On Finite 0-Simple Semigroups and Graph Theory

by

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1. Introduction

This paper is concerned with the study of a certain basic type of finite semigroup, known as a 0-simple semigroup. In particular, the structure of several important classes of subsemigroups of 0-simple semigroups will be determined and a new canonical form for 0-simple semigroups will be found (cf. Theorems 1′, 2, 3, 4). The basic technique employed involves the transformation of the algebraic problems into equivalent graph-theoretic problems by means of a natural correspondence between subsets of the semigroup and subgraphs of a certain directed bipartite graph.

The remainder of this section is intended to be a brief review of some standard elementary facts from the theory of semigroups which will be used in the remainder of the paper. Proofs and much fuller discussions of the following facts may be found in Clifford and Preston [1]. The reader who is familiar with these may proceed to Section 2.

A *semigroup* is an ordered pair $(S, \circ)$ where S is a nonempty set and $\circ$ is an associative binary operation, i.e., a function mapping $S \times S$ into S by $(s_1, s_2) \to s_1 \circ s_2$ such that for all $s_1, s_2, s_3 \in S$, $s_1 \circ (s_2 \circ s_3) = (s_1 \circ s_2) \circ s_3$. We shall usually just say that S is the semigroup. An element $x \in S$ is said to be a *zero* of S if $xs = x = sx$ for all $s \in S$. If S has a zero then it is unique and will be denoted by 0.

We say that two semigroups S_1 and S_2 are *isomorphic* if there is a map φ: $S_1 \to S_2$ which is one-to-one and onto such that $\varphi(s_1 \circ s_2) = \varphi(s_1) \circ \varphi(s_2)$ for all $s_1, s_2 \in S_1$. A subset $T \subseteq S$ is said to be a *subsemigroup* of S if $t_1 \circ t_2 \in T$ for all $t_1, t_2 \in T$. A nonempty subset $I \subseteq S$ is said to be an *ideal* of S if for all $i \in I$ and $s \in S$, $s \circ i \in I$ and $i \circ s \in I$. If $A \circ B$ denotes $\{a \circ b: a \in A, b \in B\}$, then I is an ideal of S if and only if $S \circ I \subseteq S$ and $I \circ S \subseteq S$. I is a *left* or *right* ideal if the first or second respectively of these conditions holds. For $s \in S$, $L(s) = S \circ s \cup \{s\}$, $R(s) = s \circ S \cup \{s\}$ and $J(s) = S \circ s \circ S \cup S \circ s \cup s \circ S \cup \{s\}$ are respectively the *principal left ideal, principal right ideal,* and *principal ideal* generated by s. For $s_1, s_2 \in S$, we say that s_1 and s_2 are $\mathscr{J}$*-equivalent*, $\mathscr{L}$*-equivalent* or $\mathscr{R}$*-equivalent*, if $J(s_1) = J(s_2)$, $L(s_1) = L(s_2)$ or $R(s_1) = R(s_2)$, respectively.

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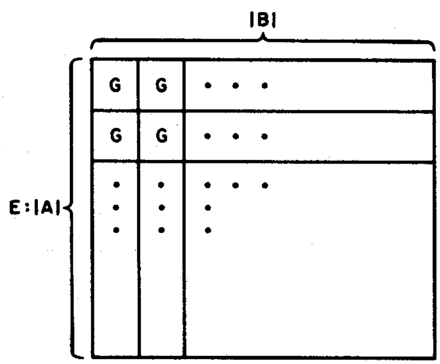
Also we say that s_1 and s_2 are *$\mathscr{H}$-equivalent* if $L(s_1)=L(s_2)$ and $R(s_1)=R(s_2)$. It is easily seen that $\mathscr{J}$, $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{H}$ define equivalence relations on S and consequently S is partitioned into equivalence classes called respectively the $\mathscr{J}$-classes, $\mathscr{L}$-classes, $\mathscr{R}$-classes and $\mathscr{H}$-classes of S.

Let A and B be nonempty sets, G a group and $C: B \times A \to G \cup \{0\}$. Then $M^0(A, B, G, C)$, *the Rees* $|A| \times |B|$ *matrix semigroup with structure group* G *and structure matrix* C, is the semigroup $(G \times A \times B, \circ)$ [or $(G \times A \times B \cup \{0\}, \circ)$ if $0 \in C(B \times A)$], where

$$(g, a, b) \circ (g', a', b') = \begin{cases} (gC(b, a')g', a, b') & \text{if } C(b, a') \neq 0 \\ 0 & \text{if } C(b, a') = 0. \end{cases}$$

It is sometimes convenient to regard (g, a, b) as an $|A| \times |B|$ matrix with g in the (a, b) position and 0 elsewhere. C is then a $|B| \times |A|$ matrix over $G \cup \{0\}$ and $(g, a, b) \circ (g', a', b') = (g, a, b)C(g', a', b')$, where the multiplication indicated is ordinary matrix multiplication and the indicated addition of group elements is determined by the rule $x + 0 = x = 0 + x$ for all $x \in G \cup \{0\}$.

The nonzero elements of $M^0(A, B, G, C)$ can be conveniently arranged into an $|A| \times |B|$ rectangular array known as the "eggbox" picture E of $M^0(A, B, G, C)$:



The rows of E are labelled by the elements of A and the columns of E are labelled by the elements of B. In each box of E is a copy of G. Hence, in the (a, b) box of E lie all the elements (g, a, b), $g \in G$, of $M^0(A, B, G, C)$. It is not hard to check that the nontrivial $\mathscr{L}$-classes, $\mathscr{R}$-classes and $\mathscr{H}$-classes of $M^0(A, B, G, C)$ are just the columns, rows and boxes of E respectively.

Suppose we have a map of $A \cup B \to G$ given by $a \to g_a$, $b \to g_b$. If C': $B \times A \to G \cup \{0\}$ satisfies $C'(b, a) = g_b C(b, a) g_a$ for all $a \in A$, $b \in B$, then the semigroup $M^0(A, B, G, C')$ is isomorphic to $M^0(A, B, G, C)$, the isomorphism being given by

$$(g, a, b) \leftrightarrow (g_a g g_b, a, b).$$

The structure matrix C of $M^0(A, B, G, C)$ is said to be *regular* (and $M^0(A, B, G, C)$ is a *regular* Rees matrix semigroup) if every row and column of C contains a nonzero entry.

A semigroup S is called *simple* if S has no proper ideals. S is called *0-simple* if $0 \in S, S \circ S \neq \{0\}$ and $\{0\}$ is the only proper ideal of S. A fundamental fact due to Rees [4] in the *finite* case is the following result.

