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EUCLIDEAN RAMSEY THEOREMS, III

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1. INTRODUCTION

In the first two parts of this study [1] [2], we investigated the general question: Let K be a set in Euclidean n-space, E^n. Let the points of E^n be r-colored (divided into r classes) in any way. Then is there a monochromatic set K' congruent to K (or similar to K, or a translate of K, etc.)? In this case we consider only the special case $n = 2$, $r = 2$, $|K| = 3$. That is if K is a triangle (= a set of three points) in E^2, we consider the following statement: $R(K)$: For any 2-coloring of E^2 there is a monochromatic K' congruent to K.

We recall from [1] that there are some K for which $R(K)$ is false. For let $E^2 = R \cup B$ where $R = \bigcup_{n=-\infty}^{\infty} \left\{(x, y) \mid nd\sqrt{3} \leqslant y < \left(n + \frac{1}{2}\right)d\sqrt{3}\right\}$,

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$B = E^2 - R$, and $d > 0$ is a real number. Then no equilateral triangle of side d is monochromatic. This is a coloring in alternate half-open strips of height $d \frac{\sqrt{3}}{2}$. If certain changes are made on the boundaries, $y = \frac{\sqrt{3}}{2} m$, there still may be no monochromatic equilateral d-triple. For instance, if the colors of each of the points $n\left(\frac{1}{2} d, \frac{\sqrt{3}}{2} d\right)$ are changed from R to B or vice-versa, there are still no monochromatic equilateral triangles of side d. We make the following conjecture.

Conjecture 1. *The only 2-colorings of E^2 for which there are no monochromatic equilateral triangles of side d are colorings in alternate strips of width $(\sqrt{3}/2)d$, as above, except for some freedom in coloring the boundaries between the strips.*

It is easy to check that if this kind of coloring is employed, only the equilateral triangle of side d fails to occur monochromatically. If $d' \neq d$, then there is a monochromatic equilateral triangle of side d'. Thus such strip colorings can avoid only one size equilateral triangle. This leads to a weaker conjecture, which may hold even if Conjecture 1 fails.

Conjecture 2. *If E^2 is 2-colored so that there is no equilateral triangle of side d, then there is a monochromatic equilateral triangle of side d', for $d' \neq d$.*

Now as we saw in [1] (and in Theorem 1 below), if there is a monochromatic equilateral triangle of side a in E^2, then there is a monochromatic triangle of sides a, b and c for every b, c such that a, b and c can be the sides of a triangle. Thus if K is any triangle which is not equilateral, and if Conjecture 2 is true, there must be some monochromatic K' congruent to K for any 2-coloring of E^2. This finally leads to a conjecture, which in view of Theorem 1 is equivalent to Conjecture 2.

Conjecture 3. *If K is a triangle which is not equilateral, then $R(K)$ is true.*

Aside from the questions of monochromatic triples K, one could ask whether various other 2-colorings of K occur. In general, a triple

$\{A, B, C\}$ with distinct distances has 4 inequivalent colorings, namely all three points the same color (monochromatic), and 2 points one color, the third point the other color (there are three choices for this coloring). For a 2-coloring f of E^2, we ask which of the four must occur. That is, for which of the following four possibilities is there a triple $\{A', B', C'\}$ congruent to $\{A, B, C\}$?

$$f(A') = f(B') = f(C')$$

$$f(A') \neq f(B') = f(C')$$

$$f(B') \neq f(A') = f(C')$$

$$f(C') \neq f(A') = f(B')$$

For isosceles triples this reduces to three possibilities, and for equilateral triples to two. Of course, if f colors the whole plane one color, then the first possibility occurs for all triples, and none of the latter three do.

It is obvious that for pairs of points distance d apart, both the monochromatic and bichromatic cases occur, unless f is a 1-coloring.

If $\{A, B, C\}$ is a collinear triple with B between A and C, then it is possible to color E^2 with two colors so that no congruent $\{A', B', C'\}$ has $f(A') = f(C') \neq f(B')$. Namely, we color (x, y) red if $y > 0$ and blue if $y \leqslant 0$. This leads to our final conjecture:

Conjecture 4. *If f is a 2-coloring of E^2 (which is not a 1-coloring), and $\{A, B, C\}$ is a triple such that there is no congruent triple $\{A', B', C'\}$ with $f(A') = f(C') \neq f(B')$, then $\{A, B, C\}$ is a collinear triple and B is between A and C.*

For convenience we use the notation $R_f(\bar{a}, b, c)$ to indicate that there exist (a, b, c)-triangles in the coloring f of E^2 for which the two endpoints of the a-side have the same color and the third point has the opposite color.

In this paper we primarily pursue Conjecture 3, and find many triples K which satisfy $R(K)$. We also obtain many bichromatic triples which must occur. Below is a partial list.

Monochromatic triangles

1. Triangles with ratios between two sides equal to r where $r = 2 \sin(\theta/2)$ and $\theta = 30°, 72°, 90°, 120°$.

2. $30°$ and $150°$ triangles.

3. Triangles whose sides are equal in length to the sides and the circumradius of an isosceles triangle.

4. Triangles with angles $(\alpha, 2\alpha, 180° - 3\alpha)$, $0° < \alpha < 60°$ and $(180° - \alpha, 180° - 2\alpha, 3\alpha - 180°)$, $60° < \alpha < 90°$.

5. Triangles (a, b, c) with $a^6 - 2a^4b^2 + a^2b^4 - 3a^2b^2c^2 + b^2c^2 = 0$ or $a^4c^2 + a^2b^4 - 5a^2b^2c^2 + b^2c^4 = 0$.

6. (Degenerate) triangles $(a, 2a, 3a)$.

7. Right triangles (a, b, c), $a^2 + b^2 = c^2$, with (i) b^2/a^2 rational, (ii) $\tan^{-1} b/a$ a rational multiple of $90°$ and many other special triangles (Theorems 6, 7).

Bichromatic triangles

1. Isosceles triangles with one base vertex opposite to the other vertices.

2. All colorings of the isosceles $120°$ triangle.

3. All colorings of the right (a, b, c)-triangle with $a^2 + b^2 = c^2$, b^2/a^2 rational, or $(\tan^{-1} b/a)/90°$ rational with even denominator.

2. CONDITIONAL THEOREMS

In this section we obtain theorems which say that if certain triples are monochromatic (or sometimes if certain triples are not monochromatic) then others are (or are not) monochromatic. It is convenient to define a new statement for each 2-coloring f of E^2:

$R_f(K)$: *Some* K' *congruent to* K *is monochromatic under* f.

The most useful conditional theorem is the following, which is a strengthening of Theorem 8 of [1].

