Unrolling residues to avoid progressions

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Mathematics is often described as the study of patterns, some of the most beautiful of which are periodic. In the integers these periodic patterns take the form of arithmetic progressions, i.e., a set of integers which are equally spaced, or have the same common difference. For example, $5, 11, 17, 23, 29$ is an arithmetic progression with 5 terms where consecutive terms have a difference of 6.

Suppose we take the numbers $1, 2, \ldots, 28$ and we divide these numbers into two sets, which we can represent by "coloring" the numbers red (r) and blue (*B*). We can then look at how many times we have three equally spaced terms all from the same set. In other words how many times can we find three equally spaced numbers which are monochromatic in either red or blue.

If we want to *maximize* the number of such occurrences then the obvious thing to do is to paint $1, 2, \ldots, 28$ either all red or all blue. Then every three equally spaced numbers is monochromatic, and in this case there are 182 such sets of these triples (see Observation 1).

We can also easily find the *average* number of such occurrences. This is the same as asking how many of these monochromatic triples would we expect if we colored by flipping a coin with one side marked r and the other *B*. If this is a fair coin then given any three equally spaced numbers the probability the coin would always land on the same side is $\frac{2}{8}$. By linearity of expectation we can conclude that on average we have $\frac{1}{4} \cdot 182 = 45.5$ of these equally spaced triples will be monochromatic.

What if we want to find the *minimum* number of these equally spaced monochromatic triples? By the random argument we know we can at least find one coloring with ≤ 45 of these, but we would hope to do better. For the case of $1, 2, \ldots, 28$ it turns out there is a unique(!) way to color these to minimize the number, namely:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
r	r	r	*B*	*B*	r	r	r	*B*	*B*	*B*	*B*	*B*	*B*	r	r	r	r	r	r	*B*	*B*	*B*	r	r	*B*	*B*	*B*

A careful count shows that there are 28 sets of three equally spaced occurrences of monochromatic triples. This pattern was found by an exhaustive computer search.

We want to reformulate this small problem in three ways. First, we want to look at longer sets of equally spaced terms. So we will look at k-term arithmetic progressions, or k-APs, which consist of the numbers $a, a+d, \ldots, a+(k-1)d$, with $d \geq 1$; in the case that $d = 0$ this becomes $a, a, \ldots, a$ which will be called a trivial k-AP. Secondly, instead of only using the two colors of red and blue we want to consider $r \geq 2$ colors. Finally, we are interested in what happens as n is large (28 is a beautiful number, but most interesting things happen for much larger numbers).

In particular, we are interested in exploring the following question where k and r are fixed and n is very large.

Question. *What is the minimum number of monochromatic k-APs that occur in a coloring of the numbers $1, 2, \dots, n$ with r colors? Further, what can be said about such a coloring achieving the minimum?*

van der Waerden's Theorem

The best case scenario would be if there was a way to color $1, 2, \dots, n$ so that there were no monochromatic k-APs. However, a classical result of van der Waerden shows that this is impossible as n gets large.

Theorem 1 (van der Waerden [6, 9])**.** *For any number r of colors and k of length of arithmetic progressions there is a threshold N so that if $n \geq N$ then* any *coloring of $1, 2, \dots, n$ using r colors must have a monochromatic arithmetic progression of length k.*

For the case $k = 3$ and $r = 2$ we can color 12345678 as BBrrBBrr and not have any three numbers which form a 3-AP all red or all blue. But for any coloring of $1, 2, \dots, 9$ there will always be some such triple of numbers which form a 3-AP and is monochromatic. So in this case $N = 9$.

As a direct consequence of Theorem 1, for large n we must have at least one monochromatic k-AP when using r colors. Actually, not only must there be one, there must be many of these k-APs which are monochromatic.

