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Erdős

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Paul Erdős and Egyptian Fractions

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One of Paul Erdős' earliest mathematical interests was the study of so-called *Egyptian fractions*, that is, finite sums of distinct fractions having numerator 1. In this note we survey various results in this subject, many of which were motivated by Erdős' problems and conjectures on such sums. This note complements the excellent treatment of this topic given by A. Schinzel in 2002.[1]

1. Introduction

The Rhind Papyrus of Ahmes [47] (see also [34, 63]) is one of the oldest known mathematical manuscripts, dating from around 1650 B.C. It contains among other things, a list of expansions of fractions of the form $\frac{2}{n}$ into sums of distinct *unit* fractions, that is, fractions with numerator 1. Examples of such expansions are $\frac{2}{35} = \frac{1}{30} + \frac{1}{42}$ and $\frac{2}{63} = \frac{1}{56} + \frac{1}{72}$. More generally, one can consider expansions of more general rational numbers into sums of unit fractions with distinct denominators such as:

$$\frac{10}{73} = \frac{1}{11} + \frac{1}{22} + \frac{1}{1606}, \quad \frac{67}{2012} = \frac{1}{31} + \frac{1}{960} + \frac{1}{2138469} + \frac{1}{10670447077440},$$

and

$$1 = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{14} + \frac{1}{15} + \frac{1}{18} + \frac{1}{20} + \frac{1}{24} + \frac{1}{28} + \frac{1}{30}.$$

There are various explanations as to why the Egyptians chose to use such representations (for example, see [63]) but perhaps the most compelling is that given to the author some years ago by the legendary mathematician

[1] See [52].

André Weil [62]. When I asked him why he thought the Egyptians used this method for representing fractions, he thought for a moment and then said, "It is easy to explain. *They took a wrong turn!*".

As is well known, Erdős' first major result (and first paper) was his beautiful 1932 proof [25] of Bertrand's postulate, namely that for any positive integer $n > 1$, there is always a prime between n and $2n$.[2] In particular, Erdős' proof was based in part on an analysis of the prime divisors of the binomial coefficients $\binom{2n}{n}$. What is perhaps less well known is that Erdős' second paper [26], also published in 1932, dealt with Egyptian fractions. In it, he generalizes an elementary result of Kürschák [41] by showing that for any choice of positive integers a, d and n, the sum $\sum_{k=1}^{n} \frac{1}{a+kd}$ is never an integer.[3]

The next paper of Erdős dealing with Egyptian fractions was his 1945 paper with I. Niven [29]. In that paper, they showed among other things that no two partial sums of the harmonic series can be equal, i.e., $\sum_{i=r}^{s} i^{-1} = \sum_{i=t}^{u} i^{-1}$ implies $r = t$ and $s = u$. In that paper they also showed that for only finitely many n can one or more of the elementary symmetric functions of $1, \frac{1}{2}, \ldots, \frac{1}{n}$ be an integer. Very recently, this was strengthened in a paper of Chen and Tang [17]. In that paper, they showed that the only pairs (k, n) for which the k^{th} elementary function $S(k, n)$ of $1, \frac{1}{2}, \ldots, \frac{1}{n}$ is an integer is $S(1, 1) = 1$ and $S(2, 3) = (1)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = 1$. Thus, for $n \geq 4$, none of the elementary functions are integers.

Perhaps the paper of Erdős dealing with Egyptian fractions which has had the greatest impact was his 1950 paper [27]. In this seminal paper, he considers the quantity $N(a, b)$, defined for integers $1 \leq a < b$ to be least value n such that the equation $\frac{a}{b} = \sum_{k=1}^{n} \frac{1}{x_k}$ has a solution with $0 < x_1 < x_2 < \ldots < x_n$. In particular, he shows that $N(b) = \max_{1 \leq a \leq b} N(a, b)$ satisfies $\log \log b \ll N(b) \ll \frac{\log b}{\log \log b}$, sharpening an earlier result of deBruijn and others. It is conjectured in [27] that $N(b) \ll \log \log b$. The best result in this direction at present is due to Vose [59] who showed that $N(b) \ll \sqrt{\log b}$.

[2] This was memoralized by Leo Moser's limerick: "Chebyshev said it and I'll say it again. There is always a prime between n and $2n$."