Theorem 1. *Let* K *be a triangle with sides* a, b *and* c, *and let* K_a, K_b *and* K_c *be equilateral triangles with sides* a, b *and* c, *respectively. Then* $R_f(K)$ *is true if and only if at least one of* $R_f(K_a)$, $R_f(K_b)$ *or* $R_f(K_c)$ *is true.*

Proof. The proof is immediate. Consider the configuration in Figure 1. The six triangles $HBC, ABD, CDE, EFH, DFG, AHG$ all have sides a, b and c. The triangles $ABH, DFE, BCD, FGH, HEC, ADG$ are equilateral with sides a, a, b, b, c, c, respectively. As in the proof of Theorem 8 of [1], we see that if one of the second six triangles is monochromatic, one of the first six must be. The converse is true by a symmetric argument. This completes the proof.

We are indebted to Raphael M. Robinson for the following remark.

Remark. Since the six triangles congruent to K in Figure 1 are like-oriented we have actually proved the stronger fact:

If a triangle K *has a monochromatic like-oriented congruent copy* K' *in a 2-coloring* f *of* E^2 *then it also has a monochromatic opposite-oriented congruent copy* K''.

The analogous statement for bichromatic colorings is not known to us.

Theorem 1 has several immediate Corollaries.

Corollary 2. *If* K *is a triple with sides* a, a, b *and if* $R_f(K)$ *is true, then* $R_f(K^*)$ *is true for any triple* K^* *with sides* a, b, c, *where* $|a - b| \leqslant c \leqslant a + b$.

Corollary 3. *If* $R_f(K_a)$ *is false* (K_a *the equilateral triple of side* a), *but* $R_f(K)$ *is true, for a triple* K *with sides* a, b, c, *then* $R_f(K^*)$ *is true for* K^* *any triple with sides* b, c, d, $|b - c| \leqslant d \leqslant b + c$.

Let $T = \{(a, b, c),\ 0 \leqslant a \leqslant b \leqslant c \leqslant a + b\}$. For each 2-coloring f

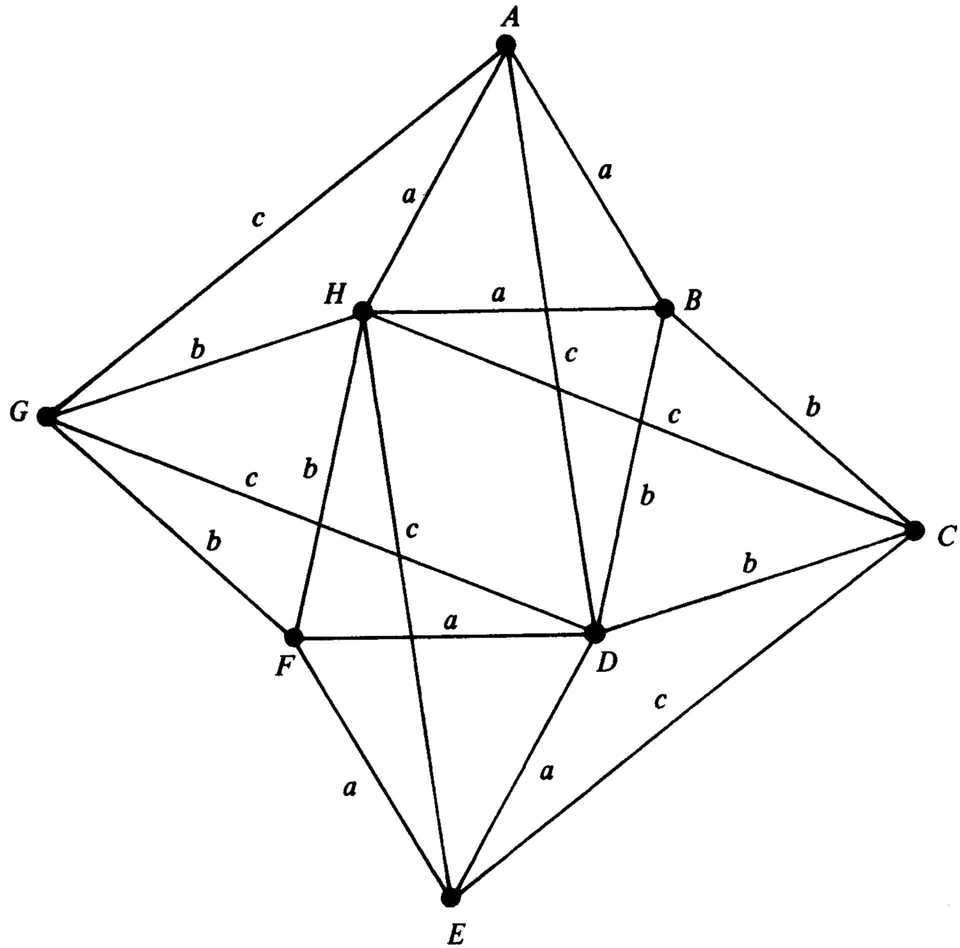

Figure 1

of E^2 let T_f be the subset of T consisting of the triples (a, b, c), such that there are no monochromatic triangles with sides a, b and c. In view of Theorem 1, Conjecture 3 is equivalent to the statement that $T_f \subset \{(a, a, a) \mid a > 0\}$ for all f.

Corollary 4. *Let K be an (a, a, b)-triangle such that $R(K)$ holds. Let f be a 2-coloring of E^2, and suppose $(c, d, e) \in T_f$. Then $(bc/a, bc/a, bc/a) \notin T_f$, and $(ac/b, ac/b, ac/b) \notin T_f$.*

Proof. If $R(K)$ is true, we observed that $R(K')$ is true for K' a (ta, ta, tb)-triangle, $t > 0$. In particular, if $t = c/a$, Theorem 1 implies then that either $(c, c, c) \notin T_f$ or $(bc/a, bc/a, bc/a) \notin T_f$. The former cannot hold by Theorem 1, since $(c, d, e) \in T_f$. Thus $(bc/a, bc/a, bc/a) \notin T_f$. A similar argument gives $((a/b)c, (a/b)c, (a/b)c) \notin T_f$. This completes the proof. We note that d or e can replace c above.

Conjecture 2 says that $T_f = \{(a, a, a)\}$ for some $a > 0$, or $T_f = \phi$. We show something not as strong below.

Theorem 5. *For every 2-coloring f of E^2, the set T_f is totally disconnected in E^3.*

Proof. Let f be a 2-coloring of E^2, and let (a', b', c') and (a, b, c) both be in the same connected component of T_f. We can assume that $a < a'$. Then by Theorem 1, $(d, d, d) \in T_f$ for all $d \in [a, a']$ or else the plane $H_d = \{(x, y, z) \mid z = d\}$ would separate (a, b, c) and (a', b', c'), and $H_d \cap T_f = \phi$.

Consider two points x, y, distance $\frac{a + a'}{2} = a''$ apart and both the same color, say red. Let z be a point such that x, y, z are an equilateral triple. Then, by Theorem 1, the disc with center z and radius $\frac{a' - a}{2}$ must be all blue.