THEOREM. *A finite semigroup S is 0-simple if and only if S is isomorphic to a regular Rees matrix semigroup.*

The importance of 0-simple semigroups in the study of finite semigroups comes from the following fact. For any subset X of an arbitrary finite semigroup $(S, \circ)$ with a zero, we define a multiplication $\circ'$ on the set $X^0 = X \cup \{0\}$ by

$$x_1 \circ' x_2 = \begin{cases} x_1 \circ x_2 & \text{if } x_1 \circ x_2 \in X \\ 0 & \text{if } x_1 \circ x_2 \notin X. \end{cases}$$

Then for any $\mathcal{J}$-class J of S, J^0 is either a *null* semigroup (i.e., $J \circ' J = \{0\}$) or J^0 is a *0-simple* semigroup.

2. The Construction of the Graph $\mathcal{G}(X)$

Let S denote an arbitrary finite 0-simple semigroup. By the Rees theorem, S is isomorphic to a regular Rees matrix semigroup $M^0 = M^0(A, B, G, C)$, where A and B are finite indexing sets, G is a finite group and C is a $|B| \times |A|$ structure matrix with entries $C(b, a) \in G \cup \{0\}$.

Definition 1. Let $X \subseteq G \times A \times B = M$. If $(a, b) \in A \times B$ we set

$$X_{a,b} \equiv \{g \in G: (g, a, b) \in X\}.$$

We shall associate to X a certain directed graph $\mathcal{G}(X)$, and a function f_X mapping $A \times B \cup B \times A$ into 2^G.

Definition 2. Let $X \subseteq M$. The graph of X, $\mathcal{G}(X)$, is the directed graph with vertices $A \cup B$ and edges

$$E(X) = \{(a, b) \in A \times B: X_{a,b} \neq \varnothing\} \cup \{(b, a) \in B \times A: C(b, a) \neq 0\}.$$

The mapping $f_X: A \times B \cup B \times A \to 2^G$ is given by

$$f_X(e) \equiv \begin{cases} X_{a,b} & \text{if } e = (a, b) \in E(X) \\ \{C(b, a)\} & \text{if } e = (b, a) \in E(X) \\ \varnothing & \text{otherwise.} \end{cases}$$

Note that f_X determines X and C.

Definition 3. *A path in $\mathcal{G}(X)$ from x to y* is a sequence $P(x, y) = (e_1, e_2, \cdots, e_r)$ with $e_i \in E(X)$ such that $e_i = (x_i, x_{i+1})$, $x_i \in A \cup B$, $1 \leqslant i \leqslant r$, with $x_1 = x$ and

$x_{r+1} = y$. If $P(x, y) = (e_1, \cdots, e_r)$ is a path in $\mathcal{G}(X)$ from x to y, we define the *path product* $\pi_X P(x, y)$ by

$$\pi_X P(x, y) \equiv \prod_{k=1}^{r} f_X(e_k),$$

where for $J, K \subseteq G$, JK denotes, as usual, the set $\{jk: j \in J, k \in K\}$.

A basic fact which will be of use is the following lemma.

LEMMA 1. *Suppose that $X \subseteq M$. Then $X \cup \{0\}$ is a subsemigroup of M^0 if and only if for all $a \in A$, $b \in B$, and for all paths $P(a, b)$ in $\mathcal{G}(X)$ from a to b,*

$$\pi_X P(a, b) \subseteq X_{a,b}.$$

Proof. We note that $X \cup \{0\}$ is a subsemigroup of M^0 if and only if $(X \cup \{0\}) \circ (X \cup \{0\}) \subseteq X \cup \{0\}$ (where we denote the semigroup multiplication by $\circ$). Assume that $X \cup \{0\}$ is a subsemigroup of M^0 and let $P(a, b) = (e_1, \cdots, e_r)$ be a path in $\mathcal{G}(X)$ from $a \in A$ to $b \in B$ of length 3. Then $e_1 = (a, b')$, $e_2 = (b', a')$ and $e_3 = (a', b)$ for some $a' \in A$, $b' \in B$. For any (g, a, b'), $(g', a', b) \in X$,

$$(g, a, b') \circ (g', a', b) = (gC(b', a')g', a, b) \in X.$$

If this product is nonzero, i.e., if $C(b', a') \neq 0$, then $gC(b', a')g' \in X_{a,b}$ and hence

$$\begin{aligned} X_{a,b'}C(b', a')X_{a',b} &= f_X(e_1)f_X(e_2)f_X(e_3) \\ &= \pi_X P(a, b) \subseteq X_{a,b}. \end{aligned}$$

The same argument can be applied to paths from a to b of lengths $\neq 3$.

On the other hand, if $\pi_X P(a, b) \subseteq X_{a,b}$ for all $a \in A$, $b \in B$, and for all paths $P(a, b)$ in $\mathcal{G}(X)$ from a to b of length 3, then $(X \cup \{0\}) \circ (X \cup \{0\}) \subseteq X \cup \{0\}$, which implies that $X \cup \{0\}$ is a subsemigroup of M^0. This proves the lemma.

It follows from Lemma 1 that if $X \cup \{0\}$ is a subsemigroup and $P(a, b)$ is a path in $\mathcal{G}(X)$ from a to b, then $(a, b) \in E(X)$. We also note that if $X = M$ then $E(X) = A \times B \cup B \times A$ and $f_X(e) = G$ for all $e \in E(X)$.

3. The Maximal Nilpotent Subsemigroups of M^0

A subsemigroup $T \subseteq M^0$ is said to be *nilpotent* if there exists an integer r such that, for an r-fold product, $T \circ T \circ \cdots \circ T = T^r = \{0\}$. In this section we determine the structure of the maximal nilpotent subsemigroups of M^0.

The translation of the problem into graph-theoretic terms is given by the following lemma.

LEMMA 2. $X \cup \{0\} \subseteq M^0$ *is nilpotent if and only if $\mathcal{G}(X)$ contains no cycles.*[1]

[1] By *cycle* we mean *directed* cycle.