[3] Interestingly, Erdős states in the German abstract of that paper: "Der Grundgedanke des Beweises besteht darin, dass ein Glied $a + kd$ angegeben wird, welches durch eine höhere Potenz einer Primzahl teilbar ist, als die übrigen Glieder. Dies ergibt sich aus der Analyse der Primteiler der Ausdrücke of $\frac{(a+d)(a+2d)\ldots(a+nd)}{n!}$ and $\binom{2n}{n}$" (The basic idea of the proof is that some term $a + kd$ is divisible by a higher power of some prime than any other terms. This follows from the analysis of the prime divisors of the expressions $\frac{(a+d)(a+2d)\ldots(a+nd)}{n!}$ and $\binom{2n}{n}$).

It is also in this paper that the celebrated Erdős-Straus "$\frac{4}{n}$ conjecture" occurs, namely that $N(4,b) \leq 3$ for every $b > 2$. This will be the subject of the next section.

2. The Erdős-Straus Conjecture

The first proof that any positive rational $\frac{a}{b}$ has an Egyptian fraction representation:

$$\frac{a}{b} = \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}, \quad 1 \leq x_1 < x_2 < \ldots < x_n, \tag{1}$$

was given by Fibonacci (= Leonardo Pisano) in 1202 [32]. His method was to apply the *greedy algorithm,* namely always subtract the largest possible unit fraction from the current remainder so that the result is nonnegative. While this ordinarily does not produce the shortest possible representation, or the one with smallest maximum denominator, it does terminate in finitely many steps since eventually the numerator of the reduced remainder must strictly decrease at each step. In particular, for fractions of the form $\frac{2}{n}$ for $n > 1$, the greedy algorithm only needs 2 steps, and for $\frac{3}{n}$, it only needs 3 steps. While this algorithm would guarantee that for the fractions $\frac{4}{n}$, a representation with 4 unit fractions is guaranteed, Erdős and Straus [27] conjectured that in fact such a fraction always had an Egyptian fraction expansion with *at most 3* terms. It is easy to see that in order to prove this, it is enough to show that it holds for prime values of n. There have been many papers published studying various aspects of this problem (for example, see [1, 40, 48, 61, 60] and especially the references in [39]). For example, it is known that if the conjectures fails for some value n then n must be congruent to one of $1^2, 11^2, 13^2, 17^2, 19^2$ or $23^2 \pmod{840}$. From a computational perspective, the conjecture has been verified for $n \leq 10^{14}$ [57]. One of the most recent treatments is in a long paper of Elsholtz and Tao [24] (extending earlier work of Elsholtz [23]). Among their many results are the following. Let $f(n)$ denote the number of different solutions to the equation

$$\frac{4}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \tag{2}$$

where here the x_i are *not* assumed to be distinct or ordered by size. It is easy to see that the Erdős-Straus conjecture is that $f(n) > 0$ for $n > 1$. In [24], it is shown that :

(i) $N \log^2 N \ll \sum_{q \le N} f(q) \ll N \log^2 N \log\log N$ where q ranges over primes;

(ii) For any prime q,

$$f(q) \ll q^{\frac{3}{5}+O\left(\frac{1}{\log\log q}\right)}.$$

(iii) For infinitely many n, one has

$$f(n) \ge \exp\left((\log 3 + o(1))\frac{\log n}{\log\log n}\right).$$

In particular, it follows from this that there are relatively few solutions to (2) for most n. However, Vaughan [58] has shown that the number of $n \le x$ for which the Erdős-Straus conjectures fails is $O(x \exp(-c(\log x)^{\frac{2}{3}}))$, $c > 0$. As of this writing, the original conjecture of Erdős and Straus is still unresolved.[4]

Motivated by the Erdős-Straus conjecture, Sierpiński [55] made the analogous conjecture[5] for the fractions $\frac{5}{n}$, namely, that for all $n \ge 5$, there is a decomposition:

$$\frac{5}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}, \quad 1 \le x_1 < x_2 < x_3.$$

This has been verified for $5 \le n \le 1057438801$ (see [39]). More generally, Schinzel (also in [55]) conjectured that for any fraction $\frac{a}{n}$, one can express it as:

$$\frac{a}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}, \quad 1 \le x_1 < x_2 < x_3,$$

provided $n > n_0(a)$. Needless to say, these conjectures are currently still unsettled.