Now we claim that no circle C of radius $r > \frac{a''}{2}$ is monochromatic. For if such a C exists, say it is red, then we choose two points distance a'' apart on C and we then move around the circle. We get a blue annulus of thickness $\frac{a' - a}{2}$ and mean radius $r' = \frac{\sqrt{3}}{2}a'' + \sqrt{r^2 - \frac{(a'')^2}{4}}$. Now repeating this argument for each blue circle in the annulus, we get a red annulus of thickness $2\left(\frac{a' - a}{2}\right)$. Continuing in this way, we get monochromatic annuli of thicknesses $n\left(\frac{a' - a}{2}\right)$ for all n, a contradiction since this would imply $T_f = \phi$.

Now let x and y be two points of different colors which are distance $\epsilon < \frac{a' - a}{2} 10^{-6}$ apart. Let x be red, y blue. Consider the circle C of radius a'' and center x. Since it cannot be monochromatic by the argument above, there must be two points x' and y' with opposite colors in C with distance $\leqslant \epsilon$. Let x' be red.

As above, if z forms an equilateral triangle with x and x', we have z blue and a disc of radius at least $\frac{a' - a}{2}$ and center z which is all blue. But there is also a z' which makes an equilateral triangle with y and y', and z' is the center of a red disc of radius at least $\frac{a' - a}{2} - 2\epsilon$.

Since the distance between z and z' is at most 2ϵ, this is a contradiction, as we get two, overlapping, differently colored discs. This completes the proof.

For right triangles we can employ certain special arguments. The first may be called the "ladder method", and is used below.

Theorem 6. *Let f be a 2-coloring of E^2 and let n be a positive integer. Let $a^2 + b^2 = c^2$, and let K be a triple with sides a, b, c. Let K' be a triple with sides $a/(2n+1), b, c'$, with $a^2/(2n+1)^2 + b^2 = (c')^2$. Let K'' be a triple with sides $a/(2n), b, c''$, with $a^2/(2n)^2 + b^2 = (c'')^2$. Then if $R_f(K)$ is true, so is $R_f(K')$. If there is a K^* congruent to K such that the two points distance b apart in K^* are the same color, while the third point is the opposite color, then $R_f(K'')$ is true. Finally, if there exists such a K^*, then there is a triple with sides $a/n, b, d$ with $(a/n)^2 + b^2 = d^2$ such that the points at distance b have the same color, and the third point has the opposite color.*

Proof. First let x and y be two points distance b apart which are the same color, say red. If $(a', b, e) \in T_f$, where $(a')^2 + b^2 = e^2$, then the points x_1 and y_1 distance a' from x and y respectively must both be blue. (See Figure 2.) Repeating this argument gives two parallel sequences $x, x_1, x_2, \ldots$ and $y, y_1, y_2, \ldots$ with alternating colors; x_i is red iff i is even, and similarly for y_i.

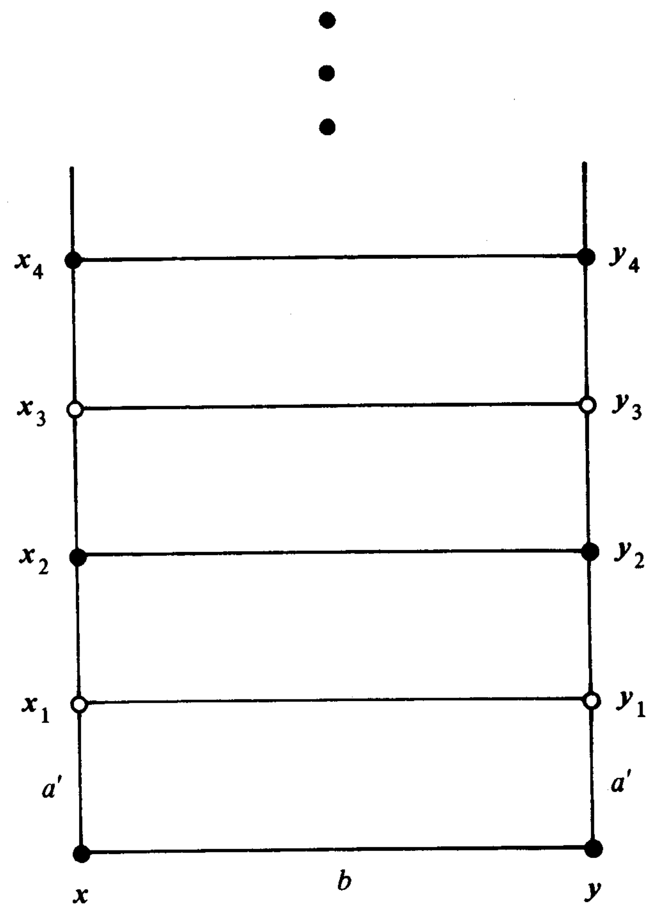

Figure 2

Now to obtain the first statement of Theorem 6, let x, y, z be three points forming a monochromatic (a, b, c)-triangle, and let x and y be distance b apart. Then z is x_{2n+1} or y_{2n+1} above, if $a' = a/(2n+1)$. This is a contradiction as x and x_{2n+1} or y_{2n+1} have opposite colors.

For the second statement, we take x and y to be the same color, and distance b apart, z the opposite color, and x, y, z the points of an (a, b, c)-triangle. If $a' = a/(2n)$, then $z = x_{2n}$ or y_{2n} above. This is a contradiction, as x and x_{2n} or y_{2n} have the same color.

For the third statement, we observe that if there were no $(a/n, b, d)$-triangle of the desired type, then by taking x and y distance b apart and the same color, we would get a "ladder" such as in Figure 2 with all x_i and y_i the same color. If we take x, y, z to be the vertices of a triangle K''' with x and y the same and z the opposite color, then $z = x_n$ or y_n, and we have a contradiction. This completes the proof.

The second method for right triangles may be called the "roulette method", and it is used below.

Theorem 7. *Let f be a 2-coloring of E^2. Let K_α be a triple with angles $90°, \alpha, 90° - \alpha$ ($\alpha = 0$ is allowed, that is, the triple may degenerate to a pair). Let K_β be a triple with angles $90°, \beta, 90° - \beta$, and with hypotenuse the same length as that of K_α. If $R_f(K_\alpha)$ is true then $R_f(K_\beta)$ is true where $(2m + 1)\beta = \alpha + n \cdot 180°$ for some integers $m \geq 0$, $n \geq 0$.*

If some triangle $K'_\alpha \cong K_\alpha$ has the two vertices on the hypotenuse one color and the third vertex the opposite color, then $R_f(K_\beta)$ is true if $2m\beta = \alpha + n \cdot 180°$.

Finally, if there is some triangle K'_α as above, then there is a triangle K'_β (with hypotenuse the same length as that of K'_α) with the $90°$ vertex colored opposite from the other two, if $m\beta = \alpha + n \cdot 180°$.

