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Irregularities in the Distributions of Finite Sequences

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Suppose $(x_1, x_2, ..., x_{s+d})$ is a sequence of numbers with $x_i \in [0, 1)$ which has the property that for each $r \leqslant s$ and for each $k < r$, the subinterval $[k/r, (k + 1/n))$ contains at least one point of the subsequence $(x_1, x_2, ..., x_{r+d})$. For fixed d, we wish to find the maximum $s = s(d)$ for which such a sequence exists. We show that $s(d) < 4^{(d+2)^2}$ for all d and that $s(0) = 17$.

Let $X = (x_1, x_2, ...)$ be a sequence of points in the interval $[0, 1)$, I a subinterval, $|I|$ its length and $X_n(I)$ the number of x_m in I with $m \leqslant n$. Let $F_n(X)$ be the least upper bound of $|X_n(I) - n|I||$ for I varying in $[0, 1)$. $F_n(X)$ was first proved to be unbounded by van Ardenne-Ehrenfest [2], [3] who showed (settling a conjecture of van der Corput [1])

$$F_n(X) > c_1 \log\log n/\log\log\log n. \tag{1}$$

This was later improved by K. F. Roth [4] who established

$$F_n(X) > c_2 \sqrt{\log n}. \tag{2}$$

In this note we consider the following finite variant of this problem: For $n \geqslant 1$, $0 \leqslant k < n$, define

$$B_{n,k} = \left[\frac{k}{n}, \frac{k+1}{n}\right). \tag{3}$$

Fix an integer $d \geqslant 0$ and *suppose* $(x_1, x_2, ..., x_{s+d})$ is a sequence with $x_i \in [0, 1)$ and with $s = s(d)$ chosen to be *maximal* such that for each $r \leqslant s$ and each $k < r$, $B_{r,k}$ contains at least one point of the subsequence $(x_1, x_2, ..., x_{r+d})$. The fact that $s(d) < \infty$ follows from (2). In fact, the results of [4] can be used to show $s(d) < 2^{2^{16}d^2}$ for sufficiently large d.

Our goal is to establish the

THEOREM.

(i) $s(0) = 17$.

(ii) $s(d) < 4^{(d+2)^2}$ *for all d.*

Proof of (i). Suppose $s(0) \geqslant 17$. Without loss of generality, we may assume that $x_1, x_2, \ldots, x_{17}$ are all irrational. Let $\{y_1, y_2, \ldots, y_8\} = \{x_1, x_2, \ldots, x_8\}$, where the y's are reordered so that

$$0 < y_1 < 1/8 < y_2 < 2/8 < y_3 < 3/8 < \cdots < 7/8 < y_8 < 1. \quad (4)$$

Heuristically, the reason that we choose to reorder the first eight points is that it results in a considerable simplification of the argument. It turns out that there are many legitimate ways of choosing seventeen points $x_1, x_2, \ldots, x_{17}$, but that most of these sequences differ from each other only in the reordering of $x_1, x_2, \ldots, x_8$. By introducing the y's, we are thus able to reduce the number of sequences that must be considered to a more tractable value. In columns 1 and 2 of Tables 1 and 2, we list the Farey sequence of fractions between 0 and 1 with denominators not exceeding 18. If we know between which two consecutive Farey fractions y_i is located, then we know in which half, which third, which fourth,..., which eighteenth of the unit interval y_i occurs. In particular, we know i. For each $j = 9, 10, 11, \ldots, 17$, we may determine whether $x_j < y_i$ or $x_j > y_i$. For example, if $y_i \in (3/10, 4/13)$, then $i = 3$. Since $y_3 \in (2/9, 3/9)$, y_3 must be the third of the first nine x's, so $x_9 > y_3$. Since $y_3 \in (3/10, 4/10)$, y_3 must be the fourth of the first ten x's, so $x_{10} < y_3$. Similarly, we may deduce that if $y_3 \in (3/10, 4/13)$, then $x_i > y_3$ if $i = 9, 11, 12, 13, 15, 16$, but that $x_i < y_3$ if $i = 10, 14$, or 17. The indices of these three smaller x_i's (10, 14, 17) are listed in Column 3. Similarly, for each Farey interval in which y_j might lie, $y_j \in (0, 1/2)$, we may determine the indices of the smaller x_i's. These indices are listed in Column 3 of Tables 1 and 2.

