A PACKING INEQUALITY FOR COMPACT CONVEX SUBSETS OF THE PLANE

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1. Introduction. Let X be a compact metric space. By a packing in X we mean a subset $S \subseteq X$ such that, for $x, y \in S$ with $x \neq y$, the distance $d(x,y) \geq 1$. Since X is compact, any packing of X is finite. In fact, the set of numbers

$$\{\text{card}(S): S \text{ is a packing in } X\}$$

is bounded. The cardinality of the largest packing in X will be called the packing number of X and will be denoted by $\rho(X)$. If $A(X)$ and $P(X)$ denote the area and perimeter, respectively, of a compact convex subset X of the plane, then a special case of a result conjectured by H. Zassenhaus [6] and proved by N. Oler [1] is the following.

THEOREM (Oler).

$$\rho(X) \leq \frac{2}{\sqrt{3}} A(X) + \frac{1}{2} P(X) + 1. \tag{1}$$

Unfortunately, Oler's proof of his general theorem requires 30 pages of rather detailed arguments. It is our purpose in this note to establish a theorem of this type for simplicial complexes in the plane. This theorem will imply (1) and, moreover, the arguments used are quite elementary.

2. Preliminaries. By a p-simplex in the plane we mean the convex hull of $p+1$ points in general position in the plane. Since there can be at most 3 points in general position in the plane, we must have $p = 0, 1$ or 2. If $x_0, \ldots, x_p$ are in general position, $(x_0, \ldots, x_p)$ will denote the p-simplex which is their convex hull. The points $x_0, \ldots, x_p$ will be called the vertices of $(x_0, \ldots, x_p)$. If σ and τ are simplexes, we say that σ is a face of τ if the vertices of σ are a subset of the vertices of τ.

By a simplicial complex in the plane, we mean a finite set K of the simplexes in the plane with the following properties:

(2) if $\sigma \in K$ then every face of σ is in K;

(2') if $\sigma, \tau \in K$ and $\sigma \cap \tau$ is nonempty, then $\sigma \cap \tau$ is a face of both σ and τ.

Let K be a simplicial complex in the plane. We denote by $|K|$ the union of the simplexes in K. If $r \geq 0$ is an integer, we denote by K^r the set of all p-simplexes in K with $p \leq r$. We let $\alpha_r(K)$ denote the number of r-simplexes in K. The Euler characteristic $\chi(K)$ is defined by $\chi(K) = \alpha_0(K) - \alpha_1(K) + \alpha_2(K)$. It is a theorem of combinatorial topology that $\chi(K)$ depends only on $|K|$ (cf. [5]).

If σ is a 1-simplex in K we let $\varepsilon(\sigma, K)$ be the number of 2-simplexes in K having σ as a face. By (2'), $\varepsilon(\sigma, K) \leq 2$. If σ is a 1-simplex or a 2-simplex in the plane, we let $m(\sigma)$ denote the length or the area, respectively, of σ. We define A(K) and P(K) by

$$A(K) = \sum_{\sigma \in K^2 - K^1} m(\sigma)$$

and

$$P(K) = \sum_{\sigma \in K^1 - K^0} (2 - \varepsilon(\sigma, K)) m(\sigma) .$$

The numbers A(K) and P(K) depend only on $|K|$ since A(K) is the area of $|K|$ while P(K) is its perimeter (suitably defined).

3. The Main Result.

THEOREM. Let K be a simplical complex in the plane. Suppose that for any $x, y \in K^0$ with $x \neq y$ we have $d(x,y) \geq 1$. Then

$$\alpha_0(K) \leq \frac{2}{\sqrt{3}} A(K) + \frac{1}{2} P(K) + \chi(K). \tag{3}$$

The proof of the theorem will be by induction using the following two lemmas.

LEMMA 1. *Let* Δ *be a triangle with area* A *and sides of length* s_1, s_2 *and* s_3. *If* $s_1 \geq s_2 \geq s_3 \geq 1$ *then* $\frac{4}{\sqrt{3}} A + s_1 \geq s_2 + s_3$.

Proof. We first note

$$(s_1 + s_2 + s_3)(s_1 + s_2 - s_3)(s_1 - s_2 + s_3) \geq s_1 + s_2 + s_3 \geq 3s_3 \geq 3(s_2 + s_3 - s_1).$$

Using Hero's formula for the area of a triangle together with the inequalities $s_1 \leq s_2 + s_3$ and $A \geq 0$ we obtain $16A^2 \geq 3(s_2 + s_3 - s_1)^2$. Hence $4A \geq \sqrt{3}\,(s_2 + s_3 - s_1)$, or $\frac{4}{\sqrt{3}} A + s_1 \geq s_2 + s_3$ as required.

