Reprinted from JOURNAL OF COMBINATORIAL THEORY
Vol. 4, No. 3, April 1968

Printed in Belgium

Note

On Edgewise 2-Colored Graphs with Monochromatic Triangles and Containing No Complete Hexagon

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Communicated by J. Riordan

The following question was raised by Erdös and Hajnal [1] recently: Construct a graph G which *does not contain a complete hexagon* such that *for every coloring of the edges by two colors there is a triangle all of whose edges have the same color*. It is easily checked that G must have more than 7 vertices. In this note we present such a graph G with 8 vertices.

Let G denote the graph formed from the complete graph on the vertices $\{1, 2,..., 8\}$ by removing the 5 edges $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}$. *Assume* that the edges of G can be partitioned into two sets A and B such that neither set contains a triangle. We can further assume that $\{6, 7\} \in A$, $\{7, 8\} \in A$, and $\{6, 8\} \in B$. Thus, for $x \in \{1, 2, 3, 4, 5\}$ we must have $\{7, x\} \in B$ since otherwise $\{7, x\} \in A$ implies *either* at least one of $\{6, x\}$, $\{8, x\} \in A$ (forming a triangle in A) *or* both $\{6, x\} \in B$, $\{8, x\} \in B$ (forming a triangle in B). This forces all the edges $\{1, 3\}, \{3, 5\}, \{5, 2\}, \{2, 4\}, \{4, 1\} \in A$. Now for any three distinct points $x, y, z \in \{1, 2, 3, 4, 5\}$ we cannot have $\{6, x\} \in A$, $\{6, y\} \in A$, and $\{6, z\} \in A$ since *some* pair $\{x, y\}, \{x, z\}, \{y, z\}$ is an edge of G in A. Hence there must exist at least three distinct points $a, b, c \in \{1, 2, 3, 4, 5\}$ such that $\{6, a\} \in B$, $\{6, b\} \in B$, $\{6, c\} \in B$. A similar argument applied to vertex 8 forces the existence of distinct points $a', b', c' \in \{1, 2, 3, 4, 5\}$ such that $\{8, a'\} \in B$, $\{8, b'\} \in B$, $\{8, c'\} \in B$. But there must exist $w \in \{a, b, c\} \cap \{a', b', c'\}$ and the triangle with vertices $\{6, 8, w\}$ is in B which is a *contradiction*. G clearly does not contain a complete hexagon and the proof is complete.

To the best of the author's knowledge, the first example of a graph satisfying the conditions of Erdös and Hajnal was given by J. H. van Lint; subsequently L. Pósa showed the existence of such a graph containing no complete *pentagon* and Jon Folkman constructed such a graph containing no complete *quadrilateral* (all unpublished).

REFERENCE

1. P. ERDÖS AND A. HAJNAL, Research Problem 2-5, *J. Combinatorial Theory* **2**, (1967), 104.