[4] As a historical note, this conjecture also occurred around the same time in a paper of Obláth [46] (submitted for publication in 1948) in which the constraint that the x_i be distinct is relaxed.

[5] It is curious why Erdős and Straus didn't make this conjecture in [27] as well.

3. Dense Egyptian Fractions

In [27], Erdős also considers various questions relating to Egyptian fraction decompositions of $1 = \sum_{k=1}^{n} \frac{1}{x_k}$. In particular, he conjectures that we must always have $\frac{x_n}{x_1} \geq 3$, with the extreme example coming from the decomposition $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$. In fact, he suggests that it may even be true that $\lim_{n\to\infty} \frac{x_n}{x_1} = \infty$. However, it is now known that this is not the case. It follows from the work of Martin [43, 44] and Croot [18, 19] that the following holds.

Theorem 1 [18]**.** *Suppose that $r > 0$ is a given rational number. Then for all $N > 1$, there exist integers $x_1, x_2, \ldots, x_k$, with*

$$N < x_1 < x_2 < \ldots < x_k \leq \left(e^r + O_r\left(\frac{\log\log N}{\log N}\right)\right) N$$

such that

$$r = \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_k}.$$

Moreover, the error term $O_r\big(\frac{\log\log N}{\log N}\big)$ is best possible.

This result settled one of the many questions raised in Chapter 4 (Unit Fractions) of the booklet [28] of Erdős and the author.

Another question raised in [28] and answered by Martin [44] deals with the quantity $L_j(s)$ defined for a positive rational s by

$$L_j(s) = \Bigg\{x \in \mathbf{Z},\ x > s^{-1} : \text{ there do not exist}$$

$$x_1, \ldots, x_t \in \mathbf{Z},\ x_1 > \ldots > x_t \geq 1 \text{ with } \sum_{i=1}^{t} \frac{1}{x_i} = s \text{ and } x_j = x\Bigg\}.$$

The largest denominator in an Egyptian fraction representation of s can be a prime only if it is a prime divisor of s. Hence the set $L_1(s)$ contains most primes and it is clearly infinite. However, $L_1(s)$ must have zero density as dictated by the following result [44]:

Let $L_1(s; x)$ denote the counting function of $L_1(s)$, i.e.,

$$L_1(s, x) = |\{1 \leq n \leq x : n \in L_1(x)\}|.$$

Then for any rational $s > 0$ and any real $x \geq 3$, we have;

$$\frac{x \log\log x}{\log x} \ll_s L_1(s,x) \ll_s \frac{x \log\log x}{\log x}.$$

However, for $j \geq 2$, the situation is quite different. In fact, for any $j \geq 2, L_j(s)$ is finite. In particular, there are only finitely many numbers which cannot be the second-largest denominator in an Egyptian fraction representation of 1. Martin suggests that perhaps the set $\{2,4\}$ is the complete list (of those greater than 1).

4. More Problems From *Old and New Problems and Results* [28]

(Many of the problems and results in this section are taken more or less directly from the above mentioned book. The reader can consult [28] for more details).

It is known that any positive rational $\frac{a}{2b+1}$ can be represented as a finite sum of the form $\sum_k \frac{1}{2q_k+1}$ (e.g., see [3, 9, 56]. An old question of Stein [53] asks if such a decomposition can always be accomplished by the greedy algorithm. In other words, if we start with an arbitrary positive rational $\frac{a}{2b+1}$ and repeatedly subtract the largest unit fraction $\frac{1}{2q+1}$ so that the remainder is nonnegative, must this process always terminate? No examples are known which provably do not terminate, although there are terminating rationals for which the denominators become very large. For example, starting with $\frac{5}{1444613}$, the greedy algorithm takes 37 terms to terminate, with the largest denominator having 384,122,451,172 decimal digits (see [45]). It is known [36] that a positive rational $\frac{a}{b}$ can be expressed as a finite sum of fractions of the form $\frac{1}{pk+q}$ if and only if $\left(\frac{b}{(b,(p,q))}, \frac{p}{(p,q)}\right) = 1$. One could ask here whether the greedy algorithm always terminates for this representation as well. Restricting the denominators even more, the author has shown [37] that a necessary and sufficient condition that a rational $\frac{a}{b}$ can be expressed as

$$\frac{a}{b} = \frac{1}{x_1^2} + \frac{1}{x_2^2} + \ldots + \frac{1}{x_k^2} \quad \text{for positive integers} \;\; 0 < x_1 < x_2 < \ldots < x_k,$$

is that

$$\frac{a}{b} \in \left[0, \frac{\pi^2}{6} - 1\right) \cup \left[1, \frac{\pi^2}{6}\right).$$

