |p_j|^2$. By a short direct calculation,

$$\sum_{j=1}^{n+2} a_j |p_i - p_j|^2 + c = 0, \qquad \Sigma a_j = 0.$$

This is a system of $n + 3$ equations and it has the non-trivial solution $(a_1, \cdots, a_n, c)$, so its determinant is zero. That is Theorem 2.

The necessity of Theorem 1 now follows from a lemma.

LEMMA 1. *Let* d_{ij}; $1 \leqq i < j \leqq n+2$ *be a set of odd integers. Then*

$$\Delta \equiv (-1)^n (n+2) (\mathrm{mod}\, 16).$$

Proof. Since d_{ij} is an odd integer we get $c_{ij} = d_{ij}^2 - 1 \equiv 0 \pmod 8$. Subtracting the last column of Δ from all other columns we have

$$\Delta = \begin{vmatrix} -1 & c_{12} & \cdots & c_{1\,n+2} & 1 \\ c_{21} & -1 & \cdots & c_{2\,n+2} & 1 \\ \vdots & & & \vdots & \\ c_{n+2\,1} & & \cdots & -1 & 1 \\ 1 & & \cdots & 1 & 0 \end{vmatrix}$$

By first adding the first $n + 2$ columns to the last column and then adding the first $n + 2$ rows to the last row, we get

$$\Delta = \begin{vmatrix} -1 & c_{12} & \cdots & c_{1\,n+2} & a_1 \\ c_{21} & -1 & \cdots & c_{2\,n+2} & a_2 \\ \vdots & \vdots & & \vdots & \\ c_{n+2\,1} & \cdot & \cdots & -1 & a_{n+2} \\ 1 & \cdot & \cdots & 1 & n+2 \end{vmatrix} = \begin{vmatrix} -1 & c_{12} & \cdots & c_{1\,n+2} & a_1 \\ c_{21} & -1 & \cdots & c_{2\,n+2} & a_2 \\ \vdots & & & & \\ c_{n+2\,1} & & \cdots & -1 & a_{n+2} \\ a_1 & & \cdots & a_{n+2} & n+2+a \end{vmatrix},$$

where $a_i = \sum_{j \neq i} c_{ij}$ and $a = \sum_{i=1}^{n+2} a_i = 2 \sum_{i<j} c_{ij}$.

Since all the terms off the main diagonal in the last expression of Δ are divisible by 8; and each product in the expansion of Δ other than the main diagonal term contains at least two off-diagonal factors; we have

$$\Delta \equiv (-1)^{n+2}(n+2+a) = (-1)^n(n+2+a) \pmod{64}.$$

But $a = 2\sum_{i<j} c_{ij} \equiv 0 \pmod{16}$ so that

$$\Delta \equiv (-1)^n(n+2) \pmod{16}.$$

We can now complete the proof of Theorem 1 by making a suitable construction.

Let $n = 16s - 2$ and choose $p_1, \cdots, p_n$ as vertices of a regular $(n-1)$-simplex of edge length $8s - 1$ in the hyperplane $x_n = 0$ so that its centroid is at the origin. Choose the remaining two points p_{n+1}, p_{n+2} as $(0, \cdots, 0, \pm(2s - \frac{1}{2}))$ on the x_n-axis. In this set there are only three distinct distances, $d(p_i, p_j) = 8s - 1$ for $1 \leqq i < j \leqq n$; $d(p_{n+1}, p_{n+2}) = 4s - 1$ and

$$(1) \qquad d(p_i, p_{n+k})^2 = |p_i|^2 + (2s - \tfrac{1}{2})^2; \qquad 1 \leqq i \leqq n;\ k = 1, 2.$$

In order to compute this last distance we need the following:

LEMMA 2. *The distance from the centroid of a unit simplex in E^k to a vertex* is $d_k = \sqrt{k/(2k+2)}$.