Theorem 2 (Frankl, Graham and Rödl [3])**.** *For fixed r and k, there is $c > 0$ so that the number of monochromatic k-APs in any r-coloring of $1, 2, \dots, n$ is at least $cn^2 + o(n^2)$.*

In this theorem and throughout the paper we will use little-"o" and big-"O" notation which are used to bound the size of the lower order terms. In general, we say that $f(n) = O\big(g(n)\big)$ to indicate that for some constants C and n_0 then for all $n \geq n_0$, $|f(n)| \leq C|g(n)|$, i.e., the size of f is bounded by the size of g; we say that $f(n) = o\big(g(n)\big)$ if for each $\varepsilon > 0$ there is a sufficiently large n_0 so that if $n \geq n_0$ then $|f(n)| \leq \varepsilon|g(n)|$, i.e., the size of f is eventually much smaller than the size of g. So the preceding corollary shows that asymptotically the number of monochromatic k-APs will be at least cn^2. (A more substantial introduction to this notation can be found in many places including Graham, Knuth and Patashnik [5, Ch. 9].)

Before we give a short sketch of the proof of the theorem, we will make an observation which we will use repeatedly throughout the paper.

Observation 1. *The number of k-APs in $1, 2, \dots, n$ is*

$$\frac{(n-a)(n-k+1+a)}{2(k-1)} = \frac{n^2}{2(k-1)} + O(n),$$

where $n = (k-1)\ell + a$ *and* $0 \le a < k-1$.

Proof. We count the number of k-APs. This is simply done by noting that the starting points for a k-AP with sizes of step d are $1, 2, \dots, n - d(k-1)$. Since $1 \le d \le \ell$ we have that the number of k-APs is

$$\begin{aligned}
\underbrace{n-(k-1)}_{d=1} + \underbrace{n-2(k-1)}_{d=2} + \cdots + \underbrace{n-\ell(k-1)}_{d=\ell} &= \frac{\ell\big(n-(k-1)+n-\ell(k-1)\big)}{2} \\
&= \frac{(n-a)(n-k+1+a)}{2(k-1)} \\
&= \frac{n^2}{2(k-1)} + \underbrace{\frac{(1-k)n+ak-a-a^2}{2(k-1)}}_{=O(n)}
\end{aligned}$$

In the first line we used the old trick of reversing the sum and adding it to itself. There are $\ell = (n-a)/(k-1)$ terms in the sum. The rest involves simple manipulations. □

Sketch of proof of Theorem 2. We will let N be as in Theorem 1 for the given r and k. Now for large n we will have inside of $1, 2, \dots, n$ *many* arithmetic progressions of length N, in particular we will have $\approx n^2/2N$. By Theorem 1, inside of each one of these progressions of length N will be a monochromatic progression of length k. We have now found many monochromatic k-APs, but need to correct for possible double counting. To do this we note that any monochromatic progression could have been counted at most $\binom{N}{2}$ times. Therefore we have at least $\approx n^2/N^3$ monochromatic progressions. □

Note that we made use of the fact that a shifted arithmetic progression is still an arithmetic progressions, one of the reasons why these are so nice to study.

By combining Observation 1 and Corollary 2 we now conclude that we must have a strictly positive portion of the k-APs in our coloring be monochromatic as n gets large.

On the other hand the bound given in Corollary 2 is a poor bound. We gave a lot away in our counting. Further, the N from Theorem 1 is hard to compute. The best bound for N in the case when we have $r = 2$ colors and are looking at arithmetic progression of length k was given by Tim Gowers [4],

$$N \le 2^{2^{2^{2^{2^{k+9}}}}}.$$

Ron Graham currently offers \$1000 to show that for this case $N \le 2^{k^2}$

Color randomly

A natural first guess for finding a coloring with a small portion of monochromatic k-APs is to color "randomly", i.e., as in our initial problem to flip a coin and assign a value. By looking over all the colorings we can find the expected number of arithmetic progressions in a typical coloring; some coloring must have at most that many. The advantage of this technique is in its simplicity to produce bounds. Moreover, there are examples of combinatorial problems when a random construction is best. Indeed for our question we are considering, the random construction was the best known construction for decades.