After completing the calculations necessary to determine the entries in Column 3, we now search for compatible intervals for y_3 and y_4. For example, if $y_3 < 2/7$, then $y_4 < 3/7$. If $y_3 \in (1/4, 4/15)$, then $x_{12} < y_3$ and $x_{16} < y_3$, so that we must also have $x_{12} < y_4$ and $x_{16} < y_4$. But $y_4 \in (3/8, 3/7)$, and there is no subinterval of (3/8, 3/7) in which y_4 might lie such that $x_{12} < y_4$ and $x_{16} < y_4$. Therefore, $y_3 \notin (1/4, 4/15)$. Similarly, $y_3 \in (4/15, 3/11)$ only if $x_{12} < y_4$ and $x_{15} < y_4$, which can happen only if $y_4 \in (5/12, 3/7)$. This match between possible intervals of y_3 and y_4 is

COLUMN 1	COLUMN 2	COLUMN 3	COLUMN 4	COL. 5	COLUMN 6	COLUMN 7	COLUMN 8
INTERVALS		INDICES OF SMALLER (LARGER) X'S	POSSIBILITIES TO ONE SIDE OF 1/2	MATCH ACROSS 1/2	ONLY SOLUTION ABOVE 1/2 (B)	REAL SOLUTION BELOW 1/2 (E)	ILLEGITIMATE SOLUTION BELOW 1/2 (D)
1/4 TO 1/2 READ DOWN	1/2 TO 3/4 READ UP						
1/4	3/4						
		12 16	**	(RELEVANT ONLY BETWEEN 3/8 AND 5/8)			
4/15	11/15						
		12 15	E			Y_3	
3/11	8/11						
		11 15	**				X_{17}
5/18	13/18						
		11 15	**				
2/7	5/7						
		11 14	A,B		Y_6		
5/17	12/17						
		11 14 17	*				
3/10	7/10						
		10 14 17	C				
4/13	9/13						
		10 13 17	D			X_{17}	Y_3
5/16	11/16						
		10 13 16	*		X_{16}		
1/3	2/3						
		9 12 15	*			X_{10}	
6/17	11/17						
		9 12 15 17	*			X_{10}	
5/14	9/14						
		9 12 14 17	*				X_{15}
4/11	7/11						
		9 11 14 17	*				
3/8	5/8						
		11 14 16	A	***	X_9	X_{17}	
5/13	8/13						
		11 13 16					
7/18	11/18						
		11 13 16					
2/5	3/5						
		10 13 15					
7/17	10/17						
		10 13 15 17	D	B'			Y_4
5/12	7/12						
		10 12 15 17	E	B'		Y_4	X_{12}
3/7	4/7						
		10 12 14 17	C	***			
7/16	9/16						
		10 12 14 16					
4/9	5/9						
		9 12 14 16					
5/11	6/11						
		9 11 14 16	B	***	Y_5		
6/13	7/13						
		9 11 13 16					
7/15	8/15						
		9 11 13 15					
8/17	9/17						
		9 11 13 15 17				X_{13}	
1/2	1/2						
REFLECTION ACROSS 1/2:							
		10 12 14 16					
9/17	8/17						
		10 12 14 16 17					
8/15	7/15						
		10 12 14 15 17					
7/13	6/13						
		10 12 13 15 17	B'				
6/11	5/11						
		10 11 13 15 17					
5/9	4/9						
		9 11 13 15 17					
9/16	7/16						
		9 11 13 15 16	C'				
4/7	3/7						

CODE:
- * NO MATE BETWEEN 3/8 TO 1/2
- ** NO MATE BETWEEN 3/8 TO 3/7
- *** NO MATE AMONG B' AND C'

TABLE 1 - POINTS BETWEEN 1/4 AND 3/4

recorded in Column 4 as possibility *E*. Similarly, we find that the only other possibilities are *A*, *B*, *C*, and *D*.

Reflecting these results across 1/2, we find that there are likewise only five possibilities for y_5 and y_6. Since either $y_4 \in (3/7, 1/2)$ or $y_5 \in (1/2, 4/7)$, we may assume, without loss of generality, that $y_5 \in (1/2, 4/7)$. Since only

COLUMN 1 0 TO 1/4 READ DOWN	COLUMN 2 3/4 TO 1 READ UP	COLUMN 3
0/1	1/1	
1/18	17/18	
1/17	16/17	17
1/16	15/16	16
1/15	14/15	15
1/14	13/14	14
1/13	12/13	13
1/12	11/12	12
1/11	10/11	11
1/10	9/10	10
1/9	8/9	9
2/17	15/17	9 17
1/8	7/8	16
2/15	13/15	15
1/7	6/7	14
2/13	11/13	13
1/6	5/6	12
3/17	14/17	12 17
2/11	9/11	11 17
3/16	13/16	11 16
1/5	4/5	10 15
3/14	11/14	10 14
2/9	7/9	9 14
3/13	10/13	9 13
4/17	13/17	9 13 17
1/4	3/4	

