COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
4. COMBINATORIAL THEORY AND ITS APPLICATIONS, BALATONFÜRED (HUNGARY), 1969.

On a class of equivalent linear and nonlinear integer programming problems

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Both nonlinear programming problems and integer programming problems tend to be extremely difficult to deal with; therefore it is a pleasant surprise to find even a simple nonlinear integer programming problem which has an effective solution algorithm.

We consider the following problem: Given a strictly convex function f of a real variable, and a real $\alpha > 0$, find integers m_i, n_i which minimize

$$\sum_{i=1}^{r} f(m_i + \alpha n_i)$$

subject to the constraints

$$\sum_{i=1}^{r} m_i = M, \qquad \sum_{i=1}^{r} n_i = N,$$

$$m_i \geq 0, \qquad n_i \geq 0.$$

We first develop some notation. We call an assignment of values to the m_i and n_i an allocation, which we write $(m_1, n_1; \dots; m_r, n_r)$. A feasible allocation is one satisfying the constraints of the problem. If A is an allocation, we give $f(A)$ the obvious meaning. If A is an allocation we define a simple transformation of A to be the replacement, for some i and j, of m_i, n_i m_j, n_j in A by m'_i, n'_i, m'_j, n'_j respectively, where $m_i + m_j = m'_i + m'_j$, $n_i + n_j = n'_i + n'_j$. Clearly, if A is feasible so is any simple transformation of A. If A' is a simple transformation of A and $f(A') < f(A)$, then A' is called a simple improvement of A. If A has no simple improvement we will say A is locally optimal. If A is a solution to the original problem, so that for no feasible A', $f(A') < f(A)$, we will say A is globally optimal.

We would clearly like to know under what conditions a locally optimal allocation is globally optimal. We will give later an example to show that if α is rational, degeneracies can occur, permitting a locally optimal allocation not to be globally optimal. The main result of this paper is to show that the irrationality of α is the only condition needed:

THEOREM: Let α be irrational. Then any feasible allocation which is locally optimal is globally optimal.

The proof of this theorem is long and will depend upon several lemmas, finally leading to the conclusion that the locally optimal allocation is essentially unique; hence the theorem. In the lemmas that follow we will not assume α is irrational unless we specifically so state. Our first lemma characterizes the term, "simple improvement".

LEMMA 1: Let A' be a simple transformation of A, with m_i, n_i, m_j, n_j replaced by m'_i, n'_i, m'_j, n'_j respectively.

Then A' is a simple improvement of A if and only if $|m'_i + \alpha n'_i - (m'_j + \alpha n'_j)| < |m_i + \alpha n_i - (m_j + \alpha n_j)|$.

Proof: Let $v_i = m_i + \alpha n_i$, $v'_i = m'_i + \alpha n'_i$, $v_j = m_j + \alpha n_j$, $v'_j = m'_j + \alpha n'_j$. We wish to prove $f(A') < f(A)$ if and only if $|v'_i - v'_j| < |v_i - v_j|$.

Without loss of generality we may assume $v_i \geq v_j$, $v'_i \geq v'_j$. Then, in view of the definition of a simple transformation, the inequality of the lemma is equivalent to $v_i > v'_i \geq v'_j > v_j$. Suppose this inequality to hold. Set $\lambda = (v_i - v'_j)/(v_i - v_j)$, so $0 < \lambda < 1$. Then by the strict convexity of f, we have

$$f(v_i) + f(v_j) = \lambda f(v_i) + (1-\lambda) f(v_j) + (1-\lambda) f(v_i) + \lambda f(v_j) >$$

$$> f(v_j + \lambda(v_i - v_j)) + f(v_i - \lambda(v_i - v_j)) =$$

$$= f(v_j + v_i - v'_j) + f(v_i - v_i + v'_j) =$$

$$= f(v'_i) + f(v'_j),$$

since $v_i + v_j = v'_i + v'_j$. Thus certainly $f(A) > f(A')$. The converse follows immediately upon exchanging A and A' in the above, and the observation that $|v'_i - v'_j| = |v_i - v_j|$ implies $F(A) = F(A')$.