Proof. The proof is analogous to that of Theorem 6. Consider the circle of radius $c/2$ in Figure 3, where x and y are two points distance c apart which are the same color, say red, and c is the length of the hypotenuse of K_α. Then if $R_f(K_\beta)$ is false, the points x_1 and y_1 must be blue, since x, y, x_1 is a triple congruent to K_β. Repeating the argument yields two sequences, $x, x_1, x_2, \ldots$ and $y, y_1, y_2, \ldots$ with alternating colors.

If, in addition, $R_f(K_\alpha)$ is true, let x, y and z be a monochromatic triangle (with x and y on the hypotenuse) congruent to K_α. But if $(2m + 1)\beta = \alpha + n \cdot 180°$, then $z = x_{2m+1}$ or y_{2m+1}, a contradiction since the colors of the x_i and y_i alternate.

The second statement follows in the same way. The third statement follows by assuming that there is no K'_β of the desired kind. This leads to sequences $x, x_1, x_2, \ldots$ and $y, y_1, y_2, \ldots$ which are all the same color. If $m\beta = \alpha + n \cdot 180°$, we get a contradiction. This completes the proof.

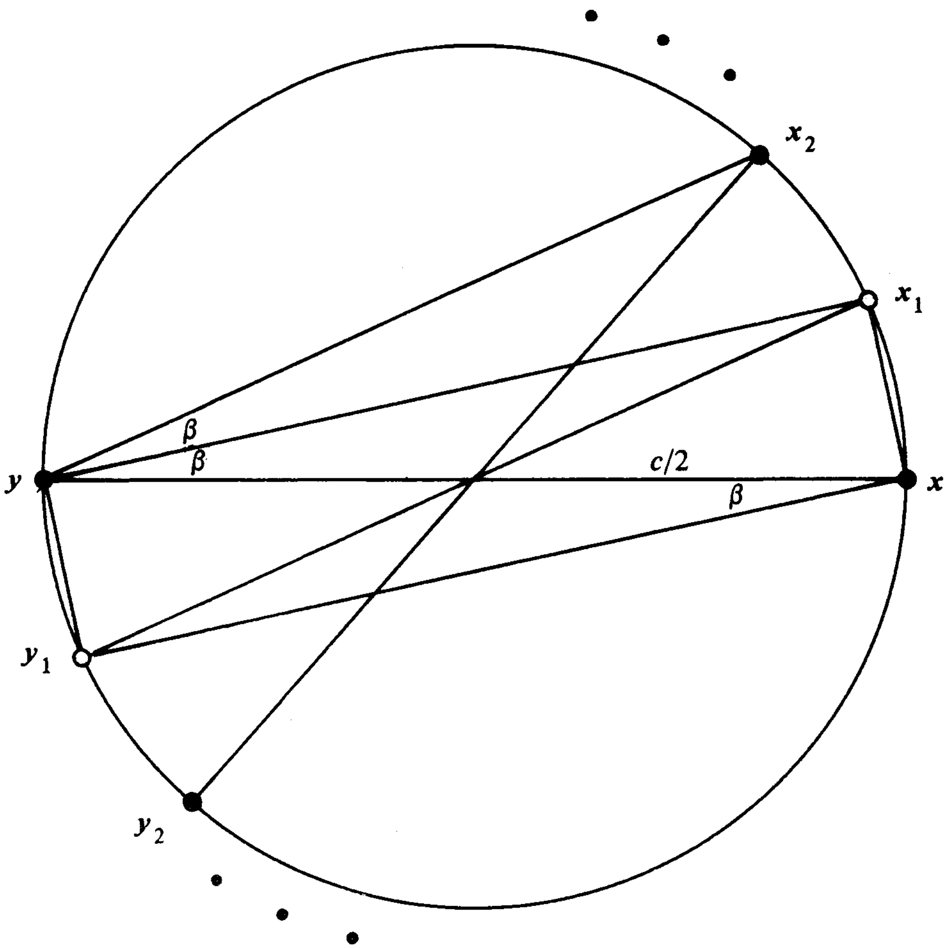

Figure 3

3. MONOCHROMATIC TRIANGLES

In this section we use the conditional theorems of Section 2, together with the fact that in every proper 2-coloring of E^2 we get both monochromatic and bichromatic pairs of points at every distance, to prove $R(a, b, c)$ for a variety of triangles with side lengths a, b, c.

We start with a theorem due to Raphael M. Robinson, which indicates what conclusions can be drawn from colorings of configurations involving only 5 points of E^2.

Theorem 8. *If five points can be found in the plane which have only the distances a, b, c, d and the distance d (not necessarily distinct from a, b, c) occurs only once and a, b, c satisfy the triangle inequality, then $R(a, b, c)$ holds.*

Proof. Let the set of five points be $P = \{p_1, p_2, \ldots, p_5\}$ with $d(p_1, p_2) = d$. We can choose the points so that p_1 and p_2 have opposite colors. Now P must contain a monochromatic triple $\{p_i, p_j, p_k\}$ whose distances are contained in $\{a, b, c\}$.

Thus, by Theorem 1 and its corollaries we have $R(a, b, c)$ whenever a, b, c satisfy the triangle inequality.

In order to check the consequences of Theorem 8 we need only classify the quadruples of points with at most 3 distinct distances and then check whether they can be augmented to a quintuple which satisfies the hypothesis of Theorem 8. Labelling the distances a, b, c in order of decreasing frequency of occurrence in the quadrilateral we get the following possibilities for the distance matrices:

$$\begin{bmatrix} 0 & a & a & a \\ a & 0 & a & b \\ a & a & 0 & c \\ a & b & c & 0 \end{bmatrix}$$

The triangle with vertices 2, 3, 4, an arbitrary 30° or 150° triangle whose circumcenter is the vertex 1 and whose circumradius is a. A reflection on a line of symmetry of the equilateral triangle 1, 2, 3 yields a 5th point.

$$\begin{pmatrix} 0 & a & a & b \\ a & 0 & c & a \\ a & c & 0 & a \\ b & a & a & 0 \end{pmatrix}$$

A rhombus. Can not in general be augmented to a quintuple satisfying Theorem 8.

$$\begin{pmatrix} 0 & a & a & a \\ a & 0 & b & b \\ a & b & 0 & c \\ a & b & c & 0 \end{pmatrix}$$

An isosceles (b, b, c)-triangle $2, 3, 4$ and its circumcenter 1 with circumradius a can be augmented to a quintuple by reflection of vertex 2 on the line 13.

$$\begin{pmatrix} 0 & a & a & c \\ a & 0 & a & b \\ a & a & 0 & b \\ c & b & b & 0 \end{pmatrix}$$

An equilateral (a, a, a)-triangle $(1, 2, 3)$ and an isosceles (b, b, a)-triangle $(2, 3, 4)$. (a, b, c) is a $30°$ or $150°$ triangle.

$$\begin{pmatrix} 0 & a & a & b \\ a & 0 & b & c \\ a & b & 0 & a \\ b & c & a & 0 \end{pmatrix}$$