Proof. Let $(e_1, \cdots, e_{2m})$ be a cycle in $\mathscr{G}(X)$. Then for some $a_i \in A$, $b_i \in B$,

$$e_{2i-1} = (a_i, b_i), \qquad e_{2i} = (b_i, a_{i+1}), \qquad 1 \leq i \leq m,$$

where $a_1 = a_{2m+1}$ and no generality is lost in assuming that $e_1 = (a_1, b_1)$. By the construction of $\mathscr{G}(X)$, $X_{a_i,b_i} \neq \varnothing$ and $C(b_i, a_{i+1}) \neq 0$. Hence, for $g_i \in X_{a_i,b_i}$ we can form arbitrarily long nonzero products of the form

$$(g_1, a_1, b_1) \circ (g_2, a_2, b_2) \circ \cdots \circ (g_m, a_m, b_m) \circ (g_1, a_1, b_1) \circ \cdots,$$

which implies that $X \cup \{0\}$ is not nilpotent.

Conversely, suppose there are no cycles in $\mathscr{G}(X)$. If s denotes the length of the longest path in $\mathscr{G}(X)$, then it is immediate that any product $x_1 \circ x_2 \circ \cdots \circ x_t$ in $X \cup \{0\}$ with $t > s/2$ must be 0. This prove the lemma.

We should note that if $X \cup \{0\}$ is a *maximal* nilpotent subsemigroup of M^0, then for any $a \in A$, $b \in B$, *either* there is path in $\mathscr{G}(X)$ from b to a *or* $(a, b) \in E(X)$, since if there is no path from b to a then the addition of the edge (a, b) to $E(X)$ cannot create any new cycles. It also follows from the maximality of $X \cup \{0\}$ that if $X_{a,b} \neq \varnothing$ then $X_{a,b} = G$. In other words, X is the union of $\mathscr{H}$-classes of M^0. Thus, for $e \in E(X)$ we have $f_X(e) = G$.

Definition 4. We call $\mathscr{G}(X)$ a *maximal tree* if $\mathscr{G}(X)$ contains no cycles but the addition of any new edge of the form (a, b) to $E(X)$ creates a cycle.

It is important to note that if $\mathscr{G}(X)$ is a maximal tree, then $X \cup \{0\}$ is automatically a subsemigroup of M^0. To see this, assume that $X \cup \{0\}$ is not a subsemigroup and (by Lemma 1) let $P(a, b)$ be a path in $\mathscr{G}(X)$ from $a \in A$ to $b \in B$ such that $\pi_X P(a, b) \not\subseteq X_{a,b}$. Then $X_{a,b} \neq G$ and by what has been said above we must have $X_{a,b} = \varnothing$, i.e., $(a, b) \notin E(X)$. But the addition of (a, b) to $E(X)$ cannot create any *new* cycles since there is already a path $P(a, b)$ in $\mathscr{G}(X)$ from a to b. Hence, $\mathscr{G}(X)$ is not a maximal tree and the assertion is proved. These results imply the following lemma.

LEMMA 3. $X \cup \{0\}$ *is a maximal nilpotent subsemigroup of* M^0 *if and only if* $\mathscr{G}(X)$ *is a maximal tree.*

We should note here that since M^0 is regular, any maximal nilpotent subsemigroup of M^0 contains 0. The main result of this section can now be stated.

THEOREM 1. *Let* $\mathscr{G}(X)$ *be a maximal tree. Then there exist unique ordered partitions*

$$A = A_1 + A_2 + \cdots + A_r, \qquad B = B_1 + B_2 + \cdots + B_r,$$

such that (i) $B_i \times A_i \subseteq E(X)$, $1 \leq i \leq r$, and (ii) $A_i \times B_j \subseteq E(X)$, $1 \leq i < j \leq r$.

Proof. The uniqueness is immediate. To show existence, first define $C_0 \subseteq B$ by

$$C_0 \equiv \{b \in B\colon (a, b) \notin E(X) \text{ for any } a \in A\}.$$

Since M^0 is regular and $\mathscr{G}(X)$ contains no cycles, we have $C_0 \neq \varnothing$. Next, for $v \in A \cup B$, define $L(v)$ to be the length of the longest path in $\mathscr{G}(X)$ from an element of C_0 to v (where we adopt the convention that $L(v) = 0$ for $v \in C_0$). By a previous remark, $L(v)$ is well-defined for all $v \in A \cup B$. Finally, for $k \geq 1$, define C_k by

$$C_k \equiv \{v \in A \cup B : L(v) = k\}.$$

Note that

$$C_k \subseteq \begin{cases} B & \text{if } k \text{ is odd} \\ A & \text{if } k \text{ is even} \end{cases}$$

and $C_{k+1} \neq \varnothing$ implies that $C_k \neq \varnothing$. Also, from the regularity of M^0, it follows that $C_{2k} \neq \varnothing$ implies that $C_{2k+1} \neq \varnothing$. We now show that if $u \in C_k$ and $v \in C_{k+1}$ then $(u, v) \in E(X)$. There are two cases:

(i) *k is even.* There can be no path $P(v, u)$ in $\mathcal{G}(X)$ since if there were, then we would have $k + 1 = L(v) < L(u) = k$. Hence, $(v, u) \notin E(X)$ and $(v, u) \in A \times B$. Thus, there must be a path $P(u, v)$ in $\mathcal{G}(X)$. However, $L(u) = k$ and $L(v) = k + 1$ imply that the length of $P(u, v)$ is 1, i.e., $(u, v) \in E(X)$.

(ii) *k is odd.* As before, since (v, u) cannot be in $E(X)$ and $(v, u) \in B \times A$, we must have $(u, v) \in E(X)$ and the assertion is proved. Thus, if r is the largest integer such that $C_{2r-1} \neq \varnothing$, then by choosing $A_i = C_{2i-1}, B_i = C_{2i-2}$, $1 \leq i \leq r$, we have partitions of A and B which satisfy (i) and (ii) (where (ii) follows from the inclusion $A_i \times B_{i+1} \subseteq E(X)$). This proves the theorem.

The preceding results can be summarized in the following theorem.

THEOREM 1′. *Let $X \cup \{0\}$ be a maximal nilpotent subsemigroup of M^0. Then there exist unique ordered partitions of A and B, $A = A_1 + \cdots + A_r$, $B = B_1 + \cdots + B_r$, such that*

$$X = \bigcup_{\substack{(a,b) \in A_i \times B_j \\ 1 \leq i < j \leq r}} H_{a,b}.$$

Consequently, by a suitable permutation of the rows and columns of the "eggbox" of the $\mathcal{H}$-classes of M^0, the $\mathcal{H}$-classes $H_{a,b}$, $(a, b) \in A_i \times B_i$, for $1 \leq i \leq r$, form a "generalized diagonal" and X consists of all the $\mathcal{H}$-classes which lie above the "diagonal" (cf. Fig. 1).

