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ON n-VALUED FUNCTIONALLY COMPLETE TRUTH FUNCTIONS

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Introduction. It is well known that the familiar Sheffer stroke function of the 2-valued propositional calculus is functionally complete (i.e., for any m, all 2^{2^m} truth functions of m variables can be defined[1] in terms of the stroke function). Indeed, it is not difficult to show that of the 16 2-valued functions of two variables, exactly two of them are functionally complete.

In this note we describe a rather large class of *n-valued* ($n \geq 2$) functionally complete functions of two variables (cf. [1]–[12]). The proofs given are short, elementary and self-contained.

Preliminary ideas. Suppose G is an n-valued truth function of two variables p and q. We shall denote the truth table of G by the linear notation

$$[G(0,0), G(0,1), \ldots, G(0, n-1), \ldots, G(n-1, n-1)]$$

where we shall let the n-values that G and the variables p and q assume come from the set $I = \{0, 1, \ldots, n-1\}$.

An important fact we point out here is that any truth function of m variables can be defined in terms of truth functions of just *two* variables. To see this,[2] let $G(p_1, p_2, \ldots, p_m)$ be an arbitrary n-valued truth function of the m variables $p_1, p_2, \ldots, p_m$ and let us assume that all truth functions of $m-1$ variables can be expressed in terms of functions of two variables. Hence, for each $i \in I$, the functions $G_i = G(i, p_2, \ldots, p_m)$ can be formed from truth functions of two variables. For $i \in I$, let J_i denote the function defined by

$$\begin{aligned} J_i(p, q) &= a \quad \text{if} \quad p = i, q = a, \\ &= 0 \quad \text{otherwise} \end{aligned}$$

and let M be the function defined by

$$M(p, q) = \max\{p, q\}.$$

Finally, let $L_i = L_i(p_1, p_2, \ldots, p_m)$ denote the function $J_i(p_1, G_i)$. It is then easily verified that the truth table[3] of

$$\overbrace{MM \ldots M}^{n-1} L_0 L_1 \ldots L_{n-1}$$

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[1] For a detailed discussion of this concept, see [6].

[2] Cf. also [6].

[3] Occasionally, we shall employ the Polish (parenthesis-free) notation of Lukasiewicz and Tarski.

is identical with that of G. This completes the induction step and since the cases $m = 1, 2$ are immediate then the assertion is proved.

In view of this observation, in order to establish the functional completeness of a truth function F, it suffices to show that all truth functions of *two* variables can be defined in terms of F.

The main results. If G is an n-valued truth function of two variables, let $P(G)$ denote the set of all truth functions of two variables which can be defined in terms of G. Thus, if G is functionally complete, $P(G)$ has cardinality n^{n^2}. Of course, the notation $[a_1, a_2, \ldots, a_{n^2}] \in P(G)$ will indicate that the function H with the truth table $[a_1, a_2, \ldots, a_{n^2}]$ belongs to $P(G)$.

In order to define the truth functions with which we shall be concerned, we introduce two mappings. Let σ and π be arbitrary fixed mappings of I into I such that for all $a \in I$:

(i) $0 < r < n$ implies $\sigma^{(r)}(a) \neq a$ (where $\sigma^{(r)}(a)$ denotes the rth iterate $\underbrace{\sigma(\sigma(\ldots\sigma}_{r}(a))\ldots)$ with $\sigma^{(0)}(a) \equiv a$).

(ii) There exists $r > 0$ such that $\pi^{(r)}(a) = 0$.

Note that σ is just a permutation of I which consists of a single cycle. Define the function $F_{\sigma,\pi}$ by

$$\begin{aligned} F_{\sigma,\pi}pq &= \sigma(a) \quad \text{if} \quad p = a, \qquad q = a, \\ &= \pi(a) \quad \text{if} \quad p = a \neq 0, q = 0, \\ &= 0 \qquad\ \text{if} \quad p = 0, \qquad q = a \neq 0, \end{aligned}$$

and is arbitrary, otherwise.