For example,

$$\frac{1}{2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{15^2} + \frac{1}{16^2} + \frac{1}{36^2} + \frac{1}{60^2} + \frac{1}{180^2}.$$

I believe that it would be a very rare event for the greedy algorithm to succeed in this situation![6]

In this vein, a number of questions were raised by Wilf [64] concerning what he called "reciprocal bases for the integers". By this he meant sets S of integers so that every positive integer can be represented as a finite sum of reciprocals of integers taken from S. For example, he asked: "Is every infinite arithmetic progression a reciprocal basis?" (Yes, by [3, 36]); "Must a reciprocal basis have positive density? "(No, by [3, 36]).

More generally, one could define a reciprocal basis for the *rationals* to be a set S of positive integers so that every positive rational $\frac{p}{q}$ is a finite sum of reciprocals of elements in S. At present, we don't know necessary and sufficient conditions for a set to be a reciprocal basis for the integers or the rationals[7]. However, a general theorem in this direction is the following.

For a set $T = \{t_1, t_2, \dots\}$ of positive integers, define $P(T)$ to be the set of all finite sums of elements taken from T. Also, define $T^{-1} = \{\frac{1}{t_i} : t_i \in T\}$. We will say that T is *complete* if every sufficiently large integer belongs to $P(T)$. Further, define $M(T)$ to be the set of all products $t_{i_1} t_{i_2} \dots t_{i_r}$ where $1 \leq i_1 < i_2 < \dots < i_r$ with $r = 1, 2, \dots$. Finally, let us say that a real number α is *T-accessible* if for all $\varepsilon > 0$, there is a $u \in T$ such that $0 \leq u - \alpha < \varepsilon$. In [36], the following result is proved.

Theorem 2. *Suppose $S = (s_1, s_2, \dots)$ is a sequence of positive integers so that $M(S)$ is complete and $\frac{s_{n+1}}{s_n}$ is bounded as $n \to \infty$.*

Then $\frac{p}{q} \in P(M(S)^{-1}$ (with $(p, q) = 1$) if and only if $\frac{p}{q}$ is $M(S)^{-1}$-accessible and q divides some element of $M(S)$.

It follows from this, for example, the set consisting of the primes together with the squares forms a reciprocal basis for the rationals. It is not known whether the condition that $\frac{s_{n+1}}{s_n}$ be bounded is needed for the conclusion of the theorem to hold.

A classical result of Curtiss [22] asserts that the closest strict under approximation R_n of 1 by a sum of n unit fractions is always given by taking $R_n = \sum_{k=1}^{n} \frac{1}{u_k+1}$, where u_n is defined recursively by: $u_1 = 1$, and $u_{n+1} = u_n(u_n + 1)$ for $n \geq 1$. The analogous fact is also known to hold [27]

[6] For similar results using n^{th} powers rather than squares, see [37].

[7] In fact, I don't know of any good conjectures here.

for rationals of the form $\frac{1}{m}$. However, it does not hold for some rationals, e.g., $R_1\left(\frac{11}{24}\right) = \frac{1}{3}$ while $R_2\left(\frac{11}{24}\right) = \frac{1}{4} + \frac{1}{5}$. Perhaps it is true that for any rational it does hold eventually. In other words, is it true that for any rational $\frac{a}{b}$, the closest strict under approximation $R_n\left(\frac{a}{b}\right)$ of $\frac{a}{b}$ is given by

$$R_n\left(\frac{a}{b}\right) = R_{n-1}\left(\frac{a}{b}\right) + \frac{1}{m}$$

where m is the least denominator not yet used for which $R_n\left(\frac{a}{b}\right) < \frac{a}{b}$ provided that n is sufficiently large? In fact, as we state in [28], this behavior might even hold for all algebraic numbers.

