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# Ramsey's Theorem for a Class of Categories

**(k-parameter sets/finite vector spaces/ranking)**

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**ABSTRACT** **Ramsey's Theorem states that for a sufficiently large set $S$, and for any splitting of the $k$-element subsets of $S$ into $r$ classes, there is a subset $T \subset S$, $|T| = l$, such that all $k$-element subsets of $T$ are in the same class. This paper establishes a theorem for certain categories that generalizes Ramsey's Theorem. In particular, it is strong enough to establish G-C. Rota's conjecture that the vector space analogue to Ramsey's Theorem is true. It also implies the Ramsey theorem for $n$-parameter sets, which has as corollaries, among others, the theorem of van der Waerden on arithmetic progressions and several results of R. Rado on regularity in systems of linear equations.**

A Ramsey theorem can be proved for certain categories which generalizes Ramsey's Theorem (4) for sets and the analogous theorem for $k$-parameter sets (1), and establishes G-C. Rota's conjectured analogue for finite vector spaces. The categories must be sufficiently like the category of $k$-parameter sets so that the proof of the Ramsey property for this category can be extended. These notions are made precise below.

We consider only categories $C$ in which the objects are the nonnegative integers $0,1,2,\ldots$, and in which for any $l > k$, the set $C(l,k)$ of morphisms from $l$ to $k$ is empty. In this situation, the subobjects of an object $l$ have an induced rank, namely, the number $k$ for which a morphism $k \xrightarrow{f} l$ is a representative of the subobject. We call a subobject of rank $k$ a $k$-subobject, and we denote by $C\begin{bmatrix} l \\ k \end{bmatrix}$ the set of all $k$-subobjects of $l$. We assume that for each $k$ and $l$ there is an integer $y_{k,l} \geq 0$ such that $\left|C\begin{bmatrix} l \\ k \end{bmatrix}\right| = y_{k,l}$, and in particular $y_{0,0} = 1$.

Let $k \xrightarrow{f} l$ be a morphism of $C$. Then $f$ induces a mapping $\bar{f}: C\begin{bmatrix} k \\ s \end{bmatrix} \to C\begin{bmatrix} l \\ s \end{bmatrix}$ for each $s \geq 0$. An $r$-coloring of $C\begin{bmatrix} l \\ s \end{bmatrix}$ is a function $c : C\begin{bmatrix} l \\ s \end{bmatrix} \to \{1,\ldots,r\}$. Then $\bar{f}$ composed with $c$ induces an $r$-coloring of $C\begin{bmatrix} k \\ s \end{bmatrix}$. If $c\bar{f}$ has only a single element in its image, we say that $c$ has a monochromatic $l$-subobject.

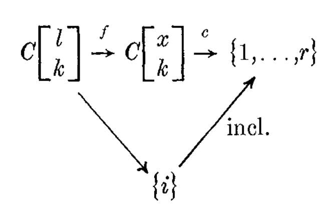
The Ramsey property for C is:

*For $x$ sufficiently large (depending on $k,l,r$) every $r$-coloring of* $C\begin{bmatrix} x \\ k \end{bmatrix}$ *has a monochromatic $l$-subobject.*

When the morphisms of C are the monomorphic functions from $\{1,\ldots,k\}$ into $\{1,\ldots,l\}$, then this is just the statement of Ramsey's Theorem. When the morphisms of $C$ are the monomorphic linear transformations from $V_k = \langle v_1,\ldots,v_k\rangle$ to $V_l = \langle v_1,\ldots,v_l\rangle$, where $v_1, v_2,\ldots$ form a basis for a vector space $V$ over $GF(q)$, then this is the statement of Rota's conjecture. Categories satisfying this property are the kind of categories referred to in (3).

We consider a stronger version of the Ramsey property more suitable for an induction argument.

