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AN IRREDUCIBILITY CRITERION FOR POLYNOMIALS OVER THE INTEGERS

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1. Introduction. If $P(x)$ is a reducible polynomial of degree $d\geqq 1$ with integer coefficients, we should not expect the sequence

$$\mathcal{S}(P) = (\cdots, P(-1), P(0), P(1), \cdots)$$

to have many noncomposite (that is, prime or unit) elements. By making this idea precise, we shall obtain an irreducibility criterion. A special case of our main result is that if $\mathcal{S}(P)$ contains p primes and u units with $p+2u>d+4$, then P is irreducible.

2. Fatness. Let $P(x)$ be any polynomial of degree $d\geqq 1$ with integer coefficients, and let u be the number of units in $\mathcal{S}(P)$. We define the *fatness* of P to be

$$f(P) = u - d,$$

and we say that P is *fat* if $f(P)>0$.

If ϵ is a unit (that is, $+1$ or -1), and if $a_1, \cdots, a_d$ are distinct integers, then the polynomial $(x-a_1)\cdots(x-a_d)+\epsilon$ has fatness at least 0. If P is fat, then clearly $\mathcal{S}(P)$ must contain units of both signs.

Note that all polynomials in the set

$$\mathfrak{I}(P) = \{\pm P(\pm x + b)\},$$

where b ranges over the integers and where all possible choices of signs are taken, have the same fatness.

3. Notation. If $P(x)$ is a polynomial, we define

$$\begin{aligned}
d &= d(P) &&= \text{degree of } P\\
p &= p(P) &&= \text{number of primes in } \mathcal{S}(P)\\
u &= u(P) &&= \text{number of units in } \mathcal{S}(P)\\
u_+ &= u_+(P) &&= \text{number of positive units in } \mathcal{S}(P)\\
u_- &= u_-(P) &&= \text{number of negative units in } \mathcal{S}(P)\\
f &= f(P) &&= \text{fatness of } P.
\end{aligned}$$

Thus $u = u_+ + u_-$, and $f = u - d$.

4. Classification of fat polynomials.

THEOREM 1. *Let* $P(x)$ *be a fat polynomial* (*with* $d \geqq 1$). *Then* $u \leqq 4$, $d \leqq 3$, $f \leqq 2$; *and one of the following holds*:

(a) $P(x) \in \mathfrak{Z}(x)$, $u_+ = 1$, $u_- = 1$, $d = 1, f = 1$

(b) $P(x) \in \mathfrak{Z}(x^2 + x - 1)$, $u_+ = 2$, $u_- = 2$, $d = 2, f = 2$

(c) $P(x) \in \mathfrak{Z}(x^3 + 2x^2 - x - 1)$, $u_\pm = 3$, $u_\mp = 1$, $d = 3, f = 1$

(d) $P(x) \in \mathfrak{Z}(2x - 1)$, $u_+ = 1$, $u_- = 1$, $d = 1, f = 1$

(e) $P(x) \in \mathfrak{Z}(2x^2 - 1)$, $u_\pm = 2$, $u_\mp = 1$, $d = 2, f = 1$.

Proof. We first prove that $u \leqq 4$. Since P is fat, we have seen that $u_+ \geqq 1$ and $u_- \geqq 1$. Clearly P may be written

$$P(x) = (x - a_1) \cdots (x - a_{u_+})Q(x) + 1,$$

where $a_1 < \cdots < a_{u_+}$. Now if $P(b) = -1$, we have $(b - a_1) \cdots (b - a_{u_+})Q(b) = -2$, $\{b - a_1, \cdots, b - a_{u_+}\} \subseteq \{-2, -1, 1, 2\}$. By the first of these relations, at least $u_+ - 1$ of the distinct integers $b - a_1, \cdots, b - a_{u_+}$ must be ± 1. Hence $1 \leqq u_+ \leqq 3$, and similarly $1 \leqq u_- \leqq 3$. If $u_+ = 3$, there is at most one integer b for which the second relation holds, so $u_- = 1$. If $u_+ = 2$, there are at most two such integers, so $u_- \leqq 2$. Thus in every case $u \leqq 4$.

Since P is fat, $d < u$, and therefore $d \leqq 3$. Since $u \leqq 4$ and $d \geqq 1$, we have $f \leqq 3$; however, we shall see that the case $f = 3$ does not occur, and therefore $f \leqq 2$.

Next we prove that $d(Q) = 0$. We may assume $u_+ \geqq u_-$, (otherwise replace P by $-P$). Since $u \leqq 4$, it follows that $u_- \leqq 2$. Since P is fat, $d(Q) < u_-$, and therefore $d(Q) = 0$ or 1. If $d(Q) = 1$, then $u_+ = u_- = 2$. Hence, for some $b_1 \neq b_2$,

$$(b_1 - a_1)(b_1 - a_2)Q(b_1) = (b_2 - a_1)(b_2 - a_2)Q(b_2) = -2,$$
$$\{b_1 - a_1, b_1 - a_2, b_2 - a_1, b_2 - a_2\} \subseteq \{-2, -1, 1, 2\}.$$

Since $\{b_1 - a_1, b_1 - a_2\}$ is a translate of $\{b_2 - a_1, b_2 - a_2\}$, it follows that $(b_1 - a_1)(b_1 - a_2) = (b_2 - a_1)(b_2 - a_2)$ and $Q(b_1) = Q(b_2)$. Hence $Q(x)$ is constant.

