RAMSEY'S THEOREM FOR n-DIMENSIONAL ARRAYS

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Communicated by Gian-Carlo Rota, November 5, 1968

Introduction. An analogue to a theorem of Ramsey [5] has been conjectured for finite vector spaces by Gian-Carlo Rota. Namely, for each choice of positive integers k, l, r, and finite field $F=GF(q)$, there exists an integer $N(k, l, r; q)$ such that if $n \geqq N(k, l, r; q)$ and the k-dimensional subspaces of an n-dimensional vector space V over F are partitioned into r classes, then some l-dimensional subspace of V has all of its k-dimensional subspaces in one class. In this note we present a very general theorem of this type, a brief outline of its proof, and general applications, including some cases of Rota's Conjecture. Complete details will appear elsewhere.

Notation. Let $A=\{a_1, \cdots, a_t\}$ be a finite set with $t>1$ and let $H_p: A \rightarrow A$ be a permutation group on A. Define $H_c=\{\sigma_a: a \in A\}$ to be the set of maps of A into A given by $x^{\sigma_a}=a$ for all $x \in A$. H will denote $H_c \cup H_p$. We can define an action of H on A^t by $(x_1, \cdots, x_t)^\sigma = (x_1^\sigma, \cdots, x_t^\sigma)$ for $x_i \in A$, $\sigma \in H$. Let l_0 denote $(a_1, \cdots, a_t) \in A^t$ and let $L_c=\{l_0^\sigma: \sigma \in H_c\}$, $L_p=\{l_0^\sigma: \sigma \in H_p\}$, $L=L_c \cup L_p$. We introduce the basic concept of a *k-parameter set*. For fixed nonnegative integers $k \leqq n$, let $\Pi=\{S_0, S_1, \cdots, S_k\}$ be a partition of the set $I_n = \{1, 2, \cdots, n\}$ with $S_i \neq \emptyset$ for $1 \leqq i \leqq k$. $S_0=\emptyset$ is possible. Let $f: I_n \rightarrow H$ have the property

$$f(i) \in H_c \text{ if } i \in S_0,$$

$$f(i) \in H_p \text{ otherwise.}$$

The set $P(\Pi, f)$ is defined by

$$P(\Pi, f) = \bigcup_{1 \leq i_0, i_1, \cdots, i_k \leq t} \{(x_1, \cdots, x_n); \quad x_j = a_{i_y}^{f(j)} \text{ if } j \in S_y\} \subseteq A^n.$$

Note that since $f(j) \in H_c$ for $j \in S_0$, $P(\Pi, f)$ consists of exactly t^k elements of A^n.

DEFINITION. P_k is *k-parameter set* of A^n if and only if $P_k=P(\Pi, f)$ for some partition Π and mapping f. Of course, we say that P_k is a *k-parameter subset* of the l-parameter set $P_l \subseteq A^n$ if $P_k \subseteq P_l$ and P_k is a k-parameter set of A^n.

The main results.

THEOREM 1. *For each choice of positive integers k, l, r there exists an* *integer $M(k, l, r)$ such that if $m \geqq M(k, l, r)$ and the k-parameter subsets of an m-parameter set $P_m \subseteq A^n$ are partitioned into r classes, then there exists an l-parameter subset $P_l \subseteq P_m$ such that all k-parameter subsets of P_l belong to the same class.*

Let us call a k-parameter set $P_k \subseteq A^n$ *normalized* if $f(j) = \sigma_{a_1}$ for all $j \in S_0$. We state the important

THEOREM 2. *The preceding theorem is valid if all parameter sets are required to be normalized.*

Before proceeding to the proof outline, we list several immediate corollaries to the theorems.

COROLLARY 1. *Given integers k and r, there exists an integer $N(k, r)$ such that if $|A| \geqq N(k, r)$ and the finite subsets of A are partitioned into r classes then there exist k disjoint nonempty subsets $A_1, \cdots, A_k$ of A such that all $2^k - 1$ unions $\bigcup_{j \in J} A_j$, $\emptyset \neq J \subseteq \{1, 2, \cdots, k\} = I_k$, are in the same class.*

This follows from Theorem 2, taking $A = \{0, 1\}$ and $H_p = \{e\}$.

COROLLARY 2 (J. FOLKMAN, J. SANDERS [6]). *Given integers k and r, there exists an integer $N(k, r)$ such that if $n \geqq N(k, r)$ and the set I_n is partitioned into r classes, then there exist k integers $a_1, \cdots, a_k$ such that all sums $\{\sum_{i=1}^{k} \epsilon_i a_i$: $\epsilon_i = 0$ or 1, not all $\epsilon_i = 0\}$ are in the same class.*

This follows for Corollary 1 by interpreting the characteristic function of A_i as the dyadic expansion of an integer a_i. For $k = 2$, Corollary 2 was first proved by Schur [7]. Schur's result can also be stated as follows:

Given r, there exists an integer $N(r)$ such that if $n \geqq N(r)$ and the set I_n is partitioned into two classes, then the equation $x + y = z$ can be solved in one class. This is also a special case of

COROLLARY 3. *Let $\mathfrak{L} = L_i(x_1, \cdots, x_m)$, $1 \leqq i \leqq n$ be a system of homogeneous linear equations with the property that for each j, $1 \leqq j \leqq m$, there exists a solution $(\epsilon_1, \epsilon_2, \cdots, \epsilon_m)$ to the system $\mathfrak{L}$ with $\epsilon_i = 0$ or 1 and $\epsilon_j = 1$. Then given an integer r there exists an integer $N(r)$ such that if $n \geqq N(r)$ and the set I_n is partitioned into r classes, then $\mathfrak{L}$ can be solved in one class.*

This is similar to a result of R. Rado [3].