$C(k; l_1,\ldots,l_r)$: There is a number $N = N_C(k; r; l_1,\ldots,l_r)$, depending only on $k,r,l_1,\ldots,l_r$, such that for any $x \geq N$ and any $r$-coloring $C\begin{bmatrix} x \\ k \end{bmatrix} \xrightarrow{c} \{1,\ldots,r\}$, there is an $i$, $1 \leq i \leq r$, and a morphism $l_i \xrightarrow{f} x$ such that the following diagram commutes:



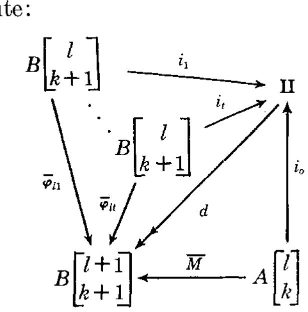
*Theorem.* Let $A$ and $B$ be categories satisfying conditions I, II and III below. If $A(k; l_1,\ldots,l_r)$ holds for all $r,l_1,\ldots,l_r$, then $B(k+1; l_1,\ldots,l_r)$ holds for all $r,l_1,\ldots l_r$.

*Corollary.* Let $\mathcal{C}$ be a class of categories $C$ such that for every $B$ in $\mathcal{C}$ there is an $A$ in $\mathcal{C}$ such that $A$ and $B$ satisfy conditions I, II, and III. Then $C(k; l_1,\ldots,l_r)$ holds for all $C$ in $\mathcal{C}$ and all $k,l_1,\ldots,l_r$.

With this corollary, we can prove the Ramsey property for a category $C$ by finding a class $\mathcal{C}$ containing $C$ and satisfying the conditions of the corollary.

The conditions on A and B are as follows: There is a functor $M$ from $A$ to $B$ with $M(l) = l+1$, $l \geq 0$, a functor $P$ from $B$ to $A$ with $P(l) = l$, $l \geq 0$, an integer $t \geq 0$, and for each $l \geq 0$, $t$ morphisms $l \xrightarrow{\varphi_{lj}} l+1$, $1 \leq j \leq t$, satisfying the following:

I. For each $k+1 = 0,1,2,\ldots$ the diagonal $d$ in the following diagram is epic, where $\amalg$ (together with the indicated injections) is the coproduct of $A\begin{bmatrix} l \\ k \end{bmatrix}$ and $t$ copies of $B\begin{bmatrix} l \\ k+1 \end{bmatrix}$, $\bar{M}$ is the mapping induced on subobjects by $M$, and $d$ is the unique map determined by the coproduct to make the diagram commute:



II. For each $s \xrightarrow{g} l$ in $B$ and each $j = 1, \ldots t$, the following diagram commutes:

$$\begin{array}{ccc} l & \xrightarrow{\varphi_{lj}} & l+1 \\ g \uparrow & & \uparrow M(P(g)) \\ s & \xrightarrow{\varphi_{sj}} & s+1 \end{array}$$

III. For some $l \xrightarrow{e} l + 1$ in $A$, the following diagram commutes for each $j = 1, \ldots, t$:

$$\begin{array}{ccccc} & \overset{\varphi_{lj}}{\nearrow} & l+1 & \overset{\varphi_{l+1,j}}{\searrow} & \\ l & & & & l+2 \\ & \underset{\varphi_{lj}}{\searrow} & l+1 & \underset{M(e)}{\nearrow} & \end{array}$$

Very loosely speaking, these conditions say that $A$ and $B$ are connected (by $M$ and $P$) in such a way that: (I) each $l + 1$ contains $t$ "translates" of $l$ such that any $(k + 1)$-subobject not arising from $A$ (by $M$) must be in one of the translates, (II) this decomposition is "inherited" by subobjects, and (III) the "diagonal" composition of two such decompositions is also one. These conditions correspond closely to the properties of $k$-parameter sets given in Remarks 1, 2, and 3 of (1).

To establish Ramsey's Theorem, we let $\mathcal{C} = \{c\}$, the category with morphisms $k \xrightarrow{f} l$ the monomorphic functions from $\{1, \ldots, k\}$ into $\{1, \ldots, l\}$. We then let $M(f)$ be the extension of $f$ given by $M(f)\ (k + 1) = l + 1, P(f) = f, t = 1,$ and $\varphi_{l1}(x) = x$ for all $x$.