Lemma 1. *For n large and r, k fixed, there is a coloring of $1, 2, \ldots, n$ with r colors which has at most*

$$\frac{1}{2(k-1)r^{k-1}}n^2 + O(n)$$

monochromatic k-APs. In particular, $1/r^{k-1}$ portion of the k-APs will be monochromatic.

Proof. To establish this we will simply count the total number of monochromatic k-APs over all colorings and then take the average (i.e., divide by r^n, the number of colorings). Some coloring must have at most the average number which will then give us the bound.

For a moment, focus on a single k-AP. There are r ways to color that k-AP monochromatically and we can extend each such monochromatic coloring of that k-AP to a coloring of $1, 2, \ldots, n$ by r^{n-k} ways. Therefore each k-AP will contribute $r \cdot r^{n-k}$ to the sum total number of monochromatic k-APs over all possible colorings.

On the other hand by Observation 1 there are $\big(1/2(k-1)\big)n^2 + O(n)$ k-APs in $1, 2, \ldots, n$ so that the total number of monochromatic k-APs over all possible colorings is

$$\left(\frac{1}{2(k-1)}n^2 + O(n)\right)r^{n-k+1}.$$

Dividing this by r^n to get the average number of k-APs now establishes the result. $\square$

Beating random colorings for 3-APs

We can use the coloring found at the beginning for $n = 28$ and "blow it up" to find a coloring for larger values of n. One way to do this is to simply to repeat each color m times to get a new coloring of $28m$. So for example when $m = 2$ our initial coloring becomes

rrrrrr*BBBB*rrrrrr*BBBBBBBBBBBB*rrrrrrrrrrrr*BBBBBB*rrrr*BBBBBB*.

This gives a coloring of $28m$ which we can then simply edit, i.e., drop the last few terms, to get a coloring for n. (Note that dropping ≤ 27 terms will effect at most $27n$ arithmetic progressions which is a lower order term, i.e., it will not significantly effect the portion which are monochromatic when n is large.) A computation shows that this gives $\frac{3}{56}n^2 + O(n)$ monochromatic 3-APs, or 21.42% of the 3-APs will be monochromatic where as the random bound is 25%. (These computations are performed by turning the discrete problem into a continuous version where the coefficients are found by computing area and then doing local optimization to minimize; more information on this procedure is in Butler, Costello and Graham [2].)

The best known coloring for minimizing monochromatic 3-APs uses the same approach, but a larger starting pattern to get 21.35% of the 3-APs monochromatic. This was discovered independently by two different groups at approximately the same time.

Theorem 3 (Parrilo-Robertson-Saracino [8]; Butler-Costello-Graham [2])**.** *Start with the following coloring of* $1, 2, \ldots, 548$ *and then blow it up by repeating each term with its color* $\ell = \lceil n/548 \rceil$ *times and then deleting any excess terms to get a coloring of* $1, 2, \ldots, n$*.*

$$\underbrace{\mathtt{r}\cdots\mathtt{r}}_{28}\underbrace{B\cdots B}_{6}\underbrace{\mathtt{r}\cdots\mathtt{r}}_{28}\underbrace{B\cdots B}_{37}\underbrace{\mathtt{r}\cdots\mathtt{r}}_{59}\underbrace{B\cdots B}_{116}\underbrace{\mathtt{r}\cdots\mathtt{r}}_{116}\underbrace{B\cdots B}_{59}\underbrace{\mathtt{r}\cdots\mathtt{r}}_{37}\underbrace{B\cdots B}_{28}\underbrace{\mathtt{r}\cdots\mathtt{r}}_{6}\underbrace{B\cdots B}_{28}$$

Then the number of monochromatic three term arithmetic progressions in such a coloring is $\frac{117}{2192}n^2 + O(n)$*.*

For large n the resulting coloring will look like what is shown in Figure 1.

Figure 1: The best known 2-coloring of $1, 2, \ldots, n$ to minimize monochromatic 3-APs.

Butler, Costello and Graham [2] used a computer search to look for similar ways to color $1, 2, \ldots, n$ by subdividing into large blocks to minimize monochromatic 4-APs and 5-APs. For 4-APs they found a coloring where 10.33% of the 4-APs were monochromatic (random gives 12.5%). Another coloring gave 4.57% of the 5-APs being monochromatic (random gives 6.25%).

