Reprinted from JOURNAL OF COMBINATORIAL THEORY Vol. 4, No. 3, April 1968
Printed in Belgium

Maximal Subsemigroups of Finite Semigroups*

N. GRAHAM,[†] R. GRAHAM,** AND J. RHODES[‡]

Communicated by Gian-Carlo Rota

ABSTRACT

If M is a maximal (proper) subsemigroup of a finite semigroup S, then M contains all but one $\mathscr{J}$-class $J(M)$ of S. When $J(M)$ is non-regular $J(M) \cap M = \phi$ so $M = S - J(M)$. When $J(M)$ is regular either $J(M) \cap M = \phi$ or $M \cap J(M)$ has a natural form with respect to the Green-Rees coordinates in $J(M)$. Specifically, there exist an isomorphism $j : J(M)^0 \to \mathscr{M}^0(A, B, G, C)$ of $J(M)^0$ with a Rees regular matrix semigroup so that $j(M \cap J(M)) = G' \times A \times B$, where G' is a maximal subgroup of G or $j(M \cap J(M))$ is the complement of a "rectangle" of $\mathscr{H}$-classes of $\mathscr{M}^0(A, B, G, C)$. In the first case, $(M \cap J(M))^0$ is a maximal subsemigroup of $J(M)^0$. In the second, $(M \cap J(M))^0$ is maximal in $J(M)^0$ when $j(M \cap J(M)) = \mathscr{M}^0(A, B, G, C) - (G \times A' \times B')$ for proper subsets A' and B' of A and B, respectively, but need not be when $j(M \cap J(M)) = G \times A \times B'$ or $j(M \cap J(M)) = G \times A' \times B$.

The notation of this paper, with slight variations, follows [1]. $\mathscr{M}^0(A, B, G, C)$ denotes a Rees matrix semigroup with finite index sets A, B, finite group G and $C : B \times A \to G^0$ the structure matrix. If J is a $\mathscr{J}$-class of a semigroup S, we denote by J^0 the semigroup $J \cup \{0\}$, $0 \notin J$, with multiplication

$$j_1 \cdot j_2 = \begin{cases} j_1 j_2, & \text{if } j_1 j_2 \in J, \\ 0, & \text{otherwise.} \end{cases}$$

We use the notation $|X|$ for the cardinality of the set X. By a maximal subsemigroup M of a semigroup S we mean a proper non-empty subsemi-

* This research supported in part by the Office of Naval Research, Information Systems Branch, under Contract Nonr 4705(00) and by the United States Air Force, Air Force Systems Command, Systems Engineering Group, Wright-Patterson AFB, Ohio, under Contract AF 33(615)-3893 and by the Air Force Office of Scientific Research, Office of Aerospace Research, Applied Mathematics Division, under Contract 49(638)-1550.

† University of California, Berkeley.

** Bell Telephone Laboratories, Inc., Murray Hill, New Jersey.

‡ Krohn-Rhodes Research Institute, Inc., 2118 Milvia Street, Berkeley, California 94704, and the University of California, Berkeley.

group M of S such that, whenever $M \subseteq T \subseteq S$ for some subsemigroup T of S, we have $M = T$ or $T = S$.

It is the purpose of this note to prove the following:

PROPOSITION. Let M be a maximal subsemigroup of the finite semigroup S. Then

(1) For some $\mathscr{J}$-class $J(M)$ of S,

$$S - M \subseteq J(M).$$

(2) M meets (intersects non-trivially) each $\mathscr{H}$-class of S, or M is a union of $\mathscr{H}$-classes of S.

(3) If $J(M)$ is non-regular then $J(M) \cap M = \phi$, so $M = S - J(M)$.

(4) If $M \cap J(M) \neq \phi$ (so $J(M)$ is regular by (3) and $J(M)^0$ is isomorphic to a regular Rees matrix semigroup) two cases arise from the two possibilities in (2).

CASE 1: If M meets each $\mathscr{H}$-class of S then an isomorphism $j : J(M)^0 \to \mathscr{M}^0(A, B, G, C)$ can be so chosen that

$$j(M \cap J(M)) = G_1 \times A \times B,$$

where G_1 is a maximal subgroup of G. In this case, $(M \cap J(M))^0$ is a maximal subsemigroup of J^0.