To establish Rota's conjecture, we let $\mathcal{C} = \{C_m : m = 0, 1, \ldots\}$, where $C_m$ is defined as follows. Let $A$ and $V$ be infinite dimensional vector spaces over $GF(q)$, with bases $a_1, a_2, \ldots$ and $v_1, v_2, \ldots$, respectively, and for each $m$ let $A_m = \langle a_1, \ldots, a_m \rangle$, $V_m = \langle v_1, \ldots, v_m \rangle$. Let $C_m$ have morphisms $k \xrightarrow{(w,\varphi)} l$ where $\varphi$ is a monomorphic linear transformation from $V_k$ to $V_l$ and $w$ is an element of $A_m \otimes V_l$. Composition is effected by $(u,\psi)\ (w,\varphi) = (y, \psi\varphi)$, where $y = u + \sum_{i=1}^{m} a_i \otimes \psi(w_i)$ for $w = \sum_{i=1}^{m} a_i \otimes w_i$. We can think of these morphisms, then, as special affine transformations from $A_m \otimes V_k$ to $A_m \otimes V_l$.

For $C_{m+1} = A$ and $C_m = B$ we define $M$ and $P$. Let $k \xrightarrow{(w,\varphi)} l$ be in $C_{m+1}$, where $w = w' + a_{m+1} \otimes w_{m+1}$, $w' \epsilon A_m + V_l$, $w_{m+1} \epsilon V_l$. Then $M((w,\varphi)) = (w', \varphi')$, where $\varphi'$ is the extension of $\varphi$ given by letting $\varphi'(v_{k+1}) = v_{l+1} + w_{m+1}$. For $k \xrightarrow{(w,\varphi)} l$ in $C_m$, let $P((w,\varphi)) = (w,\varphi)$. For $a \epsilon A_m$, we let $\varphi_{al} = (a \otimes v_{l+1}, e_l)$ in $C_m$, where $e_l$ is the inclusion map from $V_l$ to $V_{l+1}$.

We note that for $m = 0$ we obtain the category $C_0$ which establishes the vector space analogue to Ramsey's Theorem. We also note that for $m = 1$ we obtain the category $C_1$ which establishes the affine space analogue to Ramsey's Theorem.

To establish the Ramsey property for $k$-parameter sets, we let $\mathcal{C} = \{C_m : m = 0, 1, \ldots\}$ where the $C_m$ are defined as follows. Let $A = \{a_1, a_2, \ldots\}$ be an infinite set, and $A_m = (a_1, \ldots, a_m)$ for each $m$. Let $G$ be a finite group. Then $C_m$ is the category with morphisms $k \xrightarrow{(f,s)} l$ where $f$ is an epimorphic function from $A_m \cup \{1, \ldots, l\}$ onto $A_m \cup (1, \ldots, k)$ acting identically on $A_m$, and $s$ is a function from $A_m \cup \{1, \ldots, l\}$ into $G$ which maps $A_m$ onto the identity element $1 \epsilon G$. Composition is given by $(g,u)\ (f,s) = (fg, sg \cdot u)$ where $fg$ and $sg$ are composition of functions, and $sg \cdot u$ is the function defined by $(sg \cdot u)\ (x) = s(g(x)) \cdot u(x) \epsilon G$.

For $B = C_m$ and $A = C_{m+1}$ we define $M, P$ and the $\varphi$'s as follows. Let $k \xrightarrow{(f,s)} l$ be in $C_{m+1}$. Then $M((f,s)) = (f', s')$, where $f'(x) = f(x)$ if $f(x) \epsilon A_m \cup \{1, .., k\}$, $f'(x) = k + 1$ if $f(x) = a_{m+1}$, $f'(l + 1) = k + 1$, and $s'(x) = s(x)$ if $x \epsilon A_m \cup (1, \ldots, l)$, $s'(l + 1) = 1 \epsilon G$. If $k \xrightarrow{(f,s)} l$ is in $C_m$, then $P(f,s) = (f'', s'')$, where $f''(x) = f(x)$ and $s''(x) = s(x)$ on $A_m \cup (1, \ldots, l)$, $f''(a_{m+1}) = a_{m+1}$, $s''(a_{m+1}) = 1$. Finally, we let $t = |A_m||G|$ and for each $l$ and any $g \epsilon G$ and $j$, $1 \leq j \leq t$, we let $\varphi_{l,jg} = (d_{jl}, 1_{gl})$, where $d_{jl}(x) = x$, $1_{gl}(x) = 1$ for $x \epsilon A_m \cup \{1, \ldots, l\}$, and $d_{jl}(l + 1) = a_j$, $1_{gl}(l + 1) = g$.

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