The main result of this paper is the following

THEOREM. *$F_{\sigma,\pi}$ is functionally complete.*

Before proceeding to the proof of the theorem we make some remarks about the notation and we prove a lemma. Abbreviate $F_{\sigma,\pi}$ by F and let $S = \{s_1, s_2, \ldots, s_r\}$ be a subset of $\{1, 2, \ldots, n^2\}$. For $b_i \in I$, the notation $[b_1, b_2, \ldots, b_r]_S \in P(F)$ will indicate there exists $[a_1, a_2, \ldots, a_{n^2}] \in P(F)$ such that $a_{s_i} = b_i$ for $1 \leq i \leq r$. Most of the time we shall be concerned with at most two of the r places in $[b_1, b_2, \ldots, b_r]_S$, say b_x and b_y (with $x < y$) and we could write this in an abbreviated form as $[B_1, b_x, B_2, b_y, B_3]_S$. However, in any particular argument given, S will be fixed and the argument will be independent of which S and which x and y were selected. Hence, we shall usually abbreviate $[b_1, b_2, \ldots, b_r]_S$ just by $[b_1, b_2, B]$ (or even $[b_1, B]$). Of course, by $[b_1, b_2, \sigma^{(k)}(B)]$ we mean $[b_1, b_2, \sigma^{(k)}(b_3), \ldots, \sigma^{(k)}(b_r)]$.

The following simple lemma will be useful.

LEMMA. *Let $S = \{s_1, \ldots, s_r\} \subseteq \{1, 2, \ldots, n^2\}$ be nonempty. If $[0, B]_S \in P(F)$ and $[0, \sigma(B)]_S \in P(F)$ then $[0, \sigma^{(k)}(B)]_S \in P(F)$ for all k.*

PROOF. Since $[0, B] \in P(F)$ by hypothesis (dropping the subscript S), then $[\sigma(0), \sigma(B)] \in P(F)$. (The argument in detail for this assertion runs as follows: By hypothesis for $[0, B]_S = [b_1, b_2, \ldots, b_r]_S \in P(F)$ there exists a truth function $G \in P(F)$ with truth table $[a_1, \ldots, a_{n^2}]$ such that $a_{s_i} = b_i$ for $1 \leq i \leq r$. Hence, the truth function $FGG \in P(F)$ has the truth table $[\sigma(a_1), \ldots, \sigma(a_{n^2})]$. Therefore

$[\sigma(b_1), \ldots, \sigma(b_r)]_S = [\sigma(0), \sigma(B)]_S \in P(F)$. (In general it will be our policy to omit the details in an argument of this type.) Since we have also assumed that $[0, \sigma(B)] \in P(F)$ then we now have $[0, \sigma^{(2)}(B)] \in P(F)$ (because $\sigma(0) \neq 0$). To see this, we observe that if $[0, \sigma^{(k-1)}(B)] \in P(F)$ and $[0, \sigma^{(k)}(B)] \in P(F)$ for some $k \geq 1$ then $[\sigma(0), \sigma^{(k)}(B)] \in P(F)$ and hence

$$F[0, \sigma^{(k)}(B)][\sigma(0), \sigma^{(k)}(B)] = [F0\sigma(0), F\sigma^{(k)}(B)\sigma^{(k)}(B)] = [0, \sigma^{(k+1)}(B)] \in P(F).$$

By continuing this argument the lemma is proved. We now present the

PROOF OF THEOREM. For $a \in I$, let s_a denote the least nonnegative integer x such that $\sigma^{(x)}(a) = 0$ and let r_a denote the least nonnegative integer x such that $\pi^{(x)}(a) = 0$. For k satisfying $1 \leq k \leq n^2$, let (A_k) denote the statement:

(A_k) For any subset $S \subseteq \{1, 2, \ldots, n^2\}$ of cardinality k and any $b_i \in I$, $1 \leq i \leq k$, $[b_1, \ldots, b_k]_S \in P(F)$.

The proof will proceed by induction on k.

To establish (A_1) we note that for any $S \subseteq \{1, 2, \ldots, n^2\}$ and any $b \in I$, we must have $[z]_S \in P(F)$ for *some* $z \in I$ and hence $[\sigma^{(i)}(z)]_S = [b]_S \in P(F)$ for some i (by the definition of σ). This proves (A_1).