Unrolling better solutions

Another method to blow up the coloring from the beginning is to simply place m copies of it over and over to get a new coloring of $28m$, then as before we can edit any excess. We will call this process *unrolling* (see Figure 3 below for an explanation of the term). So for example when $m = 2$ our initial coloring becomes

$$\mathtt{rrr}BB\mathtt{rrr}BBBBBB\mathtt{rrrrrr}BBB\mathtt{rr}BBB\mathtt{rrr}BB\mathtt{rrr}BBBBBB\mathtt{rrrrrr}BBB\mathtt{rr}BBB.$$

The surprising thing is that even though this is a great coloring of $1, 2, \ldots, 28$, the unrolling is *not* a great coloring of $1, 2, \ldots, n$. In particular, this coloring will yield $n^2/4 + O(n)$ monochromatic 3-APs, the same as random! Moreover, no coloring found by unrolling a fixed coloring of $1, 2, \ldots, \ell$ will do better than random for avoiding 3-APs.

For larger values of k the situation is dramatically different. This was first noted in the paper of Lu and Peng [7]. Their original motivation was to find a way to minimize the number of monochromatic 4-APs in colorings of $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ where we can do arithmetic operations using modular arithmetic. Lu and Peng also showed the same approach works for coloring $1, 2, \ldots, n$. Their idea was to unroll a special coloring of $\mathbb{Z}_{11}$, shown in Figure 2 with a bit that can be arbitrarily colored. This coloring had only trivial monochromatic 4-APs. As a result, when unrolled, as in Figure 3, this would cause any monochromatic 4-AP to have a large gap between consecutive terms, and hence there would be relatively few of these monochromatic 4-APs.

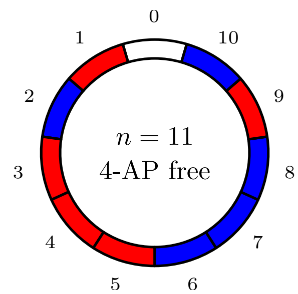

Figure 2: A coloring of $\mathbb{Z}_{11}$ with only trivial monochromatic 4-APs; the white bit can be either color.

There is further improvement in that we still have freedom to color the white squares that showed up in the unrolling. To do this we simply recurse and unroll along the white squares the same pattern we did initially. This is much easier to do than it might seem at first glance, namely the coloring is equivalent to doing the following: Given any number ℓ write it in base 11, i.e.,

$$\ell = \sum_i b_i \cdot 11^i \qquad \text{where } 0 \leq b_i \leq 10.$$

Let j be the smallest index so that $b_j \neq 0$,

$$\text{color } \ell \quad \begin{cases} \text{red} & \text{if } b_j = 1, 3, 4, 5, \text{ or } 9; \\ \text{blue} & \text{if } b_j = 2, 6, 7, 8, \text{ or } 10. \end{cases}$$

It will follow from Theorem 4 below that this coloring has $\frac{1}{72}n^2 + O(n)$ monochromatic 4-APs, so that 8.33% of the 4-APs will be monochromatic. For comparison, when we color randomly we have $\frac{1}{48}n^2 + O(n)$ monochromatic 4-APs, so that 12.5% of the 4-APs will be monochromatic.

This coloring is far superior to the block coloring referred to earlier, and also far easier to describe and to implement. Indeed we can color all of $\mathbb{N}$ and then just take the first n terms with this coloring; in a block coloring such as in Figure 1 this would be impossible.