This lemma, together with the theorem we are to prove, justifies the title of this paper. Since the lemma shows that the direction of a simple improvement does not depend on the nature of f, all such problems are equivalent with respect to this operation. In view of the theorem, they are also equivalent in the sense of having the same solution. To justify the word "linear" in the title, consider the function

$$\sum_{i<j} | m_i + \alpha n_i - (m_j + \alpha n_j) | .$$

This function is easily seen to decrease under any simple improvement; hence the linear problem with this as objective function is equivalent in the above senses to the nonlinear problems.

It will be convenient to represent an allocation $(m_1, n_1; \ldots; m_r, n_r)$ geometrically, by representing each pair (m_i, n_i) as a lattice point in the plane. In general some points will coincide. We will consider each distinct location only once; thus one point may represent many pairs. Thus an allocation A is represented by a set $\mathcal{X} = \{(x_1,y_1),\dots,(x_s,y_s)\}$; we will call this an $\mathcal{X}$-map of A. We see that A uniquely determines $\mathcal{X}$. We will call any set $\mathcal{X} = \{X_i\}$ allowable if each X_i is a lattice point in the closed first quadrant.

We will say an allowable set of points $\mathcal{X}$ is locally optimal if for no $(x_i,y_i),(x_j,y_j) \in \mathcal{X}$ do there exist nonnegative integers x'_i, y'_i, x'_j, y'_j such that $x'_i + x'_j = x_i + x_j$, $y'_i + y'_j = y_i + y_j$ and $|x'_i + \alpha y'_i - (x'_j + \alpha y'_j)| < |x_i + \alpha y_i - (x_j + \alpha y_j)|$. Clearly an allocation is locally optimal if and only if its $\mathcal{X}$-map is. We make two more definitions. Let $X_0 = (x_0,y_0)$; then the line defined by $x + \alpha y = x_0 + \alpha y_0$ is designated by $L(X_0)$ or $L(x_0,y_0)$. Note that if α is irrational, $L(X)$ can contain at most one lattice point. Second, if $X_0 = (x_0,y_0)$ and $X_1 = (x_1,y_1)$ are two points, the set of points (x,y) satisfying $0 \le x \le x_0 + x_1$, $0 \le y \le y_0 + y_1$ is called the spanning rectangle of X_0 and X_1.

We now give a lemma simplifying the criterion for local optimality.

LEMMA 2: Let $\mathcal{X}$ be allowable set of points. Suppose that for some $X_1, X_2 \in \mathcal{X}$ there exists a lattice point X in the **spanning** rectangle of X_1 and X_2, which also lies strictly between the lines $L(X_1)$ and $L(X_2)$. Then $\mathcal{X}$ is not locally optimal.

Proof: Set $(x_1,y_1) = X_1$, $(x_2,y_2) = X_2$, $(x,y) = X$. Consider also the point $X' = (x_1+x_2-x, y_1+y_2-y) = (x',y')$. Since X is in the spanning rectangle of X_1 and X_2, so is X'. Furthermore $x + x' = x_1 + x_2$ and $y + y' = y_1 + y_2$. It remains to check that $|x+\alpha y - (x'+\alpha y')| < |x_1+\alpha y_1 - (x_2+\alpha y_2)|$. Without loss of generality $x_1 + \alpha y_1 > x_2 + \alpha y_2$. Since X is between $L(X_1)$ and $L(X_2)$, $x_1+\alpha y_1 > x + \alpha y > x_2 + \alpha y_2$; but then $x_1 + \alpha y_1 > x' + \alpha y' > x_2 + \alpha y_2$. Therefore, $|x+\alpha y - (x'+\alpha y')| < |x_1 + \alpha y_1 - (x_2 + \alpha y_2)|$, and $\mathcal{X}$ is not locally optimal.