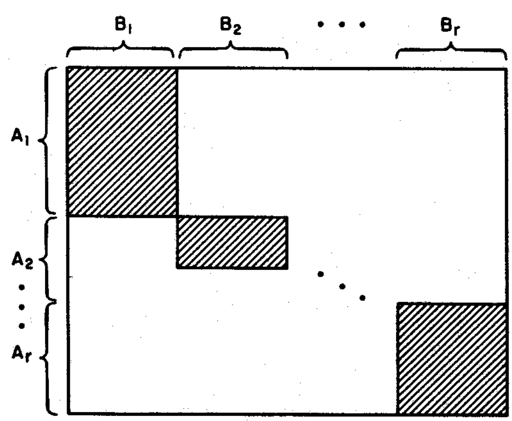
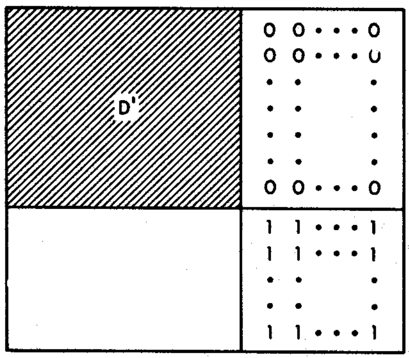


Figure 1

We give a simple algorithm which can be used to generate all the maximal nilpotent subsemigroups $X \cup \{0\}$ of M^0 in the above canonical form. This is accomplished as follows. (i) Form the transpose of the structure matrix C and replace each nonzero entry by 1, forming the binary $|A| \times |B|$ matrix $D = (d_{ij})$. The set of columns $D_j = (d_{ij})_{1 \leq i \leq |A|}$ of D can be *partially ordered* by $\leq$, where $D_j \leq D_{j'}$ if and only if $d_{ij} \leq d_{ij'}$ for all i. (ii) Choose a column D_j which is *minimal* with respect to the partial order $\leq$. Transpose D_j and all other columns identical to it to the right-hand side of D. Next, permute the rows of D so that all the 1's of D_j go into the bottom-most rows of the matrix. D now has the following form:

D:



(iii) Apply (ii) to the submatrix D'. By the assumption that M^0 is regular, this process will continue until the matrix assumes the form given in Fig. 1. The union of the $\mathscr{H}$-classes $H_{a,b}$ corresponding to the positions (a, b) of the 0's above the "diagonal" of the blocks of 1's together with $\{0\}$ form a maximal nilpotent subsemigroup $X \cup \{0\}$ of M^0. The details of the proof that this algorithm generates exactly the subsemigroups described in Theorem 1′ are omitted. We point out that the same sets of $\mathscr{H}$-classes are obtained if the algorithm is performed on the *rows* instead of the columns of the matrix.

An immediate consequence of these results is the following.

COROLLARY 1. *M^0 has a unique maximal nilpotent subsemigroup if and only if $\leq$ is a linear order on the columns (rows) of D.*

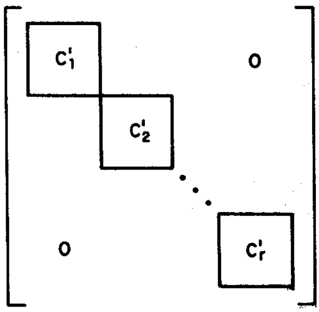
4. The Structure of the Subsemigroup Generated by the Idempotents of M^0; A Canonical Form for C

An element $x \in M^0$ is said to *idempotent* if $x \circ x = x$. The set N of idempotents of M^0 is exactly

$$\{(C(b, a)^{-1}, a, b) \colon C(b, a) \neq 0\} \cup \{0\}.$$

In this section we determine the structure of I^0, the subsemigroup generated by N. Our first goal will be to establish the following result.

THEOREM 2. *Let $M^0(A, B, G, C)$ be a regular Rees matrix semigroup. It is always possible to normalize*[2] *the structure matrix C to obtain a matrix C' with the following properties: (i) C' has the block form:*



where $C_i' = (C'(b, a))_{b\in B_i, a\in A_i}$. (ii) Let I^0 denote the subsemigroup generated by the idempotents of $M^0(A, B, G, C)$. The nonzero augmented $\mathcal{J}$-classes I_i^0 of I^0 (i.e., the classes such that I_i is a $\mathcal{J}$-class) are exactly the Rees matrix semigroups $M^0(A_i, B_i, G_i, C_i')$, where G_i is the subgroup of G generated by the nonzero entries of C_i'.

Proof. To prove the theorem it suffices to answer the following equivalent graph-theoretic question: What is the *minimal* function $f\colon A \times B \cup B \times A \to G$ such that (i) $f((a, b)) \supseteq \pi_N P(a, b)$ for any path $P(a, b)$ in $\mathcal{G}(N)$, (ii) $f((b, a)) = f_N((b, a)) = C(b, a)$ if $C(b, a) \neq 0$. By minimal we mean, of course, that if $g\colon A \times B \cup B \times A \to G$ is any function satisfying (i) and (ii), then $f(x) \subseteq g(x)$ for all $x \in A \times B \cup B \times A$. f determines a subset $I \subseteq M$ by the condition $f = f_I$ and I^0 is exactly the subsemigroup generated by the idempotents N.

It is apparent that the minimal f satisfying (i) actually satisfies (i)' $f((a, b)) = \bigcup_{P(a,b)} \pi_N P(a, b)$ taken over all paths $P(a, b)$ in $\mathcal{G}(N)$.

Definition 5. We say that two vertices $x, y \in A \cup B$ are in the same *component* of $\mathcal{G}(N)$ if and only if there is a path $P(x, y)$ in $\mathcal{G}(N)$ from x to y. Since $(a, b) \in E(N)$ if and only if $(b, a) \in E(N)$, there is a path $P(x, y)$ in $\mathcal{G}(N)$ if and only if there is a path $P(y, x)$ in $\mathcal{G}(N)$. Thus the components of $\mathcal{G}(N)$ are well-defined.