Now *assume* that (A_k) is true for some fixed k, $1 \leq k < n^2$. *Suppose* also that there exists an $S \subseteq \{1, 2, \ldots, n^2\}$ of cardinality $k + 1$ and $b_i \in I$, $1 \leq i \leq k + 1$, such that $[b_1, \ldots, b_{k+1}]_S \notin P(F)$. We shall derive a contradiction.

Case 1. Suppose there exist $i_1 \neq i_2$ such that $b_{i_1} = b_{i_2}$. Without loss of generality, we assume that $i_1 = 1$, $i_2 = 2$ and $b_1 = b_2 = b$ (since the arguments will not depend on i_1 and i_2). Thus, we are assuming there exists

$$[b, b, b_3, \ldots, b_{k+1}]_S \equiv [b, b, B] \notin P(F)$$

where B denotes the b_i, $3 \leq i \leq k + 1$ (possibly empty).

Case 1.1. Suppose there exists an m such that

$$[0, 0, \sigma^{(m)}(B)] \in P(F).$$

Case 1.1.1. Suppose $[0, 0, \sigma^{(m+1)}(B)] \in P(F)$. By (a slight modification of) the lemma we have $[0, 0, \sigma^{(i)}(B)] \in P(F)$ for all i. Therefore $[\sigma^{(i)}(0), \sigma^{(i)}(0), B] \in P(F)$ for all i. Since $\sigma^{(j)}(0) = b$ for some j then we have a contradiction to Case 1.

Case 1.1.2. Suppose $[0, 0, \sigma^{(m+1)}(B)] \notin P(F)$.

Case 1.1.2.1. Suppose there exist $x, y \neq 0$ such that $[x, y, \sigma^{(m)}(B)] \in P(F)$. Thus,

$$F[0, 0, \sigma^{(m)}(B)][x, y, \sigma^{(m)}(B)] = [F0x, F0y, F\sigma^{(m)}(B)\sigma^{(m)}(B)] = [0, 0, \sigma^{(m+1)}(B)] \in P(F)$$

which contradicts Case 1.1.2.

Case 1.1.2.2. Suppose for all $x, y \neq 0$, $[x, y, \sigma^{(m)}(B)] \notin P(F)$. By the induction hypothesis there exist $\alpha, \beta \in I$ such that $[0, \alpha, \sigma^{(m+1)}(B)] \in P(F)$ and $[\beta, 0, \sigma^{(m+1)}(B)] \in P(F)$. Therefore $[\sigma^{(n-1)}(\beta), \sigma^{(n-1)}(0), \sigma^{(m)}(B)] \in P(F)$. Since $\sigma^{(n-1)}(0) \neq 0$ then by

Case 1.1.2.2 we must have $\sigma^{(n-1)}(\beta) = 0$, i.e., $\beta = \sigma(0)$. Similarly we must have $\alpha = \sigma(0)$. We can write

$$[0, \alpha, \sigma^{(m+1)}(B)] = [0, \sigma(0), \sigma(\sigma^{(m)}(B))] \equiv [0, \sigma(A)]$$

and since we know

$$[0, 0, \sigma^{(m)}(B)] = [0, A] \in P(F)$$

by Case 1.1, then the lemma implies $[0, \sigma^{(i)}(A)] \in P(F)$ for all i. Therefore $[\sigma^{(i)}(0), A] \in P(F)$ for all i. But for some i_0, $\sigma^{(i_0)}(0) = \pi^{(r_\beta - 1)}(\beta)$ (this is well defined since $\beta \neq 0$). Consequently $[\sigma^{(i_0)}(0), A] = [\pi^{(r_\beta - 1)}(\beta), 0, \sigma^{(m)}(B)] \in P(F)$. On the other hand $[\beta, 0, \sigma^{(m+1)}(B)] \in P(F)$ implies

$$[\sigma^{(n-1)}(\beta), \sigma^{(n-1)}(0), \sigma^{(n+m)}(B)] = [0, \sigma^{(n-1)}(0), \sigma^{(m)}(B)] \in P(F).$$