COROLLARY 4 (VAN DER WAERDEN [2]). *Given integers k and r there exists an integer $N(k, r)$ such that if $n \geqq N(k, r)$ and the set I_n is partitioned into r classes, then at least one class contains an arithmetic progression of length k.*

This result is implied by the stronger

COROLLARY 5 (HALES-JEWETT [1]). *Let* $A=\{a_1, \cdots, a_t\}$ *be a finite set. Given an integer* r *there exists an integer* $N(r, t)$ *such that if* $n \geqq N(r, t)$ *and the set* A^n *is partitioned into* r *classes, then there exists a set of* t *elements of the form*

$$X_i = (x_{11}, \cdots, x_{1u}, a_i, x_{21}, \cdots, x_{2v}, a_i, \cdots, a_i, x_{d1}, \cdots, x_{ds}) \in A^n, \quad 1 \leqq i \leqq t,$$

all of which belong to one class.

This follows from Theorem 1 by taking $A=\{a_1, \cdots, a_t\}$, $k=0$, $l=1$, $H_p=\{e\}$.

COROLLARY 6. *Given integers* l *and* r *and a finite field* $GF(q)$ *there exists an integer* $N(l, r, q)$ *such that if* $n \geqq N(l, r, q)$ *and the* 1-*dimensional subspaces of an* n-*dimensional vector space* V *over* $GF(q)$ *are partitioned into* r *classes, then* V *contains an* l-*dimensional subspace* V' *all of whose* 1-*dimensional subspaces are in one class.*

This follows from Theorem 2 by taking $A=GF(q)$, $H_p=$ mult. group of $GF(q)$, and $k=0$. The corresponding result for *affine* spaces over $GF(q)$ follows from Theorem 1. Corollary 6 was first proved for $q=2$ by D. Kleitman (unpublished) and $q=3, 4$ by B. L. Rothschild [**4**]. From the result for 1-dimensional *affine* subspaces, techniques of Rothschild [**4**] can be used to prove the result corresponding to Corollary 6 when 1-*dimensional* subspace is replaced by 2-*dimensional* subspace. It was conjectured by G.-C. Rota that Corollary 6 holds for k-dimensional subspaces in general.

Finally, as a more powerful application, let C^n denote an n-dimensional cube in E^n. Let us say that a set S_k of 2^k vertices of C^n forms a k-*subspace* of C^n if S_k is contained in some k-dimensional euclidean subspace of E^n.

COROLLARY 7. *Given integers* k, l, r *there exists an integer* $N(k, l, r)$ *such that if* $n \geqq N(k, l, r)$ *and the* k-*subspaces of* C^n *are partitioned into* r *classes, then there exists an* l-*subspace of* C^n *all of whose* k-*subspaces are in one class.*

BRIEF OUTLINE OF PROOF OF THEOREM 1. Let $S(k; t_1, \cdots, t_r)$ denote the statement:

There exists an integer $M(k; t_1, \cdots, t_r)$ such that if $m \geqq M(k; t_1, \cdots, t_r)$ and the k-parameter subsets of an m-parameter set P_m are partitioned into r classes $C_1, C_2, \cdots, C_r$, then there exists an t, $1 \leq t \leq r$ and an t-parameter subset P_{tt} of P_m such that all the k-parameter subsets of P_{tt} belong to class C_t.

We prove $S(k; t_1, \cdots, t_r)$ by multiple induction on k and $t_1+t_2+\cdots+t_r$. We can assume $0 \leq k$, $r \geq 1$ and $t_i \geq 1$ for all i. The first step in the induction is $S(0; t_1, \cdots, t_r)$. Once certain notational difficulties have been overcome, the proof of this statement is relatively straightforward. We assume $S(i; t_1, \cdots, t_r)$ has been established for $0 \leq i < k$ and all t_i. Since $S(k; t_1, \cdots, t_r)$ is certainly valid if $t_1+t_2+\cdots+t_r \leq rk$, we further assume that for some $t > rk$, $S(k; t_1, \cdots, t_r)$ is valid for all choices of t_i with $t_1+\cdots+t_r < t$.

A critical step in the proof rests on the following fact. It is possible to define a map $M: L^n \to 2^{A^n}$ such that for each l-parameter set $P_l \subseteq A^n$ there exists an $(l-1)$-parameter set $P^*_{l-1} \subseteq L^n$ with $M(P^*_{l-1}) = P_l$ such that for "certain" k-parameter subsets $P_k \subseteq P_l$, there exists a $(k-1)$-parameter subset $P^*_{k-1} \subseteq P^*_{l-1}$ which makes the following diagram commutative:

$$\begin{array}{ccc} P^*_{k-1} & \subseteq & P^*_{l-1} \\ M \downarrow & & \downarrow M \\ P_k & \subseteq & P_l \end{array}$$

Thus, the original partition of the k-parameter sets P_k into r classes induces a partition of $(k-1)$-parameter sets P^*_{k-1} to which we can apply the induction hypothesis. It turns out that the "remaining" k-parameter sets can be naturally embedded in a large parameter set to which we can again apply the preceding argument. After a large number of iterations of this procedure, we are left with a configuration of blocks of "remaining" k-parameter sets which in a certain sense is isomorphic to a large parameter set in which the blocks are identified with points. By then partitioning these point-blocks according to the way in which the corresponding constituent k-parameter subsets have been partitioned and applying $S(0; t_1', \cdots, t_r')$ for suitable $t_1', \cdots, t_r'$ we can extract a configuration of k-parameter sets from which the induction step can be completed fairly directly. Theorem 2 follows from Theorem 1 with little difficulty. As might be expected, the bounds provided on $M(k, l, r)$ by this proof are extremely large.

References

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