It follows from (i)' that if x and y are in the same component of $\mathcal{G}(I)$, then they are in the same component of $\mathcal{G}(N)$. Hence, the components of $\mathcal{G}(I)$ and $\mathcal{G}(N)$ each contain the same sets of vertices. Let us denote the components of $\mathcal{G}(N)$ and $\mathcal{G}(I)$ by $\mathcal{G}(N_i)$ and $\mathcal{G}(I_i)$, respectively, $1 \leq i \leq r$, and let A_i and B_i denote the vertices of $\mathcal{G}(I_i)$ in A and B, respectively. From the

[2] I.e., permute rows and columns and make transformations of the type $C(b, a) \to g_b C(b, a) g_a$.

regularity of M^0 it follows that $A_i \neq \varnothing \neq B_i$. Of course, I is correspondingly partitioned into "components" I_i, $1 \leqslant i \leqslant r$, in the obvious way.

It is important to note that if (g, a, b) and (g', a', b') are elements of I which lie in *different* components, then $(g, a, b) \circ (g', a', b') = 0$. This follows from (i)′ and the fact that there are *no* paths $P(a, b')$ in $\mathscr{G}(I)$ from a to b'. Hence, in order to analyze I, it is sufficient to determine the structure of each component I_i.

Consider an arbitrary fixed component $\mathscr{G}(I_i)$ of $\mathscr{G}(I)$. For any $(a, b) \in A_i \times B_i$, it follows from (i)′ that $(I_i)_{a,b} \neq \varnothing$ since $\mathscr{G}(I_i)$ is a component of $\mathscr{G}(I)$. Consequently, for any $(a', b') \in A_i \times B_i$, there are paths $P(a, a')$, $P(a', b')$ and $P(b', b)$ in $\mathscr{G}(I_i)$. Thus, there is a path $P(a, b)$ in $\mathscr{G}(I_i)$ which includes the edge (a', b'), and, by (i)′, we have $|f_{I_i}((a, b))| \geqslant |f_{I_i}((a', b'))|$. By symmetry we find that $|f_{I_i}((a, b))| = |f_{I_i}((a', b'))|$ for all (a, b), $(a', b') \in A_i \times B_i$ or equivalently that $|(I_i)_{a,b}| = |(I_i)_{a',b'}|$. Since $|f_{I_i}((a, b))| > 0$, we see that $\mathscr{G}(I_i)$ is the *complete* bipartite graph (cf. [3]) on the sets of vertices A_i and B_i.

Now assume that $(a, b) \in E(N)$. We know that $f_{I_i}((a, b)) = (I_i)_{a,b}$ forms a group under the semigroup multiplication which is isomorphic to the group formed by the set $C(b, a)(I_i)_{a,b}$ under the *group* multiplication. By (i)′, $(I_i)_{a,b}$ is just the set of all path products $\pi_N P(a, b)$ for all paths $P(a, b)$ in $\mathscr{G}(N)$ (and therefore in $\mathscr{G}(N_i)$) from a to b. (Of course π_{N_i} and π_N are identical in $\mathscr{G}(N_i)$.) Consequently, $C(b, a)(I_i)_{a,b}$ is just the set of all products $\pi_N P(b, b)$ where $P(b, b)$ ranges over all *cycles* in $\mathscr{G}(N_i)$ which start with the edge (b, a). Now consider any cycle $P(b', b')$, where $b' \in B_i$. By hypothesis, there is a path $P(a, b')$ in $\mathscr{G}(N_i)$. Let $P(b, b)$ denote the cycle formed by going first from b to a along the edge (b, a), then from a to b' along $P(a, b')$, then from b' to b' around the cycle $P(b', b')$, next back to a along the *inverse* of $P(a, b')$ (which is possible since $P(a, b')$ is a path in $\mathscr{G}(N_i)$), and finally back to b along the edge (a, b). Direct calculation shows that

$$\pi_N P(b, b) = C(b, a)\pi_N P(a, b')\pi_N P(b', b')(\pi_N P(a, b'))^{-1} C(b, a)^{-1},$$

where the meaning of $(\pi_N P(a, b))^{-1}$ is obvious since in this case $\pi_N P(a, b')$ has a single element. Hence, for any $(a', b') \in A_i \times B_i$, there exists $g = g(a, b, b') \in G$ such that

$$C(b, a)(I_i)_{a,b} \supseteq gC(b', a')(I_i)_{a',b'}g^{-1}.$$

Since both sides have the same cardinality, we can write

$$C(b, a)(I_i)_{a,b} = gC(b', a')(I_i)_{a',b'}g^{-1},$$

i.e., all the groups $C(b, a)(I_i)_{a,b}$ are conjugate in G.

We should point out that the group $C(b, a)(I_i)_{a,b}$ does not depend on a. To show this, consider two edges (a, b), $(a', b) \in E(N_i)$. Letting $P(a, b)$ denote the path $((a, b), (b, a'), (a', b))$ in $\mathscr{G}(N_i)$, we have

$$\begin{aligned} f_N((a, b)) = \pi_N P(a, b) &= f_N((a, b)) f_N((b, a')) f_N((a', b)) \\ &= f_N((a, b)) C(b, a') f_N((a', b)). \end{aligned}$$

Therefore

$$C(b, a)f_N((a, b)) = C(b, a)f_N((a, b))C(b, a')f_N((a', b)),$$

which can be stated equivalently as

$$C(b, a)(I_i)_{a,b} = C(b, a)(I_i)_{a,b}C(b, a')(I_i)_{a',b}.$$

Since both $C(b, a)(I_i)_{a,b}$ and $C(b, a')(I_i)_{a',b}$ are subgroups of G, they are equal.

Let us choose an arbitrary fixed $b_i^* \in B_i$ and denote $C(b_i^*, a)(I_i)_{a,b_i^*}$ by G_i. We have seen that for any $(a', b') \in E(N_i)$, $C(b', a')(I_i)_{a',b'}$ is conjugate to G_i in G. We have also shown that if $P(b_i^*, b_i^*)$ is any cycle in $\mathcal{G}(I_i)$ from b_i^* to b_i^*, then $\pi_N P(b_i^*, b_i^*) \subseteq G_i$.

Since it is easily seen that I_i is a $\mathcal{J}$-class of I^0, I_i^0 is a 0-simple semigroup (since it is certainly not null). The question naturally arises how I_i^0 can be expressed as a regular Rees matrix semigroup. We would like to write $I_i^0 = M^0(A_i, B_i, G_i, C_i)$, where C_i is the $|B_i| \times |A_i|$ submatrix $(C(b, a))_{b \in B_i, a \in A_i}$ of C. However, we cannot do this since, in general, the entries of C_i lie in $G \cup \{0\}$ and not just in $G_i \cup \{0\}$. Nevertheless, there is something we can do which resolves this difficulty nicely. It is well known (cf. [1]) that if the structure matrix C of a Rees matrix semigroup $M^0(A, B, G, C)$ is modified by multiplying the rows of C on the left and the columns of C on the right by elements of G to form the matrix C', i.e., $C'(b, a) = g_b C(b, a) g_a$, where $a \to g_a$, $b \to g_b$ are maps of A and B respectively into G, then $M^0(A, B, G, C) \cong M^0(A, B, G, C')$. We shall show that it is indeed possible to perform such a normalization of C_i in such a way that *all the elements of* C_i' *lie in* $G_i \cup \{0\}$. In fact, it will then be easy to see that it is possible to normalize C so that this happens simultaneously to C_i for all i. We state this as a lemma.