We note that the converse is clear; but we do not need this. The advantage of the above lemma is that we need only consider the relation between the pair of points X_1, X_2 and a single point X, not a pair of points. If X is in the rectangle spanned by X_1 and X_2 and is between $L(X_1)$ and $L(X_2)$ we will say X is derivable from X_1 and X_2.

LEMMA 3: Let $\mathcal{X}$ be an allowable set of points. Suppose that for some $X_1, X_2 \in \mathcal{X}$ it happens that $x_1 > x_2$, $y_1 > y_2$. Then $\mathcal{X}$ is not locally optimal.

Proof: Consider the point $X = (x_1, y_2)$. Clearly X is in the spanning rectangle of X_1 and X_2. Furthermore, $x_1 + \alpha y_1 > x_1 + \alpha y_2 > x_2 + \alpha y_2$, so by Lemma 2, $\mathcal{X}$ is not locally optimal.

LEMMA 4: If $\mathcal{X} = \{(x_1, y_1), \dots, (x_s, y_s)\}$ is locally optimal, then $\mathcal{X}' = \{(y_1, x_1), \dots, (y_s, x_s)\}$ is locally optimal if in the original problem, α is replaced by $1/\alpha$.

Proof: If $|x_i + \alpha y_i - (x_j + \alpha y_j)| \le |x_i' + \alpha y_i' - (x_j' + \alpha y_j')|$,
then $|y_i + x_i/\alpha - (y_i + x_j/\alpha)| \le |y_i' + x_i'/\alpha - (y_j' + x_j'/\alpha)|$.

This lemma is useful in reducing the number of cases to be considered in a proof.

LEMMA 5: Let α be irrational and let $\mathcal{X}$ be an allowable set of points. Suppose that for some $X_1, X_2 \in \mathcal{X}$ there is another lattice point on the line segment joining X_1 and X_2. Then $\mathcal{X}$ is not locally optimal.

Proof: Clearly such a point is derivable from X_1 and X_2.

LEMMA 6: Let α be irrational and let $\mathcal{X}$ be an allowable set of points. Let $X_1, X_2, X_3 \in \mathcal{X}$ be not collinear, and suppose that there is a lattice point in the interior of the triangle formed by these three points. Then $\mathcal{X}$ is not locally optimal.

Proof: Assume $\mathcal{X}$ to be locally optimal. Let the longest side of the triangle (or one of them if it is isosceles) be that joining X_1 and X_2. We first observe that this side cannot be either vertical or horizontal. Assume, for instance, that it were vertical, say $x_1 = x_2$, $y_1 > y_2$. Then the third point (x_3, y_3) would be in one of four regions defined by: $x_3 > x_1$, $y_3 > y_2$; $x_3 < x_1$, $y_3 < y_1$; $x_3 < x_1$, $y_3 \geq y_1$; and $x_3 > x_1$, $y_3 \leq y_2$. But the first two cases are impossible by Lemma 3, and the second two violate the assumption that X_1X_2 is the longest side of the triangle.

Therefore we may assume that $x_1 < x_2$, $y_1 > y_2$. By Lemma 4 we may also assume that $x_1 + \alpha y_1 > x_2 + \alpha y_2$, since the transformation involved does not alter distances. We now consider the possible location of X_3. We cannot have $x_3 \leq x_1$, $y_3 \geq y_1$ or $x_3 \geq x_2$, $y_3 \leq y_2$, since then X_1X_2 would not be the longest side of the triangle. On the other hand, by Lemma 3, X_3 cannot be in any of the four (overlapping) regions defined by: $x_3 > x_1$, $y_3 > y_1$; $x_3 > x_2$, $y_3 > y_2$; $x_3 < x_1$, $y_3 < y_1$; and $x_3 < x_2$, $y_3 < y_2$. Thus $x_1 \leq x_3 \leq x_2$ and $y_1 \geq y_3 \geq y_2$. This rectangular region is clearly contained within the spanning rectangle of X_1 and X_2; thus by Lemma 2 either $x_3 + \alpha y_3 > x_1 + \alpha y_1$ or $x_3 + \alpha y_3 < x_2 + \alpha y_2$.