Thus,

$$F[\pi^{(r_\beta - 1)}(\beta), 0, \sigma^{(m)}(B)][0, \sigma^{(n-1)}(0), \sigma^{(m)}(B)] = [\pi^{(r_\beta)}(\beta), 0, \sigma^{(m+1)}(B)]$$
$$= [0, 0, \sigma^{(m+1)}(B)] \in P(F)$$

since $0 \neq \sigma^{(n-1)}(0)$ and $\pi^{(r_\beta)}(\beta) = 0$ by definition. This is impossible though, since it contradicts Case 1.1.2.

Case 1.2. Suppose for all m, $[0, 0, \sigma^{(m)}(B)] \notin P(F)$. By the induction hypothesis there exist α and $\beta \in I$ such that $[0, \alpha, B] \in P(F)$ and $[\beta, 0, B] \in P(F)$. Thus, we have:

$$[0, \pi(\alpha), \sigma(B)] \in P(F), \qquad [\pi(\beta), 0, \sigma(B)] \in P(F),$$
$$[0, \pi^{(2)}(\alpha), \sigma^{(2)}(B)] \in P(F), \quad [\pi^{(2)}(\beta), 0, \sigma^{(2)}(B)] \in P(F),$$
$$\cdots$$
$$[0, \pi^{(i)}(\alpha), \sigma^{(i)}(B)] \in P(F), \quad [\pi^{(i)}(\beta), 0, \sigma^{(i)}(B)] \in P(F).$$

We can continue this argument until, for the first time, i reaches one of the values r_α, r_β. When this happens, say for r_α first (the argument for r_β is identical), we have

$$[0, \pi^{(r_\alpha)}(\alpha), \sigma^{(r_\alpha)}(B)] = [0, 0, \sigma^{(r_\alpha)}(B)] \in P(F)$$

which contradicts Case 1.2. We are left with

Case 2. Suppose $b_i \neq b_j$ for $i \neq j$. Let us write $[b_1, b_2, \ldots, b_{k+1}]_S$ as $[\beta, B]$. Since we are assuming $[\beta, B] \notin P(F)$ then we must have

$$[\sigma^{(s_\beta)}(\beta), \sigma^{(s_\beta)}(B)] = [0, \sigma^{(s_\beta)}(B)] \notin P(F).$$

By the induction hypothesis there exists $\alpha \in I$ such that $[\alpha, \sigma^{(s_\beta)}(B)] \in P(F)$. (Note that $\alpha \neq 0$.) Consequently,

$$[\sigma^{(s_\alpha)}(\alpha), \sigma^{(s_\alpha + s_\beta)}(B)] = [0, \sigma^{(s_\alpha + s_\beta)}(B)] \in P(F).$$

Let t be the greatest integer $<n$ for which $[0, \sigma^{(t+s_\beta)}(B)] \in P(F)$. We note that $1 \leq s_\alpha \leq t < n$ and $[0, \sigma^{(t+1+s_\beta)}(B)] \notin P(F)$.

Case 2.1. Suppose there exists $z \neq 0$ such that $[z, \sigma^{(t+s_\beta)}(B)] \in P(F)$. Then

$$F[0, \sigma^{(t+s_\beta)}(B)][z, \sigma^{(t+s_\beta)}(B)] = [0, \sigma^{(t+1+s_\beta)}(B)] \in P(F),$$

which is a contradiction.

Case 2.2. Suppose that for all $z \neq 0$, $[z, \sigma^{(t+s_\beta)}(B)] \notin P(F)$. If we write $[z, \sigma^{(t+s_\beta)}(B)]$ as $[z, c_2, \ldots, c_{k+1}]$ then the c_i are *distinct* (since the b_i are). However, if $[z, c_2, \ldots, c_{k+1}] \notin P(F)$ for *all* $z \neq 0$ then *all the* c_i *must be* 0 (since by Case 1, we know that for all S of cardinality $k + 1$ and all $a_i \in I$, $1 \leq i \leq k + 1$, such that $a_i = a_j$ for some $i \neq j$, we have $[a_1, a_2, \ldots, a_{k+1}]_S \in P(F)$). Hence *we must have* $k = 1$ *and* $c_2 = 0$.