This idea of unrolling works for any pattern. What we need is a pattern where the unrolling produces few monochromatic k-APs. We will focus on colorings of $\mathbb{Z}_m$ which have

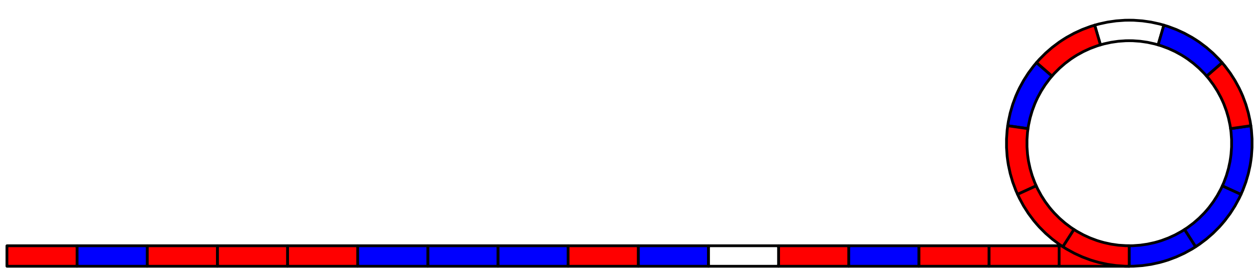

Figure 3: Unrolling the coloring of $\mathbb{Z}_{11}$ to find a coloring of $1, 2, \ldots, n$.

only trivial monochromatic k-APs. In this case we get the following count for the number of monochromatic k-APs in an unrolling.

Theorem 4. *If there is a coloring of $\mathbb{Z}_m$ with r colors which have only trivial monochromatic k-APs, then for n large there is an r-coloring of $1, 2, \ldots, n$ which has*

$$\frac{1}{2m(k-1)}n^2 + O(n)$$

monochromatic k-APs.

If there are r different colorings of $\mathbb{Z}_m$ with r colors which have only trivial monochromatic k-APs and these differ only in the coloring of 0, then for n large there is an r-coloring of $1, 2, \ldots, n$ which has

$$\frac{1}{2(m+1)(k-1)}n^2 + O(n)$$

monochromatic k-APs.

Proof. Any monochromatic k-AP can be rolled back into a monochromatic k-AP of $\mathbb{Z}_m$. Since we only have trivial k-APs in our coloring of $\mathbb{Z}_m$ it follows that the difference between any consecutive term in our arithmetic progression is a multiple of m.

In the first case, when unrolling we color according to the residue class in the coloring of $\mathbb{Z}_m$. In particular, the residue completely determines the color and so we can simply count our coloring by counting what happens on each of the m residue classes. Each residue consists of a solid coloring of length $\frac{n}{m} + O(1)$. Using Observation 1, each of these m residue classes will contribute $\frac{n^2}{2m^2(k-1)} + O(n)$ monochromatic k-APs. This establishes the first case.

In the second case, we will do recursive unrolling (similar as in the 4-AP case). Namely for $i = 1, 2, \ldots, r$ let C_i be the nonzero elements of $\mathbb{Z}_m$ colored with color i. For each ℓ we find the first nonzero digit in the base m expansion of ℓ and color ℓ the ith color if the digit is in C_i. Initially when we subdivide by residue classes then as before $m-1$ of the residue classes will be colored monochromatically. The residue class for 0 will look like the unrolling for the coloring of $\frac{n}{m} + O(1)$.

If we let $F(n)$ be the number of monochromatic k term arithmetic progressions then the above analysis shows that

$$F(n) = F\left(\frac{n}{m}\right) + (m-1)\frac{1}{2(k-1)}\left(\frac{n}{m}\right)^2 + O(n).$$

Iteratively applying this we have

$$\begin{aligned} F(n) &= \frac{(m-1)n^2}{2(k-1)} \sum_{i \geq 1} \left(\frac{1}{m^2}\right)^i + O(n) \\ &= \frac{(m-1)n^2}{2(k-1)} \cdot \frac{1}{m^2-1} + O(n) \\ &= \frac{1}{2(m+1)(k-1)}n^2 + O(n). \end{aligned}$$

□

Using residues to color

To be able to apply Theorem 4 we must find a coloring of $\mathbb{Z}_m$ that only contains trivial arithmetic progressions of length k. Furthermore, given a k we want m to be as large as possible.