A parallelogram with one of the diagonals equal to one of the sides can be augmented by reflection of 4 on the line of symmetry of $(1, 2, 3)$. We have $c^2 = a^2 + 2b^2$.

$$\begin{pmatrix} 0 & a & a & b \\ a & 0 & b & a \\ a & b & 0 & c \\ b & a & c & 0 \end{pmatrix}$$

An isosceles trapezoid with one base equal to the sides can be augmented by rotating about the circumcenter when $a \neq c$. The (a, b, c)-triangle has angles α, 2α, $180° - 3\alpha$ for $0 < \alpha < 60°$, $\alpha \neq 45°$, $180° - \alpha$, $180° - 2\alpha$, $3\alpha - 180°$; $60° < \alpha < 90°$ or is a (degenerâte $(a, 2a, 3a)$-triangle. Satisfies $a^2 \pm ac - b^2 = 0$.

$$\begin{pmatrix} 0 & a & a & c \\ a & 0 & b & a \\ a & b & 0 & b \\ c & a & b & 0 \end{pmatrix}$$

Isosceles (a, a, b)-triangle with b-side common with isosceles (b, b, a)-triangle, can be augmented by reflection on line of symmetry of one of the isosceles triangles. It satisfies $a^6 - 2a^4b^2 + a^2b^4 - 3a^2b^2c^2 + b^2c^4 = 0$.

$$\begin{pmatrix} 0 & a & a & b \\ a & 0 & c & b \\ a & c & 0 & c \\ b & b & c & 0 \end{pmatrix}$$

Contains 3 isosceles triangles (a, a, c) (b, b, a) and (c, c, b). Can be augmented by reflection on the line of symmetry of one of these triangles. Satisfies
$a^4c^2 + a^2b^4 - 5a^2b^2c^2 + b^2c^4 = 0.$

$$\begin{pmatrix} 0 & a & a & b \\ a & 0 & b & c \\ a & b & 0 & c \\ b & c & c & 0 \end{pmatrix}$$

Can not in general be augmented.

$$\begin{pmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{pmatrix}$$

Rectangle, can not in general be augmented.

To sum up we have:

Theorem 9. *$R(K)$ holds for all triangles $K = (a, b, c)$ which*

(i) *have a* $30°$ *angle,*

(ii) *have a* $150°$ *angle,*

(iii) *are the sides and the circumradius of an isosceles triangle. Satisfies* $4a^2b^2 - a^2c^2 - b^4 = 0.$

(iv) *satisfy* $c^2 = a^2 + 2b^2$,

(v) *satisfy* $a^2 \pm ac - b^2 = 0$, $a \neq c$. *Includes* K *with angles* $(\alpha, 2\alpha, 180° - 3\alpha)$, $0 < \alpha < 60°$, $\alpha \neq 45°$ *and angles* $(180° - \alpha, 180° - 2\alpha, 3\alpha - 180°)$; $60° < \alpha < 90°$ *and* $K = (a, 2a, 3a)$.

(vi) *satisfy* $a^6 - 2a^4b^2 + a^2b^4 - 3a^2b^2c^2 + b^2c^4 = 0.$

(vii) *satisfy* $a^4c^2 + a^2b^4 - 5a^2b^2c^2 + b^2c^4 = 0.$

The above list includes isosceles triangles with vertical angles of $\theta = 30°, 72°, 108°, 120°, 150°$.

Corollary 10. *We have* $R(K)$ *for all triangles for which the ratio of two sides is* $r = 2\sin(\theta/2)$ *where* θ *is one of the above mentioned angles.*

Corollary 11. *If* $(a, a, a) \in T_f$ *and* $(b, b, b) \in T_f$ *and the triangle* $K = (a, b, x)$ *belongs to one of the family of triangles for which* $R(K)$ *holds (e.g. according to Theorem 9 or Corollary 10), then* $R_f(x, x, x)$ *and hence* $R_f(x, y, z)$ *for all possible values of* y, z.

The hypothesis of Corollary 11 would follow from $(a, b, c) \in T_f$ *for any* c *or from* $(a, ., .) \in T_f$ *and* $(b, ., .) \in T_f$ *for any choices of the unnamed sides.*

4. BICHROMATIC TRIANGLES

In this section we consider the question of which triangles must occur bichromatically for any f which is not a 1-coloring (i.e., a *proper* 2-coloring). It is clear that out of the 4 possible colorings of a triangle at least 3 must occur in every proper 2-coloring.

Theorem 12. *Let* f *be a proper 2-coloring of* E^2. *Then for every* a *and* b, $2a \geqslant b$, *the isosceles* (a, a, b)*-triple satisfies* $R_f(\bar{a}, a, b)$.

Proof. Since f is proper, there are two points of opposite color distance b apart. The third point of an (a, a, b)-triangle agrees with one of these in color, and the proof is complete.

We now turn to right triangles and use an argument by Raphael M. Robinson improving an earlier argument.

Lemma 13. *Let* $L = \{k + l\sqrt{-d} \mid k, l \in Z\}$, $d > 0$, $d \in Q$, *be a lattice in* E^2 *(considered here as the complex plane). Then there is a lattice* L' *obtained by rotating* L *about* 0 *by an angle which is not a multiple of* $90°$ *such that* $L \cap L'$ *is a two-dimensional sublattice of* L.

Proof. Pick $\lambda = u + v\sqrt{-d} \in Q(\sqrt{-d})$ such that λ^2 is neither real nor purely imaginary. Let $\mu = \lambda^2/N(\lambda) = (u + v\sqrt{-d})^2/(u^2 + dv^2)$. Then $L' = \mu L$ satisfies the lemma.

Theorem 14. *If* f *is a proper* 2*-coloring of* E^2 *and* (a, b, c) $(c^2 = a^2 + b^2)$ *is a right triangle with* b^2/a^2 *rational, then the* (a, b, c)*-triangle occurs in all four possible colorings.*

Proof. By Theorem 6 we know that $R_f(\bar{a}, 2b, \sqrt{a^2 + 4b^2})$ implies $R_f(a, b, c)$. It therefore suffices to prove the theorem for bichromatic colorings. We may assume without loss of generality that $a = 1$. Consider a rectangular lattice with sides 1 and b. Consider three consecutive points x, y, z of L with $|x - y| = |y - z| = 1$. Unless all pairs x and z of such triples agree in color, there must be such a triple with, say, x and y of one color and z of opposite color. We now consider the three possible bichromatic colorings of (a, b, c). If $R_f(1, b, \bar{c})$ is false, each line of L perpendicular to $z - y$ must be monochromatic. If $R_f(\bar{1}, b, c)$ is false then, equally, each line of L perpendicular to $x - y$ must be monochromatic. If $R_f(1, \bar{b}, c)$ is false then the colors must alternate on each line of L perpendicular to $z - y$.