CASE 2: If M is a union of $\mathscr{H}$-classes of S, then an isomorphism $j : J(M)^0 \to \mathscr{M}^0(A, B, G, C)$ can be so chosen that $j(M \cap J(M))$ is the complement of a "rectangle" of $\mathscr{H}$-classes of $\mathscr{M}^0(A, B, G, C)$. Precisely, $j(M \cap J(M))$ has one of the following three forms:

(a) $G \times (A\text{-}A') \times B$, A' a proper non-empty subset of A,

(b) $G \times A \times (B\text{-}B')$, B' a proper non-empty subset of B,

(c) $(G \times A \times B) - (G \times A' \times B')$, A', B' proper non-empty subsets of A and B, respectively.

In Case 2, $(M \cap J(M))^0$ is a maximal subsemigroup of $J(M)^0$ if $j(M \cap J(M))$ has form (c) but need not be in the other two cases.

PROOF: For (1), let J be minimal (in the usual ordering $J_1 \leqslant J_2$ iff $S^1J_1S^1 \subseteq S^1J_2S^1$) among the $\mathscr{J}$-classes of S not contained in M. Then $M \cup J$ is a subsemigroup of S properly containing M, so that $M \cup J = S$. Thus $S - M \subseteq J \equiv J(M)$.

For (2), let $J = J(M)$. Define M' to be the union of all $\mathscr{H}$-classes that M meets. We will show M' to be a subsemigroup of S containing M, so by the maximality of M, either $M' = M$ or $M' = S$. The former implies M is a union of $\mathscr{H}$-classes; the latter shows that M meets every $\mathscr{H}$-class of S.

To show M' a subsemigroup, let $h_1, h_2 \in M'$. If $h_1h_2 \in M \subseteq M'$, done; so suppose not. Then $h_1h_2 \in J$ and at least one of $h_1, h_2 \in J$. By the definition of M', there exist $m_1, m_2 \in M$ such that $h_i\mathscr{H}m_i$, $i = 1, 2$. There are two cases:

CASE 1: Suppose $h_1 \in M$, $h_2 \in J$. Since (by assumption) $h_1h_2 \in J$, left multiplication by h_1 moves the $\mathscr{H}$ class containing h_2 onto the $\mathscr{H}$-class containing h_1h_2. (See [2].) Thus $h_2\mathscr{H}m_2$ implies $h_1h_2\mathscr{H}h_1m_2 \in M$, so $h_1h_2 \in M'$. (The case $h_1 \in J$, $h_2 \in M$ is handled dually.)

CASE 2: Suppose $h_1, h_2 \in J$. Then using the Rees Theorem, $h_i\mathscr{H}m_i$, $i = 1, 2$ implies $h_1h_2\mathscr{H}m_1m_2 \in M$, so $h_1h_2 \in M'$. This exhausts the possibilities, so M' is a semigroup and (2) is proved.

For (3), let $J = J(M)$. We first recall that J^0 is null (i.e., $(J^0)^2 = \{0\}$) iff J is non-regular. See [1]. Let J^0 be null and let $n_1, n_2 \in J$. Then $n_1 = s_1n_2s_2$ for some $s_1, s_2 \in S^1$ by the definition of $\mathscr{J}$, and $s_1, s_2 \notin J$, since, by the definition of null, the product of two or more elements of J is in $S^1JS^1 - J$. Thus, $s_1, s_2 \in M^1$, so that $n_2 \in M$ implies $n_1 \in M$. Hence $J \cap M = \phi$.

For Case 1 of (4), assume that M meets every $\mathscr{H}$-class of S. Let j' be an isomorphism of J^0 onto $T = \mathscr{M}^0(A, B, G, C')$. See [1]. Then $T_1 = j'(M \cap J)^0$ is a subsemigroup of T meeting each $\mathscr{H}$-class $H(a, b) = (G, a, b) = \{(g, a, b) : g \in G\}$. Let

$$T_1 \cap H(a, b) = M(a, b) = (X(a, b), a, b) = \{(x, a, b) : x \in X(a, b) \subseteq G\}.$$