A close examination of the coloring of $\mathbb{Z}_{11}$ reveals that the nonzero elements colored red, $\{1, 3, 4, 5, 9\}$, are precisely the nonzero *quadratic residues* of $\mathbb{Z}_{11}$, i.e., values r for which there is some $x \in \mathbb{Z}_{11}$ satisfying $x^2 = r$. Similarly, the nonzero elements colored blue, $\{2, 6, 7, 8, 10\}$ are the non-residues.

This suggests that we can look at colorings of $\mathbb{Z}_p$ where p is a prime and we use the residues to help us determine the coloring. This has two large advantages.

- Suppose that $a, a+d, \ldots, a+(k-1)d$ is a nontrivial monochromatic k-AP, i.e., $d \neq 0 \pmod{p}$. Then we can multiply each term by d^{-1} to get the arithmetic progression $ad^{-1}, ad^{-1}+1, \ldots, ad^{-1}+(k-1)$ consisting of k consecutive elements of $\mathbb{Z}_p$. Further since we started with a sequence which consisted either of only residues or non-residues the resulting sequence must also consist of only residues or non-residues. (This is because if d^{-1} is a residue then multiplication by d^{-1} sends residues to residues and non-residues to non-residues; a similar situation occurs if d^{-1} is not a quadratic residue.)

  This shows that to determine the length of the longest arithmetic progression in a coloring by quadratic residues it suffices to find the longest run of residues or non-residues. This is a tremendous speedup and makes large computer searches possible.

- It is known that for $\mathbb{Z}_p$ that there are no long runs of consecutive residues or non-residues. In particular, Burgess [1] showed that the maximum number of consecutive quadratic residues or non-residues for a prime p is $O\big(p^{1/4}(\log p)^{3/2}\big)$.

  (The result of Burgess is not strong enough to guarantee the type of colorings we need for large k. If we compare the random bound with Theorem 4 then we need the length of the longest monochromatic progression in $\mathbb{Z}_p$ to be $\leq \log_2(p)$ to yield a coloring which is better than random for some k. On the other hand the result of Burgess is a general bound and does not preclude the possibility that for some (possibly most) primes p the length of the longest run might be smaller than $\log_2(p)$.)

When we want to use more than two colors we can use higher order residues in place of quadratic residues to look for colorings which have only trivial monochromatic k-APs. In general, suppose p is a prime, so that $\mathbb{Z}_p^*$ (the invertible elements of $\mathbb{Z}_p$) is a group of order $p-1$. Further if $r \big| (p-1)$ then there is a unique subgroup of index r, namely

$$S = \{x^r \mid x \in \mathbb{Z}_p, x \neq 0\}.$$

If there are only trivial monochromatic k-APs in $S \cup \{0\}$ then there can only be trivial k-APs in $yS \cup \{0\}$ for any $y \in \mathbb{Z}_p^*$ (i.e., otherwise if we multiply the progression by y^{-1} we find a nontrivial k-AP in $S \cup \{0\}$, a contradiction). In particular by choosing $y_1 = 1, y_2, \ldots, y_r$ so that the collection of $y_i S$ form the cosets of $\mathbb{Z}_p^*/S$ we form a coloring of $\mathbb{Z}_p$ with r colors which has no k-AP regardless of the color assignment of 0. We have now established the following.

Theorem 5. *Suppose that p is an odd prime and $r\big|(p-1)$. If $\{x^r : x \in \mathbb{Z}_p\}$ contains only trivial k-APs, then there is a coloring with r colors where* 0 *can be colored arbitrarily and which contains only trivial monochromatic k-APs.*

In some cases we can combine two good colorings to form a larger coloring.