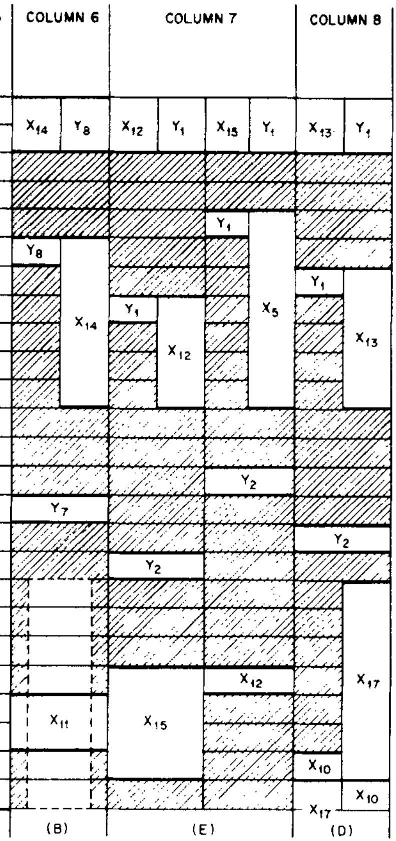

TABLE 2 - POINTS BETWEEN 0 AND 1/4 OR 3/4 AND 1

two of the subintervals of (1/2, 4/7) have matching intervals for y_6, we deduce that either $y_5 \in (7/13, 6/11)$ or $y_5 \in (9/16, 4/7)$. This latter possibility is incompatible with all of the possible intervals for y_4 as noted by the "***" in Column 5. We conclude that $y_5 \in (7/13, 6/11)$, and either $y_4 \in (5/12, 3/7)$ or $y_4 \in (7/17, 5/12)$. Since $y_5 \in (7/13, 6/11)$, we conclude that $y_6 \in (12/17, 5/7)$ and $y_7 \in (11/13, 6/7)$.

It is also evident that $y_5 < x_9 < y_6 < x_{11} < y_7 < x_{14}$, and that $y_5 < x_{16} < y_6$. It follows that the Farey interval occupied by x_9 must contain 11 and 14 in Columns 3, but it cannot have any other number higher than 9 in Column 3 with the possible exception of 16. Inspection reveals that $x_9 \in (8/13, 5/8)$ is the only possibility. Since the Farey interval (8/13, 5/8) contains a 16 in Column 3, $x_9 < x_{16}$. The Farey interval containing x_{16} must not contain any number higher than 16 in Column 3. Inspection reveals that $x_{16} \in (11/17, 11/16)$.

Similar arguments constrain x_9, x_{16}, x_{11}, and x_{14} to the intervals shown in Column 6. Since x_{10}, x_{12}, x_{13}, x_{15}, and x_{17} all lie below 1/2, Column 6 shows that none of the first 17 points can occur in the interval (10/18, 11/18) and x_{11} is the only one of the first 17 points which can occur in the interval (13/18, 15/18). We therefore deduce that $s(0) < 18$.

To prove that $s(0) = 17$, we list in Column 7 the possible intervals if $y_4 \in (4/7, 7/12)$ and we list in Column 8 the possible intervals if $y_4 \in (7/12, 10/17)$. In each case, there are several possibilities, which are shown in different subcolumns. Finally, we must check to see whether we can order the set $\{y_1, y_2, \ldots, y_8\}$. It turns out that the solution of Columns 6 and 8 is unorderable, for we have $2/7 < y_3 < y_4 < 3/7$, so either $y_3 = x_8$ or $y_4 = x_8$. But only $y_3 \in (1/6, 1/3)$ and only $y_4 \in (1/3, 1/2)$. Hence, we cannot have either $y_3 = x_8$ or $y_4 = x_8$. Therefore, the solution of Columns 6 and 8 is illegitimate. However, all of the solutions of Columns 6 and 7 are legitimate; the y's can be sequenced in many different ways. For example, we may choose $x_1 = y_6$, $x_2 = y_1$, $x_3 = y_4$, $x_4 = y_7$, $x_5 = y_3$, $x_6 = y_5$, $x_7 = y_8$, $x_8 = y_2$.