Thus if one of the bichromatic colorings fails to occur, then in every lattice congruent to $2L$ there is one of its coordinate directions in which each line is monochromatic. Now consider the lattice $2L'$ of Lemma 12. The lattice $(2L) \cap (2L')$ contains a two-dimensional sublattice generated by the monochromatic coordinate directions of $2L$ and $2L'$ which in turn contains a sublattice kL, $(k \in 2Z)$. We thus get a monochromatic sublattice for each of the four possible choices of the monochromatic directions of $2L$ and $2L'$ and therefore a sublattice KL, $(K \in 2Z^+)$ which is common to all three lattices. Thus, since the position of L is arbitrary, it follows that all points with distance K are like colored, contrary to the hypothesis that f is proper.

Since Theorem 14 includes the case of isosceles right triangles we get the following.

Corollary 15. *We have* $R(a, b, c)$ *whenever the ratio of two of the sides is* $\sqrt{2}$.

Theorem 16. *If* f *is a proper* 2*-coloring of* E^2 *there is a* $(1, 1, \sqrt{3})$*-triangle with the* 120° *vertex colored oppositely from the other two. Thus* *all* 3 *colorings of an isosceles* 120° *triangle occur in any proper* 2-*coloring of* E^2.

Proof. This proof is similar to the last proof, but we employ the triangular lattice rather than the rectangular lattice.

Let f be a proper 2-coloring of E^2, and let x be a red point. Let u and v be unit vectors forming a 60° angle, and consider the lattice $x + L$, where $L = \{iu + jv \mid i, j \text{ integers}\}$.

We suppose that the desired kind of triangle does not occur. We will show that $x + 120u$ must also be red. This, then, is a contradiction, as x and u were arbitrary, and f is proper.

Suppose $x + L$ is not all red. Then somewhere in the lattice are three points q, r and s forming a bichromatic triangle with $\{\pm(s-q), \pm(r-q), \pm(r-s)\} = \{u, v, \pm(v-u)\}$. Let s be red and q and r blue. Then all points $r + i(q-r)$ are blue, and all points $s + i(q-v)$ are red. Now by induction on k, we see that each "line" of points $s + k(s-q) + i(q-r)$, $i = \ldots, -1, 0, 1, 2, \ldots$ is a single color, depending on k, k any integer, positive or negative.

What we have shown, then, is that in at least one of the three directions u, v or $u - v$ the colors are constant. Without loss of generality we can assume that colors are constant in the u direction.

Let $L' = \{iu' + jv' \mid i, j \text{ integers}\}$, where $u' = \frac{1}{7}(5v + 3u)$, $v' = \frac{1}{7}(3v - 5u)$. Then applying the argument above to the lattice $x + L'$, we see that colors are constant in one of the directions $u', v', u' - v'$. Thus starting at x, one of the three points $x + 7 \cdot 24u' = 120v + 72u + x$, $x + 7 \cdot 15v' = x + 120v - 75u$, $x - 7 \cdot 40(u' - v') = x + 120v - 320u$, must be red.

But since colors are constant in the u direction in $x + L$, we see that $x + 120v$ must be red in all cases. Since x and v were arbitrary, we have every two points distance 120 apart are the same color, contradicting the fact that f is proper, and completing the proof.

Theorem 17. *If K is a right (a, b, c)-triangle with $a^2 + b^2 = c^2$, and the angle opposite the side of length a is a rational multiple of* $180°$, *then $R(K)$ is true.*

Proof. It is sufficient to assume $c = 1$. Let f be a 2-coloring of E^2. We wish to show $(b, \sqrt{1 - b^2}, 1) \notin T_f$ for every b such that $\sin^{-1} b$ is rational. Let $\beta = \sin^{-1} b$. If $\beta = n \cdot 90°$ for any n, it is true, since K has only two points. Then by the roulette argument (Theorem 7), it is also true for $\beta = \frac{n}{2m + 1} 90°$. By Theorem 14 it is true for $\beta = \frac{2n + 1}{2} 90°$, and again by the roulette argument, it is true for $\beta = \frac{2n + 1}{2(2m + 1)} 90°$. Finally, by Theorem 14, either f is a 1-coloring, in which case we are done, or there are points forming a bichromatic $(1/\sqrt{2}, 1/\sqrt{2}, 1)$-triangle with the $90°$ vertex the opposite color from the other two. But now using the roulette argument here yields the desired result for any $\frac{2n + 1}{2(2m)} 90°$. This completes the proof, as all cases are exhausted.

Theorem 18. *Let K be a right triangle with angles* $(\alpha, 90° - \alpha, 90°)$ *where $\alpha/90°$ is rational with even denominator. Then in every proper 2-coloring of E^2 there exists a triangle ABC congruent to K whose* $90°$ *vertex C is colored opposite to A and B.*

Proof. It suffices to prove the theorem for triangles whose hypotenuse is of length 1. By Theorem 14 there exists an isosceles right $(1/\sqrt{2}, 1/\sqrt{2}, 1)$-triangle $A'B'C'$ with C' colored opposite to A' and B'. By the roulette argument (Theorem 7) there therefore exist right triangles ABC with C colored opposite to A and B and $\sphericalangle A = \frac{1 + 2n}{2m} 90°$ for any integer $m > n \geqslant 0$.

Theorem 19. *Let K be a right triangle with angles* $(\alpha, 90° - \alpha, 90°)$ *where α is a rational multiple of* $90°$ *and the numerator and denominator of $\alpha/90$ are not both odd. Then in every proper 2-coloring of E^2 there exists a triangle ABC congruent to K with $\sphericalangle A = \alpha$ and A colored opposite to B and C.*

Proof. It again suffices to prove the theorem for triangles of hypotenuse 1. We can pick points A_0, B_0 of opposite colors and distance 1. If there are no triangles congruent to K with the desired coloring then by the roulette argument we get a sequence of points $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ on the circle with diameter A_0B_0 so that the arcs A_iA_{i+1} and B_iB_{i+1} are all 2α and the A_i and B_i have alternating colors. This leads to a contradiction if $\alpha = \frac{1+2n}{2m} 90°$ where m, n are integers with $2m > 2n + 1 > 0$; or if $\alpha = \frac{2m}{1+2n} 90°$ where m, n are integers and $0 < 2m < 1 + 2n$.

Corollary 20. *In every proper 2-coloring of E^2 all right triangles with angles α, $90° - \alpha$, $90°$ where $\alpha/90°$ is rational with even denominator occur in all 4 possible colorings.*

The situation summarized in Theorem 14 and Corollary 20 is by no means complete since we can apply the ladder and roulette methods alternately to get increasing families of right triangles with prescribed colorings. Both methods lead to new side-lengths which are obtained from the old side-lengths by rational operations and solutions of cyclotomic equations. This leads us to the following conjecture about the closure under the two operations.

Conjecture 5. *In any proper 2-coloring of E^2 all right (a, b, c)-triangles with $a^2 + b^2 = c^2$ and a/b in the cyclotomic closure of Q occur in all 4 possible colorings.*

We outline briefly some of the wealth of additional results on right triangles.