We consider the former case first. Let $X = (x, y)$ be a lattice point in the interior of the triangle. We observe that $x_3 + \alpha y_3 > x_1 + \alpha y_1 > > x_2 + \alpha y_2$, so that no point in the entire triangle is above $L(X_3)$. Thus $x + \alpha y < x_3 + \alpha y_3$; certainly $x + \alpha y > x_1 + \alpha y_1$ also. Now, either $x \leq x_3$ or $x > x_3$. But certainly $y_1 \geq y$ and $x_2 \geq x$. Thus if $x \leq x_3$, X is in the spanning rectangle of X_1 and X_3 and is certainly between $L(X_1)$ and $L(X_3)$, which is impossible. If $x > x_3$, then $y < y_3$, so X is derivable from X_2 and X_3, which is impossible.

Now suppose that $x_3 + \alpha y_3 < x_2 + \alpha y_2$. Let $X = (x, y)$ again be a lattice point in the interior. By the above argument $x_3 + \alpha y_3 < x + \alpha y < x_2 + \alpha y_2$, $y_1 \geq y$, and $x_2 \geq x$. Now, there are three possibilities: $x \leq x_3$; $y \leq y_3$; or $x > x_3$ and $y > y_3$. If $x \leq x_3$, X is derivable from X_1 and X_3. If $y \leq y_3$, X is derivable from X_2 and X_3. Finally, let $x > x_3$ and $y > y_3$. Since X and X_3 are lattice points, so is $X' = (x, y_3)$. But then X' satisfies the condition of the second of the three possibilities, which we have just shown to be impossible. Thus the lemma is proved.

LEMMA 7: Let α be irrational and let $\mathcal{X}$ and $\mathcal{U}$ be two allowable sets of points. Let $X_1, X_2 \in \mathcal{X}$ and $U_1, U_2 \in \mathcal{U}$. Then if the line segments X_1X_2 and U_1U_2 intersect, either $\mathcal{X}$ or $\mathcal{U}$ is not locally optimal.

Proof: Assume $\mathcal{X}$ and $\mathcal{U}$ to be locally optimal. Set $X_i = (x_i, y_i)$, $U_i = (u_i, v_i)$, $i = 1, 2$. Without loss of generality we may assume that $x_1 + \alpha y_1 < x_2 + \alpha y_2$, $u_1 + \alpha v_1 < u_2 + \alpha v_2$, $x_1 + \alpha y_1 < u_1 + \alpha v_1$.
Then $u_1 + \alpha v_1 < x_2 + \alpha y_2$, since otherwise X_1X_2 and U_1U_2 could not intersect. We will distinguish two cases.

Case I: $u_2 + \alpha v_2 > x_2 + \alpha y_2$. Thus X_2 lies between $L(U_1)$ and $L(U_2)$. By Lemma 4, we may assume without loss of generality that $x_1 \leq x_2$, $y_1 \geq y_2$. Then we cannot have $u_1 \leq x_2$, $v_1 \leq y_1$, since then U_1 would be derivable from X_1 and X_2. Suppose that $u_1 > x_2$; then $v_1 < y_2$. But then $v_2 > y_2$, since the lines X_1X_2 and U_1U_2 intersect. Therefore X_2 is in the spanning rectangle of U_1 and U_2, which is impossible. Now suppose that $v_1 > y_1 \geq y_2$. Consider the triangle formed by X_1X_2, $L(X_1)$, and the perpendicular dropped from X_2. This triangle lies below $L(X_2)$, so that U_2 is not in the triangle. Since U_1U_2 crosses the side X_1X_2 , it must also cross one of the other two sides. It cannot cross $L(X_1)$, since U_2 must be above $L(X_1)$. Therefore it crosses the perpendicular, and hence $u_2 > x_2$. Hence X_2 is again in the spanning rectangle of U_1 and U_2, which is impossible.