LEMMA 4. *There exists a map of* $A_i \cup B_i$ *into* G, *where* $a \to g_a$, $b \to g_b$, *such that*

$$C'(b, a) = g_b C(b, a) g_a \in G_i \cup \{0\}$$

for all $a \in A_i$, $b \in B_i$.

Proof. Consider the graph $\mathcal{G}(N_i)$ and let T_i be a *maximal* directed subgraph of $\mathcal{G}(N_i)$ which contains *no undirected cycles.* In other words, if the directions of the edges of T_i are ignored, forming $\overline{T}_i$, then $\overline{T}_i$ is a *tree* (cf. [3]). Since $\mathcal{G}(N_i)$ is a component of $\mathcal{G}(N)$, the set of vertices of T_i is $A_i \cup B_i$ and T_i has $|A_i| + |B_i| - 1$ edges. Let us call a path $P = (e_1, \cdots, e_r)$ in $\mathcal{G}(N_i)$ *unreduced* if there is an i such that $e_i = (x_i, x_{i+1})$ and $e_{i+1} = (x_{i+1}, x_i)$. Otherwise we say that P is *reduced.* $\overline{T}_i$ has the important property that for any $x \in A_i \cup B_i$, there is a *unique* reduced path $P^*(b_i^*, x)$ in $\overline{T}_i$ from b_i^* to x, where b_i^* was chosen previously to define G_i. For $x \in A_i \cup B_i$ define $g_x \in G$ by

$$g_x \in \begin{cases} (\pi_N P^*(b_i^*, x))^{-1} & \text{if } x \in A_i \\ \pi_N P^*(b_i^*, x) & \text{if } x \in B_i, \end{cases}$$

where $\pi_N P^*(b_i^*, b_i^*) = 1$ and $(\pi_N P^*(b_i^*, x))^{-1}$ has the obvious meaning since $\pi_N P^*(b_i^*, x)$ consists of a single element of G.

We assert that this choice of g_x satisfies the lemma. To show this, let $(a, b) \in E(N_i)$. Consider the paths $P^*(b_i^*, b) = (d_1, \cdots, d_r)$ and $P^*(b_i^*, a) = (e_1, \cdots, e_s)$ in T_i. The path

$$P(b_i^*, b_i^*) = (d_1, \cdots, d_r, (b, a), e_s^{-1}, \cdots, e_1^{-1})$$

is a *cycle* in $\mathscr{G}(N_i)$ from b_i^* to b_i^* where, if $e_i = (x, y)$, then $e_i^{-1} = (y, x)$, i.e., this just reverses the direction of the edge. Hence, $\pi_N P(b_i^*, b_i^*) \subseteq G_i$. But

$$\begin{aligned}\pi_N P(b_i^*, b_i^*) &= f_N(d_1) \cdots f_N(d_r) f_N((a, b)) f_N(e_s^{-1}) \cdots f_N(e_1^{-1}) \\ &= \pi_N P^*(b_i^*, b) C(b, a) (\pi_N P^*(b_i^*, a))^{-1} \\ &= \{g_b C(b, a) g_a\} = \{C_i'(b, a)\},\end{aligned}$$

and the lemma is proved.

The same calculation shows that if (a, b) or (b, a) is an edge of T_i, then $C_i'(b, a) = 1$. We note here that since $C_i'(b, a) \in G_i \cup \{0\}$ for all $a \in A_i$, $b \in B_i$, and since G_i is the set of all path products of cycles in $\mathscr{G}(N_i)$ starting at b_i^*, G_i is exactly the subgroup of G generated by the nonzero entries of C_i'. Since the components $\mathscr{G}(I_i)$ of $\mathscr{G}(I)$ are disjoint, we obtain, by piecing together these normalizations C_i' of C_i for each i, a normalization C' of the whole matrix C such that for all i, $C_i'(b, a) \in G_i \cup \{0\}$. This completes the proof of Theorem 2.

We isolate here several useful facts which have been established in the preceding proof:

(i) If a and a' are in the same component of $\mathscr{G}(N)$ then $C(b, a)I_{a,b}$ and $C(b', a')I_{a',b'}$ are conjugate subgroups of G. In fact, $C(b, a)I_{a,b} = C(b, a')I_{a',b}$.

(ii) For any $b_i \in B_i$, the union of all path products of cycles in $\mathscr{G}(I_i)$ which start at b_i forms a group isomorphic to G_i.

The preceding results also yield the following theorem.

THEOREM 3. *For a subgroup $G' \subseteq G$, a necessary and sufficient condition that there exists a normalization C' of C such that all entries of C' lie in $G' \cup \{0\}$ is that there exist $g_i \in G$ such that $g_i G_i g_i^{-1} \subseteq G'$ for all i.*

Proof. For any normalization C' of C by $C'(b, a) = g_b' C(b, a) g_a'$, the group of path products of cycles in a component $\mathscr{G}(I_i') = \mathscr{G}(I_i)$ starting at b_i^* is just $G_i' = x_i G_i x_i^{-1}$, where $x_i = g_{b_i^*}$. Hence $x_i G_i x_i^{-1}$ is certainly contained in the subgroup generated by the nonzero entries of C_i'. Therefore, if the entries of C_i' are in $G' \cup \{0\}$, then $x_i G_i x_i^{-1} \subseteq G'$ for all i. On the other hand, suppose there exist $g_i \in G$ such that $g_i G_i g_i^{-1} \subseteq G'$ for all i. By Lemma 4, there exist $g_a, g_b \in G$ such that for all i, $g_b C_i(b, a) g_a \in G_i \cup \{0\}$, for all $a \in A$, $b \in B$. Therefore

$$g_i g_b C_i(b, a) g_a g_i^{-1} \in g_i G_i g_i^{-1} \cup \{0\} \subseteq G' \cup \{0\}$$

for all $a \in A$, $b \in B$ and for all i. This proves the theorem.