If we have $(a, b, c) \in T_f$ with $a^2 + b^2 = c^2$, then according to Theorems 6 and 7 we get an infinity of right triangles in T_f. From now on we we use s, t, u to denote arbitrary odd positive integers. We get

$$(sa, tb, \sqrt{s^2a^2 + t^2b^2}) \in T_f \qquad \text{(Theorem 6);}$$

$$c(\cos\alpha, \sin\alpha, 1) \in T_f, \quad \tan(\alpha/s) = b/a \qquad \text{(Theorem 7);}$$

$$c(t\cos\alpha, u\sin\alpha, \sqrt{t^2\cos^2\alpha + u^2\sin^2\alpha}) \in T_f, \quad \tan(\alpha/s) = b/a \qquad \text{(Theorem 6).}$$

and so forth, applying alternately the ladder and the roulette method. We thus get an infinite number of equilateral triangles in T_f and by Corollary 11 an infinite number of equilateral triangles for which R_f holds. Before listing these triangles we get an additional tool.

Lemma 21. *If* $(a, b, c) \in T_f$, $a^2 + b^2 = c^2$ *then* $R_f\left(\frac{c}{2}(1, 1, 1)\right)$ *holds. Hence* $2sx(\cos\alpha, \sin\alpha, 1) \notin T_f$ *where* $x = a, b$ *or* c.

Proof. In any coloring of five points which form the vertices and the center of an $a \times b$ rectangle we get a monochromatic triangle whose sides have lengths contained in the set $\{a, b, c, c/2\}$. Since our hypothesis implies $(a, a, a), (b, b, b), (c, c, c) \in T_f$ we must have $R_f(c/2, c/2, c/2)$.

Theorem 22. *Let* $(a, b, c) \in T_f$, $a^2 + b^2 = c^2$. *Then we get the following partial list of side lengths for equilateral triangles in* T_f *and triangles not in* T_f.

We make two columns headed T_f *and* R_f, *listing sample side lengths of equilateral triangles which occur in either category, using the following shorthand:* x, y *distinct elements of* $\{a, b, c\}$; $r =$ *one of the ratios in Corollary* 10 *or* 15; $m =$ *arbitrary positive integer;* $s, t =$ *arbitrary positive odd integers.*

T_f		R_f	
sx	(Theorem 6)	$sc/2$	(Lemma 21)
$\sqrt{s^2a^2 + t^2b^2}$	(Theorem 6)	$rsx, sx/r$	(Corollary 10, 15)
$tc \cos\alpha$, $\tan(\alpha/s) = b/a$ or a/b	(Theorem 7)	$2x\sqrt{m}$	(Theorem 9, (iv) with $c = sx$, $a = tx$, $8m = s^2 - t^2$)
		$x\sqrt{s^2 + t^2 \pm \sqrt{3}\, st}$	(Theorem 9, (i), (ii))
		$x\sqrt{2m}$	(Theorem 9, (v))
		$\sqrt{s^2x^2 \pm 2t^2y^2}$	(Theorem 9, (iv))
		$\sqrt{sx(sx \pm ty)}$	(Theorem 9, (v))

It is clear that we get a contradiction whenever a number appears in both columns. Thus, for example we must have $R((1, \sqrt{10+3\sqrt{3}}, \sqrt{11+3\sqrt{3}}))$ since otherwise the number $\sqrt{10+3\sqrt{3}}$ would occur in both columns.

As an interesting special case we mention the following:

Corollary 23. *If we have* $R(s, t, u)$ *for some triple of odd integers* s, t, u *then we have* $R(K)$ *for all right triangles* K.

Proof. If $K = (a, b, c) \in T_f$ for some 2-coloring f of E^2 then we have $sK, tK, uK \in T_f$ by Theorem 6 and hence $s(a, a, a), t(a, a, a), u(a, a, a) \in T_f$ by Theorem 1 and thus $(sa, ta, ua) \in T_f$ by the same Theorem. Since the triangles for which R holds are closed under similarities it follows that $R(s, t, u)$ is false.

5. ADDITIONAL REMARKS AND PROBLEMS

One might ask whether for any 2-coloring of E^2 there is one color so that all triangles K which occur monochromatically occur in that color. We cannot answer that question, but we can give affirmative answers to related questions.

Theorem 24. *For any 2-coloring of* E^3 *there is one color so that all equilateral triangles occur in that color.*

Proof. If there is some distance, say a, so that no pair of red points has distance a, then every monochromatic set K is either blue, or every translate of K by a vector of length a is blue. We may therefore assume that there are pairs of points of either color with any prescribed distance.

Now assume that there is no red (a, a, a)-triangle and that A, B are red points at distance a. Then the entire circle $\mathscr{C}$ lying in the perpendicular bisecting plane of AB with center at the midpoint of AB and radius $a\sqrt{3}/2$ must be blue. If there is any (b, b, b)-triangle with $b < a\sqrt{3}$ which does not occur in blue, then to every pair of points C, D on $\mathscr{C}$ with distance $\overline{CD} = b$ there corresponds a red circle of radius $b\sqrt{3}$ in the perpendicular bisecting plane of CD centered at their midpoint. The union of these circles is a red torus $\mathcal{T}$ whose outer radius is $R = b\sqrt{3}/2 + \sqrt{3a^2 - b^2}/2$ and whose inner radius is $r = \max\{0, \sqrt{3a^2 - b^2}/2 - b\sqrt{3}/2\}$. If $r \leqslant a/\sqrt{3}$, the circumradius of an (a, a, a)-triangle, then $\mathcal{T}$ would contain the vertices of a red (a, a, a)-triangle contrary to hypothesis. If $r > a/\sqrt{3}$ then $b < a/2$ and to each pair of points E, F at distance a in the inner circle of $\mathcal{T}$ there corresponds a blue circle of radius $a\sqrt{3}/2$ in the perpendicular bisecting plane of EF centered at their midpoint. The locus of all these circles is a blue torus $\mathcal{T}'$ with outer radius $R' = a\sqrt{3}/2 + \sqrt{r^2 - a^2/3}/2$ and no hole. Since $R' > b$ it follows that $\mathcal{T}'$ contains a blue (b, b, b)-triangle contrary to hypothesis.

We have thus shown that, if there exists no red (a, a, a)-triangle then there exist blue (b, b, b)-triangles for every $b < a\sqrt{3}$. Thus if there did not exist a blue (c, c, c)-triangle, we would have $c > a\sqrt{3}$, and then there would exist a red (a, a, a)-triangle contrary to hypothesis.

Remark. The arguments in the proof of Theorem 24 can be extended to cover all triangles which have one altitude that exceeds half the corresponding edge. If we call such triangles *nonflat* we can state the following extension of Theorem 24.

Theorem 24'. *In any 2-coloring of E^3 there is one color so that all nonflat triangles occur in that color.*

We can generalize Theorem 24 in another direction.