For each n, let $\mathbf{X}_n$ denote the set

$$\left\{ \{x_1, x_2, \ldots, x_n\} : \sum_{k=1}^{n} \frac{1}{x_k} = 1,\ 0 < x_1 < x_2 < \ldots < x_n \right\}$$

and let $\mathbf{X} = \cup_{n \geq 1} \mathbf{X}_n$. There are many attractive unresolved questions concerning these sets which were raised in [28], some of which I will now mention.

To begin, it would be interesting to have asymptotic formulas or even good estimates for $|\mathbf{X}_n|$. To the best of my knowledge, the best estimates currently known [50] are:

$$e^{c \frac{n^3}{\log n}} < |\mathbf{X}_n| < c_0^{(1+\varepsilon)2^{n-1}}$$

where $c_0 = \lim_{n\to\infty} u_n^{\frac{1}{2^n}} = 1.264085\ldots$, with u_n defined as above (see [2]). Perhaps the lower bound can be replaced by $c_0^{2^{n(1-\varepsilon)}}$.

In view of the large number of sets in $\mathbf{X}$, one would suspect that the condition that the reciprocals of a set of integers sum to 1 is not really a very stringent condition (modulo some obvious modular and size restrictions, e.g., the largest element cannot be prime). For example, it has been shown in [35] that for all $m \geq 78$, there is a set $\{x_1, x_2, \ldots, x_t\} \in \mathbf{X}$ with $\sum_{k=1}^{t} x_k = m$. Furthermore, this is not true for 77 [42]. I would conjecture that this behavior is true much more generally. Namely, it should be true that for any polynomial $p : \mathbf{Z} \to \mathbf{Z}$, there is a set $\{x_1, x_2, \ldots, x_t\} \in \mathbf{X}$ with $\sum_{k=1}^{t} p(x_k) = m$, for all sufficiently large m, provided p satisfies the obvious necessary conditions:

(i) The leading coefficient of p is positive;
(ii) $\gcd(p(1), p(2), \ldots,) = 1$.

It is known [15] that these conditions are sufficient for expressing every sufficiently large integer as a sum $\sum_{a_i \text{distinct}} p(a_i)$.

How many integers $x_k < n$ can occur as an element of $\{x_1, x_2, \dots, x_n\} \in \mathbf{X}_n$? Are there $o(n), cn$ or $n - o(n)$?

What is the least integer $v(n) > 1$ which does not occur as an x_k, k variable, for $\{x_1, x_2, \dots, x_n\} \in \mathbf{X}_n$? It is easy to see that $v(n) > cn!$ by results in [6, 7, 8]. It may be that $v(n)$ actually grows more like $2^{2^{\sqrt{n}}}$ or even $2^{2^{n(1-\varepsilon)}}$.

Denote by $k_r(n)$ the least integer which does not occur as x_r in any $\{x_1, x_2, \dots, x_t\} \in \mathbf{X}_n$ with $x_1 < x_2 < \dots < x_t \le n$. It is not hard to show

$$k_1(n) < \frac{cn}{\log n}.$$

We have no idea of the true value of $k_r(n)$ or even $k_1(n)$.

As a related problem, suppose we define $K(n)$ to be the least integer which does not occur as x_i for any i in any $\{x_1, x_2, \dots, x_t\} \in \mathbf{X}_n$ with $x_1 < x_2 < \dots < x_t \le n$. Again,

$$K(n) < \frac{cn}{\log n}$$

is easy but at present we do not even know if $k_1(n) < K(n)$.

How many disjoint sets $S_i \in \mathbf{X}$, $1 \le i \le k$, can we find so that $S_i \subseteq \{1, 2, \dots, n\}$? As C. Sándor notes [51], applying the results of Theorem 1 iteratively, we should be able to achieve $k = (1 + o(1)) \log n$. More generally, how many disjoint sets $T_i \subseteq \{1, 2, \dots, n\}$ are there so that all the sums $\sum_{t \in T_i} \frac{1}{t}$ are equal. By using strong Δ-systems [30], it can be shown that there are at least $\frac{n}{e^{c\sqrt{\log n}}}$ such T_i. Is this the right order of magnitude? One could also ask how many disjoint sets $\{x_1, x_2, \dots, x_n\} \in \mathbf{X}_n$ are possible. It is probably true that there are only $o(\log n)$ such sets.