This proves (i). A footnote in [5] mentions that M. Warmus also verified $s(0) = 17$ by computer (unpublished).

Proof of (ii). We first note that

$$\begin{aligned} x \in B_{r,k} \quad & \text{iff} \quad x \in \left[\frac{k}{r}, \frac{k+1}{r}\right) \\ & \text{iff} \quad rx \in [k, k+1) \\ & \text{iff} \quad [rx] = k, \end{aligned}$$

where $[z]$ denotes the greatest integer not exceeding z. Thus, an alternative statement of the hypothesis is

$$\text{For each } r \leqslant x, \qquad \{0, 1, \ldots, r-1\} = \{[rx_1], \ldots, [rx_{r+d}]\}. \tag{5}$$

Let $S(x)$ denote $([x], [2x], [3x], \ldots)$. We need several facts concerning $S(x)$.

DEFINITION 1. We say that $S(x)$ *has a jump at* r if $[rx] > [(r-1)x]$.

For $0 < x < 1$, let m be the unique integer defined by

$$1/m \leqslant x < 1/(m-1). \tag{6}$$

Fact. If $S(x)$ has a jump at r then the *next* jump of $S(x)$ is either at $r + m - 1$ or $r + m$. To see this, note that

$$\begin{aligned} [(r-1)x] = u - 1, \qquad & [rx] = u \\ \Rightarrow (r-1)x < 1, \qquad & rx \geqslant u \\ \Rightarrow (r+m-2)x = (r-1)x + (m-1)x & < u + 1 \end{aligned}$$

and

$$(r + m)x = rx + mx \geqslant u + 1.$$

Thus,

$$[(r + m - 2)x] = u, \qquad [(r + m)x] \geqslant u + 1$$

and the fact follows.

DEFINITION 2. We say $S(x)$ has an *early jump at r* if the preceding jump of $S(x)$ occurs at $r - m + 1$. We say that $S(x)$ has a *late jump at r* if the preceding jump of $S(x)$ occurs at $r - m$.

If we let $\delta(x)$ denote $x - 1/m$ (so that $0 \leqslant \delta(x) < 1/m(m - 1)$), then we observe that $S(x)$ has an early jump at r iff $S(\delta(x))$ has a jump at r. Thus, if $u \leqslant t\delta(x) < u + 1$ then $S(x)$ has exactly u early jumps in the set $\{1, 2,..., t\}$. More generally, in any set $\{j, j + 1,..., j + t\}$ of $t + 1$ consecutive integers, $S(x)$ has either u or $u + 1$ early jumps since

$$u \leqslant [(j + t)\, \delta(x)] - [j\delta(x)] = [j\delta(x) + t\delta(x)] - [j\delta(x)] \leqslant u + 1. \tag{7}$$

Assume now that for some d, there is an $s \geqslant 4^{(d+2)^2}$ for which (5) holds. We shall derive a contradiction. Make the following definitions:

$$\begin{aligned}
&\text{(a)} \qquad && a = (d + 2)\, 2^d + 3. \\
&\text{(b)} \qquad && h_k = 2^k a, \qquad 0 \leqslant k \leqslant d. \\
&\text{(c)} \qquad && N_k = 2h_{k+1}h_{k+2} \cdots h_d\,, \qquad 0 \leqslant k < d. \\
&\text{(d)} \qquad && N_d = 1. \\
&\text{(e)} \qquad && M = N_0(d + 4).
\end{aligned} \tag{8}$$

A straightforward calculation shows

$$\left[\frac{Mh_d}{2}\right] \geqslant 1 + \frac{(h_{d-1} + 1)\, Mh_d}{N_{d-1} - 1}, \tag{9}$$

$$\frac{Mh_d}{N_k} \geqslant 2 + \frac{(h_{k-1} + 1)\, Mh_d}{N_{k-1} - 1}, \qquad 1 \leqslant k < d, \tag{10}$$

$$2Mh_d < 4^{(d+2)^2}. \tag{11}$$

By the hypothesis of the theorem we can find terms z_k, $0 \leqslant k \leqslant d$, of the sequence $(x_1 ,..., x_{s+d})$ such that

$$[Mh_d z_k] = \frac{Mh_d}{h_k} + N_k - 1, \qquad 0 \leqslant k \leqslant d. \tag{12}$$