Definition 6. If $M^0(A, B, G, C)$ is a regular Rees matrix semigroup, a subgroup $G' \subseteq G$ is said to be *C-admissible* if every subgroup of the subsemigroup generated by the idempotents of $M^0(A, B, G, C)$ has some conjugate contained in G'.

Theorem 3 can be restated as

THEOREM 3′. *A necessary and sufficient condition on a subgroup $G' \subseteq G$ that there exist a normalization C' of C with all the entries of C' in $G' \cup \{0\}$ is that G' be C-admissible.*

This result has applications in the next section in which the maximal subsemigroups of M^0 are treated.

It may be pointed out here that the normalization of the structure matrices of *simple* matrix semigroups $M(A, B, G, C)$ given in [5] is a special case of Lemma 4. In particular, if $C(b, a) \neq 0$ for all $a \in A$, $b \in B$, then $\mathscr{G}(I)$ is a complete bipartite graph and consists of a single component. If b_1^* is chosen to be b_1, if the edges of T_1 are chosen to be $\{(b_1, a): a \in A\} \cup \{(a_1, b): b \in B - \{b_1\}\}$, and if the constructions given in Lemma 4 are applied, the normalized matrix C' has all 1's in its first row and column.

An immediate consequence of Theorem 3 is the following corollary.

COROLLARY 2. *The following statements are equivalent: (i) I^0 is combinatorial (i.e., it contains only trivial subgroups); (ii) It is possible to normalize C to C' in such a way that $C'(b, a) \in \{0, 1\}$ for all $a \in A$, $b \in B$; (iii) All path products of cycles in $\mathscr{G}(I)$ are equal to $\{1\}$.*

This result has found recent application in the work of J. Rhodes (unpublished) on complexity of finite semigroups.

5. The Maximal Subsemigroups of M^0

As a final application of the graph-theoretic ideas discussed in this paper, we determine the structure of the (proper) maximal subsemigroups of a finite 0-simple semigroup.

THEOREM 4. *Let $M^0 = M^0(A, B, G, C)$ be a regular Rees matrix semigroup. If Q is a maximal subsemigroup and Q does not contain $\{0\}$, then $Q = M^0 - \{0\}$. If $Q = G^{(0)}$ for a simple cyclic group G, then $Q = \{0\}$ is maximal. All other maximal subsemigroups of M^0 properly contain $\{0\}$ and are given as follows:*

(i) If G' is a C-admissible maximal subgroup of G, then $Q = G' \times A \times B \cup \{0\}$ is maximal.

(ii) If $A' = A - \{a\}$ and C restricted to $B \times A'$ is regular, then $Q = G \times A' \times B \cup \{0\}$ is maximal.

(iii) If $B' = B - \{b\}$ and C restricted to $B' \times A$ is regular, then $Q = G \times A \times B' \cup \{0\}$ is maximal.

(iv) If $X \times Y \subseteq B \times A$ is a maximal submatrix of C which is identically 0, and if $A' = A - Y$ and $B' = B - X$, then $Q = M^0 - G \times A' \times B'$ is maximal.

Proof. If Q is a maximal subsemigroup and Q does not contain $\{0\}$, then $M^0 - \{0\}$ is a subsemigroup and $Q = M^0 - \{0\}$. If $|A| = |B| = 1$ and G is a simple cyclic group, then $Q = \{0\}$ is maximal. Henceforth we shall exclude these trivial M^0, and we may assume that all maximal subsemigroups of M^0 properly contain 0.

Let $Q \cup \{0\}$ denote an arbitrary maximal subsemigroup of M^0. There are two possibilities.

(i) Suppose Q intersects every $\mathscr{H}$-class of M^0. In this case, the set of edges $E(Q)$ of $\mathscr{G}(Q)$ contains $A \times B$ and by regularity, all $|Q_{a,b}|$ are equal. Note that $Q \cup \{0\}$ contains all the idempotents of M^0. Therefore, if we use arguments given in the preceding section and Theorem 3, $Q \cup \{0\} = M^0(A, B, G', C')$ for some C-admissible subgroup G' of G and for some suitable normalization C' of C, i.e., such that

$$C'(b, a) = g_b C(b, a) g_a \in G' \cup \{0\}.$$

Since $Q \cup \{0\}$ is maximal, G' is maximal in G and this case is concluded.

(ii) Suppose Q does not intersect every $\mathscr{H}$-class of M^0. Then Q is the union of $\mathscr{H}$-classes of M^0, i.e., $f_Q(e) = G$ for all $e \in E(Q)$. For $X \subseteq A \cup B$, let us define

$$S(X) \equiv \{y \in A \cup B: (x, y) \in E(Q) \text{ for some } x \in X\}.$$

Also, we define $Z(X) \subseteq A \cup B$ to be the set of all $z \in A \cup B$ such that there is a path $P(x, z)$ in $\mathscr{G}(Q)$ from x to z for some $x \in X$. In general, $S(\{x\})$ and $Z(\{x\})$ will be denoted by $S(x)$ and $Z(x)$. Finally, define $K \subseteq A$ and $L \subseteq B$ by

$$K \equiv \{a \in A: S(a) = B\}, \qquad L \equiv S(A - K).$$

We note that $K \neq A$ since $Q \cup \{0\}$ is proper. There are several cases:

(a) *Suppose* $K = \varnothing$. Then for all $a \in A$, there exists $b(a) \in B - S(a)$. If there were an $a \in A$ such that $Z(a) \not\supseteq A$, then we could choose $a' \in A - Z(a), a' \neq a$, and form the graph $\mathscr{G}(Q')$ with

$$E(Q') = E(Q) \cup \{(a', b(a'))\} \cup \{(a', x): x \in B \cap S(b(a'))\},$$

where $f_{Q'}(e) = G$ for all $e \in E(Q')$. Then $Q' \cup \{0\}$ is a subsemigroup of M^0 (by Lemma 1), it is proper since $(a, b(a)) \notin E(Q')$, and $Q' \supset Q$. This is a contradiction to the assumption of the maximality of $Q \cup \{0\}$. Hence, we may assume that $Z(a) \supseteq A$ for all $a \in A$.