Case II: $u_2 + \alpha v_2 < x_2 + \alpha y_2$. Thus both U_1 and U_2 lie between $L(X_1)$ and $L(X_2)$. By Lemma 4 we may assume that $u_1 \leq u_2$, $v_1 \geq v_2$. We will relabel the points X_1 and X_2. Let $P_1 = (p_1, q_1)$ be that point of $\{X_1, X_2\}$ which lies below the line U_1U_2, and $P_2 = (p_2, q_2)$ be that point which lies above. Since P_2 is above U_1U_2, we have

$$p_2(v_1 - v_2) + q_2(u_2 - u_1) > u_2v_1 - u_1v_2 .$$

Suppose that $p_1 + p_2 \geq u_2$. But $v_2 < v_1$, so certainly

$v_2 \le \max(q_1, q_2)$ and U_2 would be in the spanning rectangle of P_1 and P_2, which is a contradiction. Thus $p_1 + p_2 < u_2$. By the same argument $q_1 + q_2 < v_1$. Since all values concerned are integers $p_1 + p_2 \le u_2 - 1$, $q_1 + q_2 \le v_1 - 1$.

From the preceding inequality for P_2, we find

$$u_2 v_1 - u_1 v_2 < (u_2 - p_1 - 1)(v_1 - v_2) + (v_1 - q_1 - 1)(u_2 - u_1).$$

From this we deduce

$$p_1(v_1 - v_2) + q_1(u_2 - u_1) <$$

$$< u_2 v_1 - u_1 v_2 - 2u_2 v_1 + 2u_1 v_2 + u_2 v_1 - u_2 v_2 + u_2 v_1 - u_1 v_1 - (v_1 - v_2) - (u_2 - u_1)$$

$$= u_2 v_1 - u_1 v_2 - u_1(v_1 - v_2) - v_2(u_2 - u_1) - (v_1 - v_2) - (u_2 - u_1).$$

Suppose that $u_1 = u_2$; then $v_1 > v_2$. Then from the above we have

$$p_1(v_1 - v_2) < u_1(v_1 - v_2) - u_1(v_1 - v_2) - (v_1 - v_2),$$

so $p_1 < 0$, which is impossible. So we may assume $u_1 > u_2$.

Therefore

$$q_1 < -p_1 \frac{v_1 - v_2}{u_2 - u_1} + \frac{u_2 v_1 - u_1 v_2}{u_2 - u_1} - (u_1 + 1)\frac{v_1 - v_2}{u_2 - u_1} - v_2 - 1 \le$$

$$\le -p_1 \frac{v_1 - v_2}{u_2 - u_1} + \frac{u_2 v_1 - u_1 v_2}{u_2 - u_1} - v_2 - 1.$$

Since one form of the equation of the line $U_1 U_2$ is

$$q = -p \frac{v_1 - v_2}{u_2 - u_1} + \frac{u_2 v_1 - u_1 v_2}{u_2 - u_1},$$

we see that P_1 lies below U_1U_2 by more than v_2+1 in the vertical direction. From this we see also that $p_1 < u_2$, since otherwiese q_1 would be negative. In like manner we see that P_1 lies to the left of U_1U_2 by more than u_1+1 and that $q_1 < v_1$. Therefore P_1 lies in the pentagon bounded by the axes, the horizontal line through U_1, the vertical line through U_2, and the line U_1U_2. For any point $P=(p,q)$ in this pentagon, $p+\alpha q \le \max(u_1+\alpha v_1, u_2+\alpha v_2)$.