Hence we have

$$\frac{1}{h_k} + \frac{N_k - 1}{Mh_d} \leqslant z_k < \frac{1}{h_k} + \frac{N_k}{Mh_d} \tag{13}$$

or equivalently,

$$\frac{N_k - 1}{Mh_d} \leqslant z_k - \frac{1}{h_k} < \frac{N_k}{Mh_d}. \tag{13'}$$

For $k = 0$,

$$\begin{aligned} \frac{h_k(h_k - 1) N_k}{h_d} &= \frac{(h_0 - 1) N_0}{2^d} = \frac{(a - 1) N_0}{2^d} \\ &= \frac{(d + 2) 2^d + 2}{2^d} \cdot N_0 = ((d + 2) + 2^{-d+1}) N_0 \\ &\leqslant (d + 4) N_0 = M. \end{aligned} \tag{14}$$

For $0 < k \leqslant d$,

$$\frac{h_k(h_k - 1) N_k}{h_d} = \frac{h_k - 1}{h_d} \cdot N_{k-1} \leqslant \frac{h_d - 1}{h_d} N_0 \leqslant M. \tag{14'}$$

Thus by (13)

$$z_k - \frac{1}{h_k} < \frac{N_k}{Mh_d} \leqslant \frac{1}{h_k(h_k - 1)} \tag{15}$$

and

$$\frac{1}{h_k} \leqslant z_k < \frac{1}{h_k - 1} \tag{16}$$

for $0 \leqslant k \leqslant d$. Consider z_d. By (13′) and (8)

$$\delta(z_d) = z_d - \frac{1}{h_d} < \frac{1}{Mh_d}. \tag{17}$$

Thus, in the set $I = \{Mh_d, Mh_{d+1}, \ldots, 2Mh_d - 1\}$ there is at most *one* point at which $S(z_d)$ has an early jump. Therefore, there is a subset $I_d \subseteq I$ consisting of at least $[(Mh_d/2)]$ consecutive integers such that $S(z_d)$ has *only late jumps* on I_d. Since the difference between consecutive late jump points of $S(z_d)$ is h_d, then for some integer q, the elements of I_d at which $S(z_d)$ jumps are *exactly* the elements of I_d which are congruent to q modulo h_d. By (9), (13′), and a previous remark, $S(z_{d-1})$ has at least $h_{d-1} + 1$ early jumps on I_d. Since

$$\delta(z_{d-1}) = z_{d-1} - \frac{1}{h_{d-1}} < \frac{N_{d-1}}{Mh_d}, \tag{18}$$

then any (Mh_d/N_{d-1}) consecutive integers contain at most one early jump point of $S(z_{d-1})$. Also we note the important fact that if $S(z_{d-1})$ jumps at r, r' and r'' with $r < r' < r''$, r' an early jump point and r'' a late jump point, then

$$r \equiv r'' + 1 \pmod{h_{d-1}}. \tag{19}$$

Since $S(z_{d-1})$ has at least $h_{d-1} + 1$ early jump points on I_d then the late jump points of $S(z_{d-1})$ occur in all residue classes modulo h_{d-1} on I_d. In fact, we can assert that there must exist an interval $I_{d-1} \subseteq I_d$ of $(Mh_d/N_{d-1}) - 1$ consecutive integers such that the elements of I_{d-1} on which $S(z_{d-1})$ jumps are exactly the elements of I_{d-1} which are congruent to q modulo h_{d-1}. Now, by (10) and (13′), we can argue as before to conclude that there exists an interval $I_{d-2} \subseteq I_{d-1}$ of $(Mh_d/N_{d-2}) - 1$ consecutive integers such that the elements of I_{d-2} on which $S(z_{d-2})$ jumps are exactly the elements of I_{d-2} which are congruent to q modulo h_{d-2}.

This argument can be continued until we reach I_0, an interval of $(Mh_d/N_0) - 1$ consecutive integers. By construction $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_d \subseteq I$ and $S(z_0)$ has jumps exactly on the elements of I_0 which are congruent to q modulo h_0. By (8)(e), I_0 consists of

$$(Mh_d/N_0) - 1 = (d+4)\,h_d - 1 > (d+3)\,h_d \tag{20}$$

consecutive integers. Note that if $S(z_k)$ jumps at $r \in I_0$ then $S(z_l)$ jumps at r for all $l < k$. By (20), $S(z_d)$ has at least $d + 2$ jump points on I_0; let us denote them by $\{j_0, j_1, \ldots, j_{d+1}\}$ where

$$j_k = j_0 + kh_d, \qquad 0 \leqslant k \leqslant d+1, \tag{21}$$

and let I' denote the set $\{j_0, j_0 + 1, \ldots, j_{d+1}\}$.