We now have

$$P(x) = c(x - a_1) \cdots (x - a_{u_+}) + 1.$$

Since $u_- \geqq 1$, we may assume $P(0) = -1$; that is, $(-1)^{u_+} c a_1 \cdots a_{u_+} = -2$. It follows that $|c| = 1$ or 2.

If $|c| = 1$, then $a_1 \cdots a_{u_+} = \pm 2$, so either $a_1 = -2$ or $a_{u_+} = 2$. We may assume $a_1 = -2$. (Otherwise replace $P(x)$ by $P(-x)$.) If $u_+ = 1$, then $ca_1 = 2$, $c = -1$, and $P(x) = -(x+2) + 1 = -(x+1)$, so $-P(x-1) = x$. If $u_+ = 2$, then $ca_1a_2 = -2$, $ca_2 = 1$, $a_2 = c = \pm 1$, and $P(x) = c(x+2)(x-c) + 1$. If $c = 1$, then $P(x) = x^2 + x - 1$. If $c = -1$, then $P(x) = -(x+2)(x+1) + 1$, so $-P(x-1) = x^2 + x - 1$. Finally, if $u_+ = 3$, then $ca_1a_2a_3 = 2$, $ca_2a_3 = -1$, $a_2 = -1$, $a_3 = 1$, $c = 1$, and $P(x) = x^3 + 2x^2 - x - 1$.

If $|c|=2$, then $a_1 \cdots a_{u_+}=\pm 1$, so $u_+=1$ or 2. If $u_+=1$, then $ca_1=2$, $c=\pm 2$, $a_1=\pm 1$, and $P(\pm x)=2x-1$. If $u_+=2$, then $ca_1a_2=-2$, $a_1=-1$, $a_2=1$, $c=2$, and $P(x)=2x^2-1$. This completes the proof.

COROLLARY 1. *If P is a fat polynomial with $d=1$ or 2, then there is an integer b such that $P(-x)=(-1)^d P(x-b)$.*

5. Irreducibility criterion.

THEOREM 2. *Let $P(x)$ be a polynomial with $p+2u>d\geqq 2$. Then either P is irreducible or $P=QR$ with $f(Q)+f(R)\geqq p+2u-d$.*

Proof. If P is reducible, we can write $P=QR$ with $f(Q)\geqq f(R)$. Now for each integer n such that $P(n)$ is prime, either $Q(n)$ or $R(n)$ must be a unit, while for each n such that $P(n)$ is a unit, both $Q(n)$ and $R(n)$ must be units. Therefore $u(Q)+u(R)\geqq p+2u$, and $f(Q)+f(R)\geqq p+2u-d$, as was to be shown.

COROLLARY 2: *If $p+2u>d+4$, then P is irreducible.*

6. Example. Let $P(x)=x^5-x^4+2x^3-x^2+x-1$. Then

$$\begin{aligned} P(0) &= -1 \\ P(1) &= 1 \\ P(2) &= 29 \\ P(4) &= 883 \\ P(-1) &= -7 \\ P(-2) &= -71 \\ P(-4) &= -1429. \end{aligned}$$

Thus $p\geqq 5$, $u\geqq 2$, and $p+2u-d\geqq 4$. Hence if P is reducible, we have $P=QR$ with $f(Q)=f(R)=2$. But this implies $d=4$, which is a contradiction, so P is irreducible.

If we fail to notice that $P(4)$ and $P(-4)$ are prime, then we have $p\geqq 3$, $u\geqq 2$, and $p+2u-d\geqq 2$. In this case, if P is reducible, we have $P=QR$ with $f(Q)+f(R)\geqq 2$. Thus either $f(Q)=f(R)=1$ or $f(Q)=2$. In the first case we may assume $d(Q)=2$, and therefore $Q\in\mathfrak{Z}(2x^2-1)$. But this is impossible because P is monic. Therefore $f(Q)=2$, and $Q\in\mathfrak{Z}(x^2+x-1)$. Now by Corollary 1 we have $Q(x)=(x-b)^2+(x-b)-1$, and so x^2+x-1 divides $P(x+b)$. However the remainder of $P(x+b)$ modulo x^2+x-1 is $R_1(b)+xR_2(b)$, where

$$R_1(b) = b^5 - b^4 + 12b^3 - 17b^2 + 21b - 9$$

$$R_2(b) = 5b^4 - 14b^3 + 32b^2 - 31b + 14.$$

Since R_1 and R_2 have no common integer root, the remainder cannot vanish for any integer b. This contradiction proves that P is irreducible.

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