Now, if $S(A) = B$, then there exists $x \in A$ such that $b(a') \in S(x)$. But $x \in A \subseteq Z(a)$ implies that $b(a') \in Z(a')$ which in turn implies that $b(a') \in S(a)$ since $Q \cup \{0\}$ is a subsemigroup. But this is a contradiction since by hypothesis $b(a') \in B - S(a')$. Hence, we may assume in this case that $S(A) \neq B$. By the maximality of $Q \cup \{0\}$, we must have $B - S(A) = \{b_0\}$ for some $b_0 \in B$. Thus, $a \in A$ implies that $S(a) = B - \{b_0\}$. Now, if there is an $a' \in A$ such that $a' \notin S(B - \{b_0\})$, then the graph $\mathscr{G}(Q')$ with

$$E(Q') = E(Q) \cup \{(a', b_0)\}$$

defines a subsemigroup $Q' \cup \{0\} \supset Q \cup \{0\}$ which is *proper* if $|A| > 1$. Hence, we must have either $A = S(B - \{b_0\})$ or $|A| = 1$. However $|A| = 1$ implies $A = S(B - \{b_0\})$ provided $|B| > 1$. Since we have excluded the case $|A| = |B| = 1$, we conclude that there exists $b_0 \in B$ such that $A = S(B - \{b_0\})$ and $S(a) = B - \{b_0\}$ for all $a \in A$. Consequently, $E(Q) \cap A \times B = A \times (B - \{b_0\})$, where $A = S(B - \{b_0\})$.

(b) *Suppose* $L = B$ *and* $\varnothing \neq K \neq A$. By hypothesis we can choose $x \in K$ and $b \in B$ (by regularity) such that $x \in S(b)$. Since $L = B$, there exists $a \in A - K$ such that $b \in S(a)$. Hence,

$$Z(a) \supseteq Z(x) \supseteq S(x) = B$$

and therefore $S(a) = B$ since $Q \cup \{0\}$ is a subsemigroup, and $x \in S(a)$. This is a contradiction since $a \in A - K$ implies $S(a) \neq B$.

(c) *Suppose* $L = \varnothing$ *and* $\varnothing \neq K \neq A$. Thus $a \in A - K$ implies that $(a, b) \in E(Q)$ for *any* $b \in B$. Since $Q \cup \{0\}$ is maximal and $K \neq A$, we must have $A - K = \{a_0\}$ for some $a_0 \in A$. An argument similar to the one given at the end of (a) now shows that (excluding the case $|A| = |B| = 1$) we must have $S(b) \neq \{a_0\}$ for any $b \in B$ and $S(a) = B$ for all $a \in A - \{a_0\}$. In other words, $E(Q) \cap A \times B = (A - \{a_0\}) \times B$, where $S(b) \neq \{a_0\}$ for any $b \in B$.

We come to the last case.

(d) *Suppose* $\varnothing \neq K \neq A, \varnothing \neq L \neq B$. As in (a) we see that for all $a \in A - K$, $S(a) = L$. Suppose that there exists $b' \in B - L$ such that $S(b') \subseteq A - K$. Then there must exist $b'' \in B - L$ such that $S(b') \cap K \neq \varnothing$ since otherwise we would have $S(B - L) \subseteq A - K$; this would force $S(L) \subseteq K$ which would lead to a contradiction. But we can now form $\mathscr{G}(Q')$ by taking

$$E(Q') = E(Q) \cup \{(a, b'): a \in A - K\}$$

and $Q' \cup \{0\}$ contradicts the maximality of $Q \cup \{0\}$. Hence we must have $S(b) \nsubseteq A - K$ for any $b \in B - L$. By the argument used in (a) we see that for any $a, a' \in A - K$, we have $a' \in Z(a)$. Since $(x, y) \notin E(Q)$ for

$$(x, y) \in L \times K \cup (A - K) \times (B - L)$$

it follows that $S(L) \supseteq A - K$ must hold. However, $S(L) \subseteq A - K$ is immediate and we can write $S(L) = A - K$. In summary, we conclude in this case that

$$E(Q) \cap A \times B = A \times B - (A - K) \times (B - L) = A \times B - S(L) \times (B - L) \tag{5.1}$$

where $\varnothing \neq L \neq B$, $S(L) \neq A$ and L is the maximal subset X of B such that $S(X) \subseteq S(L)$.

Conversely, any subset $Q \cup \{0\}$ of M^0 satisfying the constraints placed on $E(Q) \cap A \times B$ by (i), (ii)(a), (ii)(c) or (ii)(d) are maximal subsemigroups of M^0. The proofs of this statement for (i), (ii)(a) and (ii)(c) are direct and will be omitted. To establish this for (ii)(d), suppose that $E(Q)$ satisfies (5.1) above. We note that these conditions on L are equivalent to the condition that $L \times (A - S(L))$ is a maximal rectangular subset of $B \times A$ which is disjoint from $E(Q)$, a fact first pointed out by D. Allen. $Q \cup \{0\}$ is certainly a proper subsemigroup of M^0. Suppose that Q' satisfies

$$E(Q') \supseteq E(Q) \cup \{(a, b)\}$$

for some $a \in S(L)$, $b \in B - L$. Let (a', b') be an arbitrary element of $S(L) \times$

$(B - L)$. By hypothesis, there exists $x \in L$ such that $(x, a) \in E(Q')$. Also there exists $y \in A - S(L)$ such that $(b, y) \in E(Q')$. Hence

$$P(a', b') = ((a', x), (x, a), (a, b), (b, y), (y, b'))$$

is a path in $\mathcal{G}(Q')$ from a' to b'. Therefore if $Q' \cup \{0\}$ is a subsemigroup of M^0, then $(a', b') \in E(Q')$. Since (a', b') was an arbitrary element of $S(L) \times (B - L)$, we have $E(Q') \supseteq A \times B$, which shows that $Q' \cup \{0\}$ is not proper. This completes the proof of maximality.

As an immediate consequence of Theorem 4, we have

COROLLARY 3. *The maximal transitive relations on a finite set U are exactly all subsets of $U \times U$ of the form $U \times U - V \times (U - V)$ for some nonempty proper subset V of U.*

The proof is a direct application of the preceding theorem in the case that $|A| = |B| = |U|$ and C is a $|U| \times |U|$ identity matrix.

6. Concluding Remarks

The preceding examples illustrate some typical results which can be obtained by a graph-theoretic approach. Other related questions such as the structure of the maximal combinatorial subsemigroups of M^0, canonical forms for arbitrary subsemigroups of M^0, and the extensions to infinite regular Rees matrix semigroups, will be discussed in a future paper.

There is a good possibility that once the "global" structure of finite semigroups is sufficiently well understood, these "local" results may then be extended to arbitrary finite semigroups. This has been successfully accomplished in the case of the maximal subsemigroups (cf. [2] or [5]).

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