LEMMA 2. *Let* Q *be a convex quadrilateral in the plane with area* A *and perimeter* P. *Suppose that:*

length of any diagonal of $Q \geq$ *length of any side of* $Q \geq 1$.

Then $\frac{4}{\sqrt{3}} A - P + 2 \geq 0$.

Proof. The sum of the interior angles of Q is 2π so one pair of diagonally opposite angles must have sum $\leq \pi$. We assume that this is the pair labelled θ and θ' in Fig. 1.

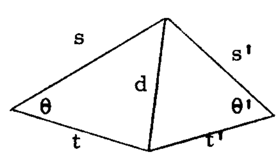

Figure 1

Now $d \geq s, t$ so θ is the largest angle in the triangle with sides labelled s, t and d. Therefore, $\theta \geq \pi/3$. Similarly $\theta' \geq \pi/3$. But $\theta + \theta' \leq \pi$, so $\pi/3 \leq \theta, \theta' \leq 2\pi/3$. It follows that

$$A = \frac{1}{2}(st \sin\theta + s't' \sin\theta') \geq \frac{\sqrt{3}}{4}(st + s't').$$

Hence,

$$\frac{4}{\sqrt{3}}A - P + 2 \geq st + s't' - (s+t+s'+t') + 2 = (s-1)(t-1) + (s'-1)(t'-1) \geq 0$$

since $s, t, s', t' \geq 1$.

Proof of Theorem. Suppose K contains only one simplex. Then that simplex is a 0-simplex and $A(K) = P(K) = 0$. The inequality (3) reduces to $\alpha_0(K) = 1 = \chi(K)$.

Now suppose that K contains more than one simplex and that the theorem holds for all complexes with fewer simplexes than K. Let $\mathcal{H}$ be the class of all complexes L in the plane such that $|L| = |K|$ and $L^0 = K^0$. Every member of $\mathcal{H}$ satisfies the hypothesis of the theorem. Furthermore, since the numbers occurring in (3) depend only on $|K|$ and K^0, to establish (3) for K it suffices to establish it for any member of $\mathcal{H}$. Henceforth we shall assume that K is chosen from $\mathcal{H}$ so that

$$\sum_{\sigma \in K^1 - K^0} m(\sigma)$$

is minimal.

Suppose K contains no 1-simplexes. Then K contains only 0-simplexes, $A(K) = P(K) = 0$, and (3) reduces to $\alpha_0(K) = \chi(K)$.

Finally, suppose K contains a 1-simplex. Let σ be a 1-simplex in K with $m(\sigma)$ as large as possible.

Case I. $\varepsilon(\sigma, K) = 0$. Then $K - \{\sigma\}$ is a complex. By the inductive assumption,

$$\alpha_0(K) = \alpha_0(K - \{\sigma\}) \leq \frac{2}{\sqrt{3}}A(K - \{\sigma\}) + \frac{1}{2}P(K - \{\sigma\}) + \chi(K - \{\sigma\})$$

$$= \frac{2}{\sqrt{3}} A(K) + \frac{1}{2}P(K) - m(\sigma) + \chi(K) + 1$$

$$\leq \frac{2}{\sqrt{3}} A(K) + \frac{1}{2}P(K) + \chi(K).$$

Case II. $\varepsilon(\sigma, K) = 1$. Let τ be the 2-simplex in K having σ as a face. Let σ' and σ'' be the other one-dimensional faces of τ, where we can assume $m(\sigma') \geq m(\sigma'')$ without loss of generality. By the hypothesis of the theorem and the choice of σ, we have

$$m(\sigma) \geq m(\sigma') \geq m(\sigma'') \geq 1. \tag{4}$$

By Lemma 1,

$$\frac{4}{\sqrt{3}} m(\tau) + m(\sigma) \geq m(\sigma') + m(\sigma'').$$

Since τ is the only 2-simplex having σ as a face, the collection $L = K - \{\sigma, \tau\}$ is a complex. By the inductive assumption and (4) we have

$$\frac{2}{\sqrt{3}} A(K) + \frac{1}{2}P(K) + \chi(K)$$

$$= \frac{2}{\sqrt{3}}A(L) + \frac{1}{2}P(L) + \chi(L) + \frac{2}{\sqrt{3}}m(\tau) + \frac{1}{2}(m(\sigma) - m(\sigma') - m(\sigma''))$$

$$\geq \frac{2}{\sqrt{3}}A(L) + \frac{1}{2}P(L) + \chi(L) \geq \alpha_0(L) = \alpha_0(K).$$

Case III. $\varepsilon(\sigma, K) = 2$. Let τ_1 and τ_2 be the 2-simplexes in K with σ as a face. We shall first show that $Q = \tau_1 \cup \tau_2$ is a convex quadrilateral satisfying the hypotheses of Lemma 2.