Now let $H(a_0, b_0)$ be a fixed non-zero $\mathscr{H}$-class of T for which $g_0 = C(b_0, a_0) \neq 0$, i.e., $H(a_0, b_0)$ is a subgroup of T isomorphic with G under the isomorphism $(g, a_0, b_0) \to g_0g$. Then

$$X(a_0, b_0) = \{g_0^{-1}g : g \in G_1\} = g_0^{-1}G_1$$

for some subgroup G_1 of G, since $M(a_0, b_0)$ is a subgroup of T contained in $H(a_0, b_0)$. For each $a \in A$, let g_a be a fixed element of $X(a, b_0)$ and for each $b \in B$, let y_b be a fixed element of $X(a_0, b)$. Then

$$\begin{aligned}(g_a, a, b_0)\, M(a_0, b_0)(y_b, a_0, b) &= (g_aG_1g_0y_b, a, b) \subseteq M(a, b)\\ &= (X(a, b), a, b).\end{aligned}$$

But by a similar argument there exist elements $t_1, t_2 \in T_1$ such that

$$t_1M(a, b)\, t_2 \subseteq M(a_0, b_0) = (X(a_0, b_0), a_0, b_0)$$

so

$$| X(a, b)| = | X(a_0 , b_0)| = | G_1 | = | g_a G_1 g_0 y_b |.$$

For each $b \in B$, let $h_b = g_0 y_b$. Then $X(a, b) = g_a G_1 h_b$. Let $C : B \times A \to G^0$ be the matrix given by

$$C(b, a) = h_b C'(b, a)\, g_a .$$

Then $\mathscr{M}^0(A, B, G, C')$ is isomorphic with $\mathscr{M}^0(A, B, G, C)$ by the isomorphism $j_1 : \mathscr{M}^0(A, B, G, C') \to \mathscr{M}^0(A, B, G, C)$ given by

$$j_1(g, a, b) = (g_a^{-1} g h_b^{-1}, a, b) \qquad \text{and} \qquad j_1(0) = 0.$$

Thus $j_1(T_1 - \{0\}) = G_1 \times A \times B$ in $\mathscr{M}^0(A, B, G, C)$; so letting $j = j_1 j'$ we have $j(M \cap J) = G_1 \times A \times B$.

Finally we will show that G_1 is a maximal subgroup of G, so that $(M \cap J)^0$ is a maximal subsemigroup of J^0. Let G_1' be a subgroup of G such that $G_1 \subseteq G_1' \subseteq G$, and let $T = j^{-1}(G_1' \times A \times B)$. Define $M' = M \cup T$. We shall show M' to be a semigroup, so by the maximality of M the assertion is proven.

Since $C(b, a) \in G_1{}^0$, $\mathscr{M}^0(A, B, G_1', C)$ is a semigroup, so

$$T \cup \{0\} = j^{-1}[(G_1' \times A \times B) \cup \{0\}]$$

is a subsemigroup of J^0.

Since T^0 is a subsemigroup of J^0, we need only show that for $m \in M$ and $x \in T$ we have $mx \in M'$ and $xm \in M'$. If $mx, xm \in M \subseteq M'$, done; so assume $mx, xm \in J$. Since J is regular there exist idempotents $e_1 , e_2 \in J$ such that $e_1 x = x$, $xe_2 = x$. Also, $e_1 , e_2 \in M$ since M meets each H class of S, so $me_1 , e_2 m \in M$. Further $me_1 , e_2 m \in J$ since $mx = (me_1)\, x \in J$ and $xm = x(e_2 m) \in J$. Thus $me_1 , e_2 m \in J \cap M \subset T$, which implies $(me_1)\, x = mx \in M'$ and $x(e_2 m) = xm \in M'$. Thus M' is a semigroup of S, which proves the assertions.

In Case 2 of (4), M is a union of $\mathscr{H}$-classes and $M \cap J \neq 0$. Let $J = J(M)$ and let $\{R(a) : a \in A\}$, $\{L(b) : b \in B\}$, and $\{H(a, b) = R(a) \cap L(b)\}$ be the $\mathscr{R}$, $\mathscr{L}$, and $\mathscr{H}$-classes, respectively, of S contained in J. Let $A' = \{a \in A : R(a) \nsubseteq M\}$ and $B' = \{b \in B : L(b) \nsubseteq M\}$. Clearly A' and B' are not empty, for then $J \subseteq M$, a contradiction.