Proof. The unit vectors in E^{k+1} form the vertices of a regular k-simplex of edge length $\sqrt{2}$ with centroid $(1/(k+1))\,(1, 1, \cdots, 1)$. Thus the distance from a vertex to the centroid is

$$\begin{aligned}\sqrt{2}d_k &= \sqrt{\left(1 - \frac{1}{k+1}\right)^2 + \left(\frac{1}{k+1}\right)^2 + \cdots + \left(\frac{1}{k+1}\right)^2} \\ &= \sqrt{\frac{k^2}{(k+1)^2} + \frac{k}{(k+1)^2}} = \sqrt{\frac{k}{k+1}}.\end{aligned}$$

Thus the value of $|p_i|^2$ in (1) is

$$|p_i|^2 = (8s-1)^2 d_{16s-3}^2 = (8s-1)^2 \cdot \frac{16s-3}{2(16s-2)} = \frac{128s^2 - 40s + 3}{4},$$

and

$$\begin{aligned}d(p_i, p_{n+k})^2 &= \tfrac{1}{4}(128s^2 - 40s + 3 + 16s^2 - 8s + 1) \\ &= (6s-1)^2.\end{aligned}$$

We have thus constructed a set with $n + 2 = 16s$ points and only three distinct distances, $4s - 1$, $6s - 1$ and $8s - 1$, all of which are odd, attained respectively once, $2n$ and $\binom{n}{2}$ times.

There are many other examples of constructing $(n+2)$-tuples of points with only three distances, all odd in case $n = 16s - 2$. For example we could construct regular simplices in complementary orthogonal subspaces E^{14s-2} and E^{2s} with edge lengths $14s - 1$ and $2s + 1$ respectively. The third distance d satisfies

$$d^2 = (14s-1)^2 d_{14s-2}^2 + (2s+1)^2 d_{2s}^2 = (10s-1)^2$$

REMARK. It is impossible to have $n+3$ points in E^n so that all distances are odd integers since by Theorem 1 this would imply both $n+2 \equiv 0 \pmod{16}$ and $(n+1)+2 \equiv 0 \pmod{16}$.

The reasoning in Lemma 1 can be applied equally well in the case of integral distances relatively prime to 3.

LEMMA 3. *Let* d_{ij}; $1 \leqq i < j \leqq n+2$ *be a set of integers relatively prime to* 3. *Then* $\Delta \equiv (-1)^n(n+2) \pmod 3$.

Proof. Since $d_{ij}^2 \equiv 1 \pmod 3$ we get

$$\Delta \equiv \left| J - I \right|_{n+3} = (-1)^n(n+2) \pmod 3.$$

THEOREM 3. *There exist* $n+2$ *points in* E^n *whose distances are integers relatively prime to* 3 *if and only if* $n \equiv 1 \pmod 3$.

There exist $n+2$ *points in* E^n *whose distances are integers relatively prime to* 6 *if and only if* $n \equiv -2 \pmod{48}$.

Proof. The necessity of the two congruences follows from Lemma 3 and Theorem 1.

For sufficiency in the second case we can use the same construction used in the proof of Theorem 1. Set $n = 48s - 2$ and construct the set of $n+2$ points with distances $12s - 1$, $18s - 1$ and $24s - 1$ respectively.

For sufficiency in the first case, set $n = 3s + 1$ and construct a regular simplex in a hyperplane E^{3s} of side length $4(3s+1)$ with centroid at the origin; then add two more points on the axis perpendicular to E^{3s} at distances $3s - 1$ from the origin. We then get three distances $4(3s+1)$, $6s - 2$, $9s + 1$ since

$$(9s+1)^2 = (3s-1)^2 + \frac{3s}{2(3s+1)} \cdot 16(3s+1)^2.$$

These distances are attained respectively $\binom{3s+1}{2}$ times, once and $6s+2$ times.

Our examples involve sets of points determining three distinct distances. One might ask whether there are examples involving $(n+2)$-tuples of points with only two distinct distances. The answer appears to be in the negative for odd distances, while there are certain dimensions in which there are examples of $(n+2)$-tuples with only two distinct distances both prime to 3.

One could generalize the above results to conditions on integral distances of the form $d_{ij}^2 \equiv 1 \pmod m$ for general moduli m. However this does not appear as attractive as the above treated problems.

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