Theorem 6. *For $i = 1, 2$, let $\mathcal{C}_i$ be a coloring of $\mathbb{Z}_{m_i}$ using r_i colors where* 0 *can be colored arbitrarily and containing no nontrivial k-APs. Then there exists a coloring $\mathcal{C}$ of $\mathbb{Z}_{m_1m_2}$ using r_1r_2 colors where* 0 *can be colored arbitrarily and containing no nontrivial k-APs.*

Proof. We show how to color $\mathbb{Z}_{m_1m_2}$ using the colors (c_1, c_2) where c_1 is a color from $\mathcal{C}_1$ and c_2 is a color from $\mathcal{C}_2$. Given $0 \le z < m_1m_2$ it can uniquely be written as $xm_2 + y$ where $0 \le x < m_1$ and $0 \le y < m_2$. This gives a map from $\mathbb{Z}_{m_1m_2} \to \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$ with the following property: If $z_1 \mapsto (x_1, y_1)$ and $z_2 \mapsto (x_2, y_2)$ then $z_1 + z_2 \mapsto (x', y_1 + y_2 \pmod{m_2})$.

The coloring is now defined by the following: Given $z \in \mathbb{Z}_{m_1m_2}$ we color it as $\big(c_1(x), c_2(y)\big)$ where $c_1(x)$ is the color of x in $\mathcal{C}_1$ and $c_2(y)$ is the color of y in $\mathcal{C}_2$.

It remains to show that this has no nontrivial monochromatic k-APs. Suppose that $z_1, z_2, \ldots, z_k$ were a monochromatic k-AP where $z_i \mapsto (x_i, y_i)$. Then $y_1, y_2, \ldots, y_k$ is a monochromatic k-AP in $\mathcal{C}_2$. But the only way this is possible is if $y_1 = y_2 = \cdots = y_k$.

We can think of a k-AP as found by sewing together several 3-APs. So we have for $2 \le i \le k-1$ (with what we have about the equality of the y_i)

$$0 = z_{i-1} - 2z_i + z_{i+1} = m_2(x_{i-1} - 2x_i + x_{i+1}) \pmod{m_1m_2};$$

which implies

$$x_{i-1} + x_{i+1} = 2x_i \pmod{m_1}.$$

This shows that we now have a sequence of 3-APs glued together in $\mathbb{Z}_{m_1}$, i.e., $x_1, x_2, \ldots, x_k$ is a monochromatic k-AP in $\mathbb{Z}_{m_1}$. But the only way this is possible is if $x_1 = x_2 \cdots = x_k$.

We now conclude that $z_1 = z_2 = \cdots = z_k$, i.e., there are no nontrivial monochromatic k-APs. Finally we note that our argument does not depend on how we choose to color 0. $\square$

By computer search we can now find several colorings by either using residues (as in Theorem 5) or combinations of colorings from residues (as in Theorem 6). The results of the search are shown in Table 1. We indicate the best known value of m. In each case the resulting number of monochromatic k-APs is $\frac{1}{2(m+1)(k-1)}n^2 + O(n)$; the resulting portion of monochromatic k-APs is thus found by looking at $\frac{1}{m+1}$. We note that for each entry this is the *best* known coloring for minimizing the number of monochromatic k-APs in a coloring of $1, 2, \ldots, n$ using r colors, i.e., we significantly beat the random bound.

m	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$
$k=3$			37		103	
$k=4$	11	97	349	751	3259	1933
$k=5$	37	241	2609	6011	14173	30493
$k=6$	139	1777	139^2	49391	$139 \cdot 1777$	317969
$k=7$	617	7309	617^2	230281	$617 \cdot 7309$	
$k=8$	1069	34057	1069^2			
$k=9$	3389	116593				
$k=10$	11497	463747				
$k=11$	17863					
$k=12$	58013					
$k=13$	136859					
$k=14$	239873					
$k=15$	608789					
$k=16$	1091339					

Table 1: Best known m for which there exists a coloring using residues of $\mathbb{Z}_m$ with r colors where 0 can be colored arbitrarily and containing no k-APs.