Theorem 25. *In any 2-coloring of E^{2n-1} there exists one color so that all regular n-simplices occur in that color.*

Proof. By induction on n. Theorem 24 is the case $n = 2$. Assume the theorem to hold for $n - 1$ and assume that there exist regular $(n-1)$-simplices of every size all of whose vertices are red. If for some $a > 0$ there does not exist a regular red n-simplex of edge-length a we can pick a regular red $(n-1)$-simplex $A_1, A_2, \ldots, A_n$ of side-length a in an $(n-1)$-plane P^{n-1}. The $(n-1)$-sphere, S, of radius $a\sqrt{(n+1)/2n}$ centered at the centroid of $A_1, \ldots, A_n$ and lying in an n-plane perpendicular to P^{n-1} must be all blue. If for some b with $0 < b < a\sqrt{(n+1)/(n-1)}$ there does not exist a regular blue n-simplex of edge-length b, then to each regular $(n-1)$-simplex $B_1, \ldots, B_n$ of edge-length b on S there corresponds a red $(n-1)$-sphere centered at the centroid of $B_1, \ldots, B_n$ and in an n-plane perpendicular to this plane. The locus of all these red spheres is a generalized torus $\mathcal{T}$ whose outer radius is

$$R = b\sqrt{\frac{n+1}{2n}} + \sqrt{a^2\,\frac{n+1}{2n} - b^2\frac{n-1}{2n}}$$

and whose inner radius is

$$r = \max\left\{0,\ \sqrt{a^2\frac{n+1}{2n} - b^2\frac{n-1}{2n}} - b\sqrt{\frac{n+1}{2n}}\right\}.$$

If $r \leqslant a\sqrt{n/2(n+1)}$ then $\mathcal{T}$ contains the vertices of a regular n-simplex of edge-length a, contrary to hypothesis. If $r > a\sqrt{n/2(n+1)}$ then $b < a/n$ and to every regular $(n-1)$-simplex $C_1, \ldots, C_n$ of edge-length a on the inner sphere of $\mathcal{T}$ there corresponds a blue $(n-1)$-sphere of radius $a\sqrt{(n+1)/2n}$. The locus of these spheres forms a blue torus $\mathcal{T}'$ which, as in the proof of Theorem 24 can be shown to contain the vertices of a regular n-simplex of side-length b, contrary to hypothesis.

Now, if there were no blue regular n-simplex of edge-length c then $c \geqslant a\sqrt{(n+1)/(n-1)}$ and hence there are regular red n-simplices of edge-length a, contrary to hypothesis.

A generalization analogous to that of Theorem 24′ holds where a nonflat – simplex is one with an altitude that exceeds the circumradius of the corresponding face.

We have already proved in [2] that for all positive integers k, l there exists an $N = N(k, l)$ so that in every k-coloring of E^N there is one color so that all bricks (vertices of a rectangular solid) with 2^l points have a congruent image all of whose vertices are in that color.

In view of the results in Section 4 one might ask whether there are analogous results concerning trichromatic triangles in a proper 3-coloring of E^2 or at least in E^n for sufficiently large n. The answer is entirely in the negative, even if we ask for triangles similar to a given triangle.

Theorem 26. *Given any triangle K there exists a proper 3-coloring of Hilbert space so that no triangle similar to K is trichromatic.*

Proof. Let F be the field generated by the ratios of the sides of K. Now color the origin white; every point whose distance from the origin is a positive number of F red, and the rest of the points blue. Then clearly the ratios of the sides of a trichromatic triangle are not in F so the triangle is not similar to K.

All the results on monochromatic triples which we have obtained involved triangles with an algebraic dependence among the 3 side lengths. Of course, in order to prove Conjecture 3 it would suffice to prove it for all non-equilateral isosceles triangles. So far we have no case of an (a, a, b)-triangle with $R(a, a, b)$ and a/b transcendental. Any such result would constitute an important advance.

Theorem 27. *If $R(1, 1, x)$ holds for some transcendental number $x < 2$ then there exists an interval I containing x in its interior such that $R(1, 1, y)$ holds for all $y \in I$.*

Proof. By a compactness result (see [1]) there exists a finite set $S(x) \subset E^2$ such that in every 2-coloring of $S(x)$ there is a monochromatic $(1, 1, x)$-triple. Without loss of generality we may assume that the origin is in $S(x)$ and that all the coordinates of the points of $S(x)$ are in the algebraic closure $\tilde{Q}(x)$ of $Q(x)$. For, if there were coordinates which were transcendental over $Q(x)$ then we could only increase the set of distances in $S(x)$ which are 1 or x by specializing those transcendentals to values in $Q(x)$.

Thus the coordinates in $S(x)$ are algebraic functions over $Q(X)$ with real values at $X = x$. The branchpoints of these algebraic functions occur at algebraic values of X. Thus there exists an interval I whose algebraic endpoints consist of the branchpoints nearest to x. For any $y \in I$ the choice $X = y$ gives us a set $S(y)$ so that in every 2-coloring of $S(y)$ there is a monochromatic $(1, 1, y)$-triple.

Even if we restrict attention to triples with commensurable distances the proof of Conjectures 3 or 4 cannot be carried out with the coloring of finite subsets of E^2 with bounded numbers of elements.

Theorem 28. *Let* $R(1, 1, x)$ *hold and let* $S(x) \subset E^2$ *be a set with a minimal number of elements such that every* 2*-coloring of* $S(x)$ *yields a monochromatic* $(1, 1, x)$*-triple. Then* $|S(x)| \to \infty$ *as* $x \to 1$.

Similarly, let $R(1, 1, \bar{x})$ *hold and let* $\bar{S}(x) \subset E^2$ *be a set with a minimal number of elements so that every proper* 2*-coloring of* $\bar{S}(x)$ *yields a* $(1, 1, x)$*-triple where the vertices adjacent to the* x*-side are colored alike and opposite to the third vertex. Then* $|\bar{S}(x)| \to \infty$ *as* $x \to 2$.

Proof. To prove the first part, assume that there exists a sequence $x_n \to 1$ so that $|S(x_n)| = N$. Assuming $\mathbf{0} \in S(x_n)$ we get $|\boldsymbol{p}| \leqslant 2N$ for all $\boldsymbol{p} \in S(x_n)$ since otherwise we could divide $S(x_n)$ into two nonempty sets, A, B, one containing $\mathbf{0}$ and the other $\boldsymbol{p}$ so that the distance between any point of A and any point of B exceeds 2. Thus all $(1, 1, x_n)$-triples are entirely in A or entirely in B, contradicting the minimality of $S(x_n)$. Thus, the $S(x_n)$ are sets with N elements in a bounded disk and as $n \to \infty$ there is a convergent subsequence $S(x_n) \to S$. Then in every 2-coloring of S there must be a monochromatic $(1, 1, 1)$-triple contrary to the fact that $R(1, 1, 1)$ is false.

The proof for the second part is entirely analogous, using the fact that $R(1, 1, \bar{2})$ is false.

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