Let X and Y be the vertices of σ and let X, Y, Z and X, Y, W be the vertices of τ_1 and τ_2 respectively. The sides of Q are the 1-simplexes (X, Z), (X, W), (Y, Z), (Y, W) which by the hypothesis of the theorem and the choice of σ all have length ≥ 1 and $\leq m(\sigma)$. Hence, Z and W must lie in the region R shown in Fig. 2. R is bounded by two circular arcs of radius $m(\sigma)$ with centers at X and Y, and R is bisected by σ. By (2'), Z and W must lie on opposite sides of σ; hence, Q is convex.

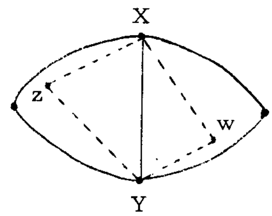

Figure 2

In order to show the hypotheses of Lemma 2 are satisfied, it remains to show that the diagonal of Q from Z to W is at least as long as any side of Q. Suppose the contrary. Then $m(Z, W) < m(\sigma)$. Let $\sigma' = (Z, W)$, $\tau_1' = (X, Z, W)$ and $\tau_2' = (Y, Z, W)$. The collection

$$L = (K - \{\sigma, \tau_1, \tau_2\}) \cup \{\sigma', \tau_1', \tau_2'\}$$

is a complex in K. But

$$\sum_{\lambda \in L^1 - L^0} m(\lambda) = \sum_{\lambda \in K^1 - K^0} m(\lambda) - m(\sigma) + m(\sigma') < \sum_{\lambda \in K^1 - K^0} m(\lambda)$$

contradicting the choice of K from the class $\mathcal{H}$. We now apply Lemma 2 to obtain

(5)

$$\frac{4}{\sqrt{3}}(m(\tau_1) + m(\tau_2)) - (m(X, Z) + m(X, W) + m(Y, Z) + m(Y\ W)) + 2 \geq 0.$$

Let $M = K - \{\sigma, \tau_1, \tau_2\}$. Then M is a complex with fewer simplexes than K. By the inductive assumption and (5) we have

$$\frac{2}{\sqrt{3}}A(K) + \frac{1}{2}P(K) + \chi(K)$$

$$= \frac{2}{\sqrt{3}}A(M) + \frac{1}{2}P(M) + \chi(M) + \frac{2}{\sqrt{3}}(m(\tau_1) + m(\tau_2))$$

$$- \frac{1}{2}(m(X, Z) + m(X, W) + m(Y, Z) + m(Y, W)) + 1$$

$$\geq \frac{2}{\sqrt{3}}A(M) + \frac{1}{2}P(M) + \chi(M) \geq \alpha_0(M) = \alpha_0(K) .$$

This completes the proof of the theorem.

To show that (3) implies (1) we argue as follows. Let X be a convex compact subset of the plane. Let S be a packing of X with card $(S) = \rho(X)$. Let Y denote the convex hull of X so $A(Y) \leq A(X)$ and $P(Y) \leq P(X)$. Let K be a complex with $K^0 = S$ and $|K| = Y$. (The existence of such a complex is easily seen, for example, by induction on the number of points in S.) Since S is a packing, K satisfies the hypotheses of the theorem. Since $|K| = Y$ is convex, $\chi(K) = 1$. By (3),

$$\rho(X) = \text{card}(S) = \alpha_0(K)$$

$$\leq \frac{2}{\sqrt{3}}A(K) + \frac{1}{2}P(K) + \chi(K)$$

$$\leq \frac{2}{\sqrt{3}}A(X) + \frac{1}{2}P(X) + 1$$

which is (1).

4. Concluding remarks. It was pointed out by Oler [2] that (1) can be used to establish the following result suggested by P. Erdös: If T_n denotes the regular 2-simplex of side n then

$$\rho(T_n) = \binom{n+2}{2} .$$

The m-dimensional analogues $(m \geq 3)$ of (3) have not yet been found. Indeed, if $T_n^{(m)}$ denotes the regular m-simplex of edge length n, it is not known that $\rho\left(T_n^{(m)}\right) = \binom{m+n}{m}$.

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