By Lemma 3, either $p_2 \ge p_1$ or $q_2 \ge q_1$. Since P_1 is below and P_2 above U_1U_2, we can say that $p_2 > p_1$ or $q_2 > q_1$. Suppose the former, so $q_2 \le q_1$ and $q_1 \ge v_2$. Consider the point $P_1' = (p_1+1, q_1)$. We have just seen that this point lies below the line U_1U_2, and hence within the pentagon, since $q_1 \ge v_2$. Thus $p_1+1+\alpha q_1 \le \max(u_1+\alpha v_1, u_2+\alpha v_2) < p_2+\alpha q_2$, so that P' lies between $L(P_1)$ and $L(P_2)$. But $p_1+1 \le p_2$ and $q_1 \le q_1$, so P' is in the spanning rectangle of P_1 and P_2; thus $\mathcal{X}$ is not locally optimal. Thus the lemma is proved.

LEMMA 8: Let α be irrational, and let $\mathcal{X}$ be an allowable set of points which is locally optimal. Then $\mathcal{X}$ contains at most three points.

Proof: By Lemma 5, no three points of $\mathcal{X}$ can be collinear. If $\mathcal{X}$ contains four points, either they are the vertices of a convex quadralateral or they form a triangle of points with the fourth point in its interior. The first case violates Lemma 8; the second violates Lemma 7.

LEMMA 9: Let A and A' be two possible allocations for the same problem. Then the convex hulls of the $\mathcal{X}$-maps of A and A' have a point X in common. Moreover, if either $\mathcal{X}$-map has three or more points, then X is in the interior of the plane figure defined by the $\mathcal{X}$-map; if it has two points, then X is in the interior of the line segment defined by the $\mathcal{X}$-map.

Proof: Indeed, they have the point $(M/r, N/r)$ in common. For let $\mathcal{X} = \{(x_1,y_1), \dots, (x_s,y_s)\}$ be the $\mathcal{X}$-map of A; then

$$\left(\frac{M}{r}, \frac{N}{r}\right) = \sum_{i=1}^{r} \frac{1}{r}(m_i, n_i) = \sum_{j=1}^{s} \frac{k_j}{r}(x_j, y_j),$$

where k_j is the number of pairs in A corresponding to (x_i, y_i). Likewise this point is in the $\mathcal{X}$-map of $\mathcal{X}'$ of A'. The rest of the lemma follows obviously from the above.

We are now ready to proceed to the proof of the theorem. Suppose that A and A' are two distinct locally optimal allocations for the problem. We first show that their $\mathcal{X}$-maps, designated by $\mathcal{X}$ and $\mathcal{X}'$ respectively, are distinct. We see that $\mathcal{X}$ has three or fewer distinct points, by Lemma 8. Suppose there are three points X_1, X_2, X_3. By the proof of Lemma 9, we have

$$(M, N) = k_1(x_1, y_1) + k_2(x_2, y_2) + k_3(x_3, y_3).$$

Also, $k_1 + k_2 + k_3 = r$. Hence

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} M \\ N \\ r \end{pmatrix}.$$

Since X_1, X_2 and X_3 are not collinear the matrix is nonsingular, so that the solution for k_1, k_2, k_3 is unique. Hence A' cannot have the same $\mathcal{X}$-map. The proof is similar but simpler if $\mathcal{X}$ has fewer than three points. (This is actually a special case of a standard result on convex set.)

We now consider a number of cases, each leading to the conclusion that either $\mathcal{X}$ or $\mathcal{X}'$ is not locally optimal. Without loss of generality we may assume that $\mathcal{X}$ has no more points than $\mathcal{X}'$. First suppose that $\mathcal{X}$ contains a single point X. Then if $\mathcal{X}'$ contains just two points, by Lemma 9 it is in the interior of the segment defined by $\mathcal{X}'$, contradicting Lemma 5. If $\mathcal{X}'$ contains three points, then by Lemma 9 it is in the interior of the triangle defined by $\mathcal{X}'$, contradicting Lemma 6. Next, suppose that $\mathcal{X}$ contains just two points X_1

and X_2. If $\mathcal{X}'$ also contains just two points, the two corresponding line segments intersect, contradicting Lemma 7. If $\mathcal{X}'$ contains three points, it either happens that X_1 or X_2 is in the interior of the triangle so defined, or that X_1X_2 intersects a side of the triangle, contradicting Lemmas 6 or 7 respectively. Finally, let $\mathcal{X}$ and $\mathcal{X}'$ both define triangles. Then either one is contained in the other, contradicting Lemma 6, or their sides intersect, contradicting Lemma 7. This completes the proof of the theorem.