Let us take stock of what has been established thus far. Assuming (A_k), we have shown that *if* there exists an $S \subseteq \{1, 2, \ldots, n^2\}$ of cardinality $k + 1$, and $a_i \in I$, $1 \leq i \leq k + 1$, such that $[a_1, \ldots, a_{k+1}]_s \notin P(F)$ *then* $k = 1$ and $[z, 0]_s \notin P(F)$ for all $z \neq 0$. However, this is impossible. For suppose $S = \{t_1, t_2\}$ and let p_i and q_i denote the values of the variables p and q, respectively, at the t_ith position, $i = 1, 2$. (Explicitly we have $p_i = [t_i/n]$ and $q = t_i - np_i$ where $[x]$ denotes the greatest integer $\leq x$.) Since $[p_1, p_2]_S \in P(F)$ and $[q_1, q_2]_S \in P(F)$ then

$$[\sigma^{(s_{p_2})}(p_1), \sigma^{(s_{p_2})}(p_2)]_S = [\sigma^{(s_{p_2})}(p_1), 0]_S \in P(F)$$

and

$$[\sigma^{(s_{q_2})}(q_1), \sigma^{(s_{q_2})}(q_2)]_S = [\sigma^{(s_{q_2})}(q_1), 0]_S \in P(F).$$

But at least one of $\sigma^{(s_{p_2})}(p_1)$ and $\sigma^{(s_{q_2})}(q_1)$ is not 0 since either $p_1 \neq p_2$ or $q_1 \neq q_2$. Thus, for any $S = \{t_1, t_2\}$, there exists $z \neq 0$ such that $[z, 0]_S \in P(F)$, and the contradiction to Case 2.2 is established. This concludes Case 2. We have shown that if $1 \leq k < n^2$ and (A_k) holds then (A_{k+1}) holds. This completes the induction step and the proof of the theorem.

Concluding remarks. It is easy to enumerate the number of distinct $F_{\sigma,\pi}$ we can form. Since in the definition of π the choice of $\pi(0)$ does not affect $F_{\sigma,\pi}$ then there is a 1:1 correspondence between possible mappings π and labeled rooted trees with n distinct points (with the root labeled 0). Hence, there are n^{n-2} choices for π (cf. [5]). There are $(n - 1)!$ choices for σ. Of course, there was nothing significant about the choice of 0 in the definition of F (i.e., we could have used "for all $a \in I$, $\pi^{(x)}(a) = 1$ for some x, $F1a = 1$ for $a \neq 1, \ldots$" etc.). Hence, if $T(n)$ denotes the number of n-valued functionally complete truth functions of two variables, we have shown

$$T(n) \geq n^{n^2-2n} \cdot n!$$

On the other hand, it is certainly necessary for F to be functionally complete that for each $a \in I$, $Faa \neq a$. For such an F, there are $n - 1$ choices for Faa for each $a \in I$. Hence there are exactly $(n - 1)^n \cdot n^{n^2-n}$ such F and we have

$$T(n) \leq (n - 1)^n \cdot n^{n^2-n} = \left(1 - \frac{1}{n}\right)^n \cdot n^{n^2}.$$

Consequently, for any $\varepsilon > 0$, if n is sufficiently large then $T(n) < ((1 + \varepsilon)/e)n^{n^2}$. The lower bound given for $T(n)$ is almost certainly quite far from best possible and, in fact, it may be true that $T(n)/n^{n^2}$ is bounded away from 0. It is known (cf. [1], [2]) that

$$T(2) = 2, \quad T(3) = 3774.$$

At this point it is tempting to conjecture that *if* for any 2-subset $S = \{t_1, t_2\} \subseteq \{1, 2, \ldots, n^2\}$ and any distinct $a, b \in I$, it is true that $[a, b]_S \in P(F)$, *then* F is functionally complete.

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