Let $a_1 \in A'$. Then $T = (M)^1 R(a_1) \cup M$ is a subsemigroup of S properly containing M. To prove this, utilize the fact that $R(a_1)\, M \subseteq R(a_1) \cup M \subseteq T$. (For let $r \in R(a_1)$, $m \in M$ and suppose $rm \notin M$. Then $rm \in J$ so $rm \mathscr{J} r$, which implies $rm \mathscr{R} r$, i.e., $rm \in R(a_1)$). Hence $T = S$. Let $a_2 \in A'$, so $R(a_2) \nsubseteq M$. Then $(M)^1 R(a_1) \cap R(a_2) \neq \phi$, i.e., there exists $m \in (M)^1$ such that $mR(a_1) \cap R(a_2) \neq \phi$. But by Green's relations $mR(a_1) = R(a_2)$ and in particular $mH(a_1, b) = H(a_2, b)$ for all $b \in B$. Similarly (using $\mathscr{L}$-classes) there exists $m \in (M)^1$ such that for $b_1, b_2 \in B'$, $H(a, b_1) m = H(a, b_2)$ for all $a \in A$.

Now to see what $\mathscr{H}$-classes of J are not in M we prove the lemma: $a \in A'$, $b \in B'$ *iff* $H(a, b) \cap M = \phi$.

Let $a \in A'$, $b \in B'$, and suppose $H(a, b) \subseteq M$. Then for each $a_i \in A'$ there exists $m_i \in (M)^1$ such that $m_i H(a, b) = H(a_i, b)$. Thus for all $a_i \in A$, $H(a_i, b) \subseteq M$, which implies $L(b) \subseteq M$, a contradiction. The converse is clear.

Thus if $B' = B$ it is easy to see that $j(M \cap J)$ has form (a). Similarly if $A' = A$, $j(M \cap J)$ has form (b). If both A' and B' are proper subsets of A and B then $j(M \cap J)$ has form (c).

Since it is easy to construct examples in which $j(M \cap J)$ has the form (a) or (b) but in which $(M \cap J)^0$ is not maximal in J^0 (see Remark 2 following this proof), we will complete the proof by showing that $(M \cap J)^0$ is maximal in J^0 if $j(M \cap J)$ has the form (c). By the above argument it suffices to show that, for each $a_1, a_2 \in A'$, there exists $m \in M \cap J$ (rather than merely $m \in M^1$ as above) such that $mR(a_1) = R(a_2)$, and that, for each $b_1, b_2 \in B'$, there exists $m' \in M \cap J$ such that $L(b_1) m' = L(b_2)$. Further, by the definition of the orderings on the $\mathscr{J}$-classes, it is equivalent to show that such m, m' can be chosen in $M \cap J^*$, where $J^* = \cup \{J' : J'$ is a $\mathscr{J}$-class of S and $J' \leqslant J\}$ since $J^* - J$ is an ideal of S.

Let $R(A') = \cup \{R(a) : a \in A'\}$. Now for all $a \in A'$, we have shown above that $R(A') \subseteq M^1 R(a)$. Also, by the definition of J^*, we have $(M \cap J^*) M^1 = M \cap J^* = M^1(M \cap J^*)$. Now for any $a \in A'$, $R(A') \subseteq (M \cap J^*) R(a)$, or $R(A') \cap (M \cap J^*) R(a) = \phi$, since, if $mR(a) \cap R(a') \neq \phi$ for some $a' \in A'$ and $m \in M$, then $mR(a) = R(a')$, so $R(a') \subseteq (M \cap J^*) R(a)$ and

$$R(A') \subseteq M^1 R(a') \subseteq M^1(M \cap J^*) R(a) = (M \cap J^*) R(a).$$

If $R(A') \cap (M \cap J^*) R(a) = \phi$, then

$$\begin{aligned}(M \cap J^*) R(A') \cap R(A') &\subseteq (M \cap J^*) M^1 R(a) \cap R(A') \\ &= (M \cap J^*) R(a) \cap R(A') = \phi.\end{aligned}$$

Now $j(J^0) = \mathscr{M}^0(A, B, G, C)$ and

$$j(M \cap J)^0 = ((G \times A \times B) - (G \times A' \times B')) \cup \{0\}$$

is a subsemigroup, so $C(b, a) = 0$ for all $(a, b) \in (A - A') \times (B - B')$. If for $a \in A'$, we have $R(A') \cap (M \cap J^*) R(a) = \phi$, then, by the above,