Another set of attractive questions concerns what might be called *Ramsey* properties of the $\mathbf{X}_n$. It was asked in [28] whether for any partition of $\{2, 3, 4, \dots\}$ into finitely many blocks, some block must contain an element of $\mathbf{X}$. Put another way, is it true that if the integers greater than 1 are arbitrarily r-colored, then at least one of the color classes contains a finite set of integers whose reciprocals sum to 1? Erdős and I liked this problem so much that we posted a reward \$500 for its solution. As it turned out, the problem was settled in the affirmative by a beautiful argument of Ernie Croot [20].[8]

[8] As it happened, Erdős did not live to see the solution. When I asked Ernie whether he would like a check for the \$500 signed by Erdős, he said he would pleased to be paid this

A stronger conjecture is that any sequence $x_1 < x_2 < \dots$ of positive upper density contains a subset whose reciprocals sum to 1. Perhaps this can be proved if we assume that the differences $x_{k+1} - x_k$ are bounded. It is not enough to just assume that $\sum_k \frac{1}{x_k}$ is unbounded as the set of primes shows. (The letter in Figure 1 from Erdős' mathematical notebook from 1963 shows our interest in these questions going back some 50 years. In the appendix, we show some additional notes of Erdős on these problems). However, perhaps the sum $\sum_{k=1}^{n} \frac{1}{x_k}$ cannot grow much faster than this (i.e., $\log\log n$) for the x_k to fail to form some $\overline{x} \in \mathbf{X}$.

Let $A(n)$ denote the largest value of $|S|$ such that $S \subseteq \{1, 2, \dots, n\}$ contains no set in $\mathbf{X}$. Probably $A(n) = n - o(n)$ but this is not known. A related question is this. What is the smallest set $S' \subseteq \{1, 2, \dots, n\}$ which contains no set in $\mathbf{X}$ and which is maximal in this respect. Very little is known here. More generally, one could ask for the largest subset $S_n^* \subseteq \{1, 2, \dots, n\}$ so that for any distinct elements $s, s_1, s_2, \dots, s_m \in S_n^*$, we have $\frac{1}{s} \neq \sum_{k=1}^{m} \frac{1}{s_k}$ where $m > 1$? We can certainly have $|S_n^*| > cn$ as the set $\{i : \frac{n}{2} < i < n\}$ shows. Can $|S_n^*| > cn$ for $c > \frac{1}{2}$? Is it true that if $S \subseteq \{1, 2, \dots, n\}$ with $|S| > cn$ then S contains x, y, z with $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$? It has been shown by Brown and Rödl [10] that the partition version of this question holds, i.e., for any partition of $\mathbf{Z}$ into finitely many classes and for any fixed value of n, one of the classes must contain a solution to $\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} = \frac{1}{z}$.

There are many interesting unresolved questions which involve restricting the denominators of the elements in the $\mathbf{S}_n$. For example, Burshtein [11] gives an example of $\{x_1, x_2, \dots, x_n\} \in \mathbf{X}_n$ with no x_i dividing any other x_j. Even more striking, Barbeau [5] finds an example in which each x_i is the product of exactly 2 distinct primes. A smaller such example was given by Burshtein [12, 13], The smallest such example known is that of Allan Johnson (see [39]) with the denominators shown in the table below.

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way. (I kept a number of checks pre-signed by Erdős for just such contingencies.) After sending Ernie the Erdős check, I subsequently sent Ernie a *real* check for \$500, which he certainly earned. However, unknown to me, Ernie *cashed* the Erdős check. That is, it was sent to my bank and it was honored. This was unexpected since Erdős never had an account at my bank! I am guessing that the bank tellers were so used to seeing Erdős' checks countersigned by me that they just assumed this was one of those and they cashed it. When I discovered this, I wrote to Ernie that he owed me \$500. He agreed to send back the \$500 overpayment but on the condition that I send him back the canceled Erdős check (which I did).