It is not hard to show that (8) implies

$$\frac{Mh_d}{h_{k-1}} + N_{k-1} - 1 > 1 + \frac{Mh_d}{h_{k-1}}, \qquad 1 \leqslant k \leqslant d, \tag{22}$$

and

$$\frac{Mh_d}{h_k} + N_k - 1 < \frac{Mh_d}{h_k - 1}. \tag{23}$$

Thus, we have

$$\frac{Mh_d}{h_k - 1} < \left[1 + \frac{Mh_d}{h_k - 1}\right] < \frac{Mh_d}{h_{k-1}} + N_{k-1} - 1, \qquad 1 \leqslant h \leqslant d. \tag{24}$$

By the hypothesis of the theorem there exist terms y_k, $0 \leqslant k \leqslant d$, of the sequence $(x_1, \ldots, x_{s+d})$ such that

$$[Mh_d y_k] = \left[1 + \frac{Mh_d}{h_k - 1}\right]. \tag{25}$$

It follows from (23) and (24) that

$$\begin{aligned} [ty_k] &\geqslant [tz_k] \quad \text{for any} \quad t \geqslant 0, \quad 0 \leqslant k \leqslant d, \\ [tz_{k-1}] &\geqslant [ty_k] \quad \text{for any} \quad t \geqslant 0, \quad 1 \leqslant k \leqslant d. \end{aligned} \tag{26}$$

By the definition of y_k we have

$$\delta(y_k) = y_k - \frac{1}{h_k - 1} < \frac{1}{Mh_d}. \tag{27}$$

Since I' consists of $(d+1)\,h_d + 1$ consecutive integers then $S(y_k)$ can have at most *one* early jump point in I'. Note that if r is an early jump point of $S(y_k)$ then $r - h_k + 2$ is the preceding jump point of $S(y_k)$. Also $S(z_k)$ has exactly $1 + (d+1)\,2^{d-k}$ jump points on I'. By (8) it follows that

$$(d+1)\,2^{d-k}(h_k - 1) + h_k - 2 > (d+1)\,h_d, \qquad 0 \leqslant k \leqslant d. \tag{28}$$

Hence, $S(z_k)$ and $S(y_k)$ can have *at most one common jump point* in I'. Since the j_i, $0 \leqslant i \leqslant d+1$, are jump points of all the $S(z_k)$, then, for some w, j_w is a *jump point of each* $S(z_k)$ and *not a jump point of any* $S(y_k)$.

The critical point to observe now is the following. Suppose

$$\begin{aligned} [(j_w - 1)\,z_k] &= m, \\ [(j_w - 1)\,u_1] &= m + 1, \\ &\;\vdots \\ [(j_w - 1)\,u_{n-m-1}] &= n - 1, \\ [(j_w - 1)\,y_k] &= n, \end{aligned} \tag{29}$$

where the u_i are suitable terms from the sequence $(x_1, \ldots, x_{j_w+d-1})$ which exist by the hypothesis of the theorem. [By (26), $n > m$.] It follows that

$$\begin{aligned} [j_w z_k] &= m + 1 \\ [j_w u_1] &= m + 1 \quad \text{or} \quad m + 2 \\ &\;\vdots \\ [j_w u_{n-m-1}] &= n - 1 \quad \text{or} \quad n \\ [j_w y_k] &= n \end{aligned} \tag{30}$$

since $S(z_k)$ jumps at j_w and $S(y_k)$ does not jump at j_w. Thus, some integer v_k, $m + 1 \leqslant v_k \leqslant n$, must occur at least twice in this list. This argument implies the existence of integers v_k, $0 \leqslant k \leqslant d$, such that

$$\begin{aligned} [j_w z_d] \leqslant v_d \leqslant [j_w y_d] < [j_w z_{d-1}] \leqslant v_{d-1} \leqslant [j_w y_{d-1}] < \cdots \\ < [j_w x_0] \leqslant v_0 \leqslant [j_w y_0] \end{aligned} \tag{31}$$

and such that each v_k, $0 \leqslant k \leqslant d$, occurs at least twice in the set $T = \{[j_w x] : 1 \leqslant x \leqslant j_w + d\}$. By (31) all the v_k are distinct. Consequently $|T| \leqslant j_w - 1$ which *contradicts* the hypothesis of the theorem. This completes the proof of (ii).

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