The results we have shown raise some interesting combinatorial questions about the configurations of locally optimal allowable sets. Every rational point $(M/r, N/r)$ in the first quadrant represents some problem, so every such point is in the convex hull of some such set. Thus the entire first quadrant is filled with nonoverlapping triangles representing such sets. It would be of considerable interest to study such configurations and their behavior as α is varied. For example, the triangles necessarily have area $\frac{1}{2}$, so that it is possible to estimate their number in a large region. Furthermore, it is clear that the configuration in a finite region in general remains constant under a small change in α. It should be possible to say a good deal more than these trivial results.

The theorem we have proved yields a rather satisfactory method of solving problems of the type considered. Choose some starting allocation (preferably in in an artful manner), and perform simple improvements until the process terminates at the solution, as it must if α is irrational. A problem can arise if α is rational. A continuity argument shows that is still true that all the problems discussed are equivalent in the same sense as before. However, it can happen that a locally optimal allocation is not globally optimal. Let $\alpha = 2$ and consider the allocation $A_1 = (0,1;0,2;3,0)$. It is easy to see that A_1 is locally optimal. However, a simple transformation of A_1 leads to $A_2 = (2,0;0,2;1,1)$, with $f(A_1) = f(A_2)$. This is not locally optimal and leads to $A_3 = (1,1;1,1;1,1)$ which is clearly globally optimal.

Define a neutral transformation of A to be a simple transformation of A into A' such that $f(A) = f(A')$. (We rule out merely exchanging values: $m'_i = m_j$, $n'_i = n_j$, $m'_j = m_i$, $n'_j = n_i$.) By a continuity argument, the globally optimal allocation is always obtainable by simple improvements and neutral transformations alone. Therefore, one possible method of solving the problem with α rational is by generating simple improvements until a locally optimal allocation is found, and testing all possible allocations related to it by sequence of neutral transformations. This, however, is likely to be tedious; perturbation techniques might also be effective. The authors have reason to believe that a simple, systematic way of dealing with rational α exists; this should be an interesting avenue to explore.

In practice, α may be irrational, but it must be represented by a rational number if the problem is to be solved by a computer. In this case one may proceed as before to find a locally optimal allocation; if no neutral transformation of it exists, it is globally optimal. If the original α is irrational this will almost certainly happen.

There are two interesting directions in which the result of this paper might be extended. First, one might generalize the given objective function to

$$\sum_{i=1}^{r} f(m_i + \alpha n_i + \gamma_i),$$

where the γ_i, $i = 1, \ldots, r$ are given constants. In this case we conjecture that the theorem we have proved is still true. Clearly the proof we have given would have to be considerably modified; in fact, if the generalization is true it is likely that an altogether different proof would be necessary.

Another direction of generalization would be to increase the dimension. Thus, one could ask to minimize

$$\sum_{i=1}^{r} f(k_i + \alpha m_i + \beta n_i),$$

subject to

$$\sum_{i=1}^{r} k_i = K, \qquad \sum_{i=1}^{r} m_i = M, \qquad \sum_{i=1}^{r} n_i = N .$$

It seems very likely that a result analogous to that of this paper is true in this case. It is not clear, however, what definition of a simple transformation to take. In particular, perhaps such a transformation should alter three triplets, not two. If such a generalized result exists, it is probable that it can be proved in a manner similar to that of this paper.