$R(A') \cap (M \cap J^*) R(A') = \phi$, so in particular, $(G \times A' \times (B - B')) \cdot (G \times A' \times B) = \{0\}$, showing that $C(b, a) = 0$ for all $(a, b) \in A \times (B - B')$, contradicting the regularity of J. It follows that $R(A') \subseteq (M \cap J^*) R(a)$ for any $a \in A'$, i.e., for all $a_1, a_2 \in A'$, there exists $m \in M \cap J$ (we replace J^* by J since no element of $J^* - J$ could satisfy the condition) such that $mR(a_1) = R(a_2)$. The proof for $\mathscr{L}$-classes is analogous. This proves the proposition.

The following reformulation of the theorem for 0-simple semigroups is due to Dennis Allen, Jr.

REMARK 1. Let $S = \mathscr{M}^0(A, B, G, C)$ be a regular Rees matrix semigroup. If M is a maximal subsemigroup of S, then $J(M) = \{0\}$ or $J(M) = S - \{0\}$. In the first case $S - \{0\}$ is a subsemigroup and $M = S - \{0\}$. In the second case, $M \cap J(M) = \phi$ iff $S - \{0\}$ is a simple Abelian group, (i.e., $(Z_p, +)$ for some prime p). Otherwise $M \cap J(M)$ has one of the following forms in some coordinate system:

(1) $(G' \times A \times B)$, G' a maximal subgroup of G.

(2) $(G \times A \times B')$, where $B' = B - \{b\}$ for some $b \in B$ and C restricted to $B' \times A$ is regular (i.e., non-zero at least once in each row and column).

(3) $(G \times A' \times B)$ where $A' = A - \{a\}$ for some $a \in A$ and C restricted to $B \times A'$ is regular.

(4) $G \times A \times B - (G \times A' \times B')$, where $A' = A - Y$, $B' = B - X$, and $X \times Y$ is a *maximal* "rectangle" on which C is identically zero.

Furthermore, each subsemigroup M of S containing all but one $\mathscr{J}$-class $J(M)$ and such that $M \cap J(M)$ has one of the above forms is a maximal subsemigroup of S.

REMARK 2. A counterexample to show that $(M \cap J)^0$ need not be a maximal subsemigroup of J^0 when $j(M \cap J)$ has form (a) of (4), Case 2 can be constructed as follows.

Let $F(X_n)$, $n \geqslant 2$, be the semigroup of all functions on n letters $x_1, \ldots, x_n$ under ordinary composition. Let $\{x_1, \ldots, x_n, z\}^l$ be the semigroup defined by $xy = x$ for all $x, y \in \{x_1, \ldots, x_n, z\}$. Form the semigroup $S = F(X_n) \cup \{x_1, \ldots, x_n, z\}$ by defining the multiplication as follows: Let $F(X_n)$ and $\{x_1, \ldots, x_n, z\}$ be subsemigroups, and for all $f \in F(X_n)$,

$$\begin{aligned} f \cdot x_i &= f(x_i), && \text{for all } x_i \in X_n, \\ x_i \cdot f &= x_i, && \text{for all } x_i \in X_n, \\ f \cdot z &= z \cdot f = z. \end{aligned}$$

Then $M = F(X_n) \cup \{z\}$ is a maximal subsemigroup of S and

$$J(M) = \{x_1, \ldots, x_n, z\}.$$

Since each element of $J(M)$ is an $\mathscr{R}$-class, $j(M \cap J(M))$ is of form (a). But, by Remark 1 above, $j(M \cap J(M))^0$ is not a maximal subsemigroup of $j(J(M)^0)$.

A counterexample for form (b) is constructed dually.

References

1. A. H. Clifford and G. B. Preston, Algebraic Theory of Semigroups, (Vol. 1, Math. Surveys No. 7), American Mathematical Society, Providence, R. I., 1962.
2. J. A. Green, On the Structure of Semigroups, *Ann. Math.* **54** (1951), 163–172.
3. D. Rees, On Semigroups, *Proc. Cambridge Philos. Soc.* **36** (1940), 387–400.

PRINTED IN BRUGES, BELGIUM, BY THE ST. CATHERINE PRESS, LTD.