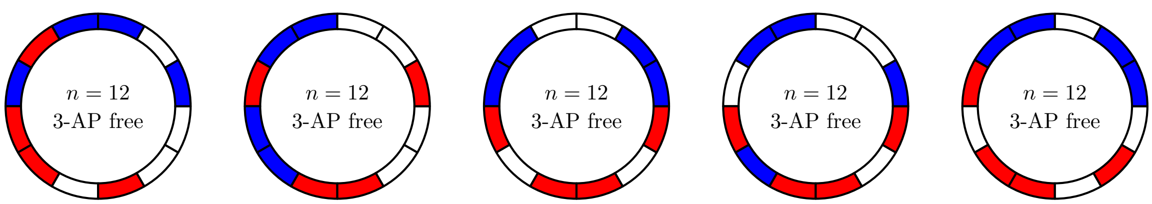

Figure 4: Colorings of $\mathbb{Z}_{12}$ with no nontrivial monochromatic 3-AP using three colors.

Open problems and variations

There is no reason why we must use residues for our colorings. As an example if we work with $r = 3$ then -1, 0 and 1 are always cubic residues and so we cannot avoid 3-APs. On the other hand there are colorings of $\mathbb{Z}_{12}$ using three colors (shown in Figure 4) which by Theorem 4 when unrolled have $\frac{1}{48}n^2+O(n)$ monochromatic 3-APs, so 8.33% of the colorings will be monochromatic, whereas in a random coloring we would expect 11.11% of the 3-APs to be monochromatic. We still have a lot to learn about ways to color $\mathbb{Z}_m$ beyond residues which give efficient unrollings.

Further, all of our work in answering our original question about finding the minimum number of monochromatic arithmetic progressions has been in constructing colorings which give upper bounds. Much less is known about the lower bounds, and in particular *no* coloring is currently known to be optimal. For the simplest nontrivial case when $r = 2$ and $k = 3$ the best known lower bound is due to Parrilo, Robertson and Saracino [8] who showed that the number of monochromatic 3-APs in a 2-coloring of $1, 2, \ldots, n$ lies in the interval

$$\frac{1675}{32768}n^2 + o(n^2) \approx 0.05111n^2 < 0.05338n^2 \approx \frac{117}{2192}n^2 + O(n).$$

Conjecture 1. *Each coloring of $1, 2, \ldots, n$ with two colors has at least $\frac{117}{2192}n^2 + O(n)$ monochromatic three term arithmetic progressions.*

In general we have provided some evidence in support of showing that we can always beat the random bound for minimizing the number of monochromatic k-APs. While this evidence is compelling we must be careful not to be misled by these small cases. We will need a more general machinery to establish the following.

Conjecture 2. *For each $k \geq 3$ and $r \geq 2$ there is a coloring of $1, 2, \ldots, n$ with r colors so that the minimum portion of monochromatic k-APs that occur is $cn^2 + o(n^2)$ where $c < \frac{1}{2(k-1)r^{k-1}}$.*

Besides considering ways to minimize the number of monochromatic k-APs one can also try to work with other patterns. For example, we can try to avoid $a, a+2d, a+3d, a+5d$ (which can be thought of as a 6-AP with some terms skipped over). In this case the best known coloring with two colors for unrolling is found by a coloring of $\mathbb{Z}_{13}$ and is shown in Figure 5 (as before the white bit can be arbitrary). Similar calculations to Theorem 4 will show that this unrolling has $\frac{1}{140}n^2 + O(n)$ monochromatic patterns; so that 7.14% of the patterns are monochromatic; a random coloring would give 12.5% of the patterns monochromatic.

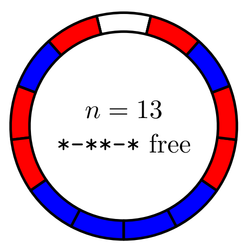

Figure 5: A coloring of $\mathbb{Z}_{13}$ with no nontrivial monochromatic `*-**-*` pattern; the white bit can be either color.

We have explored a small corner of the colorful world of avoiding monochromatic arithmetic progressions. There are many beautiful open problems and variations left to explore. We look forward to seeing more progress in this area.

Summary: We look at the problem of coloring $1, 2, \ldots, n$ with r colors to minimize the portion of monochromatic k-term arithmetic progressions. By using residues to color $\mathbb{Z}_m$ and then unrolling we produce the best known colorings for several small values of r and k.

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