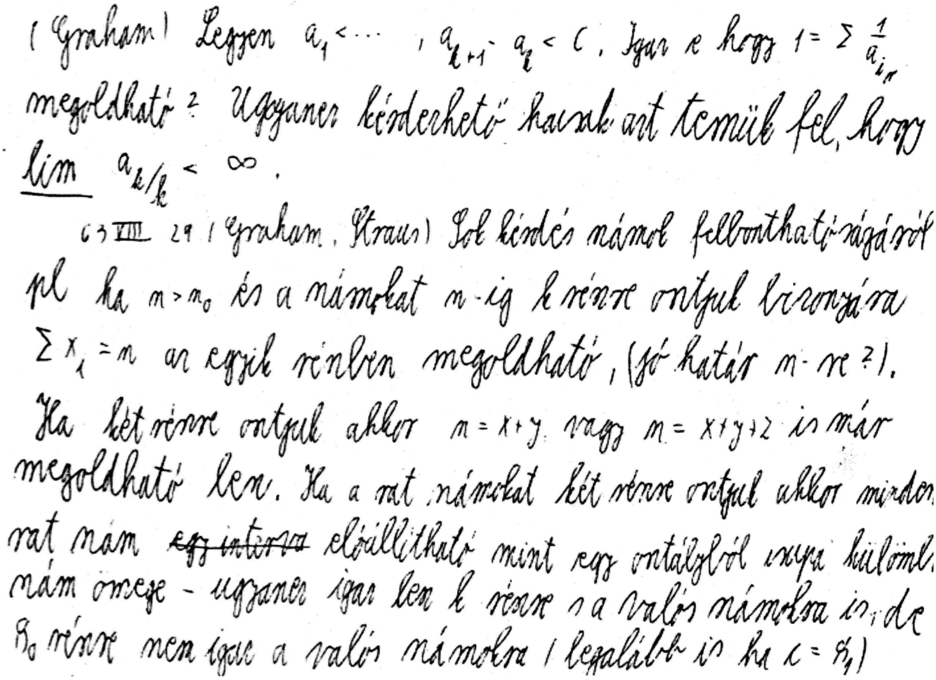
(Graham) Legyen $a_1 < \cdots$, $a_{k+1} - a_k < C$. Igaz e hogy $1 = \sum \frac{1}{a_{i_r}}$
megoldható? Ugyanez kérdezhető hacsak azt tesszük fel, hogy
$\lim a_k/k < \infty$.
63 VIII 29 (Graham, Straus) Sok kérdés számok felbonthatóságáról
pl ha $n > n_0$ és a számokat n-ig k részre osztjuk bizonyára
$\sum x_i = n$ az egyik részben megoldható, (jó határ n-re?).
Ha két részre osztjuk akkor $n = x+y$ vagy $n = x+y+z$ is már
megoldható lesz. Ha a rac. számokat két részre osztjuk akkor minden
rac szám ~~egy intervallum~~ előállítható mint egy osztályból vegyes különb.
szám összege - ugyanez igaz lesz k részre s a valós számokra is, de
$\aleph_0$ részre nem igaz a valós számokra (legalább is ha $c = \aleph_1$)

Fig. 1. A page from Erdős' 1963 notebook

6	21	34	46	58	77	87	114	155	215	287	391
10	22	35	51	62	82	91	119	187	221	299	689
14	26	38	55	65	85	93	123	203	247	319	731
15	33	39	57	69	86	95	133	209	265	323	901

Table 1. Denominators for Johnson's decomposition of 1

However, as Barbeau notes in [4], it is not known if 1 can be represented as the product of two sums of the form $\frac{1}{q_1} + \frac{1}{q_2} + \ldots + \frac{1}{q_r}$ where the q_i are distinct primes. Perhaps this can be done if we just assume that the q_i are pairwise relatively prime. (Related results can be found in [33].) In a (still) unfinished manuscript of Erdős and the author[9], it is shown that any integer can be represented as a sum of reciprocals of distinct numbers which each have exactly three prime factors (see [39]). Whether this can be accomplished with just two prime factors is not clear.

In [54], Shparlinski answers a question of Erdős and the author by proving the following result.

[9] I'm still working on it!

Theorem 3. *For any $\varepsilon > 0$ there is a $k(\varepsilon)$ such that for any prime p and any integer c there exist $k \le k(\varepsilon)$ pairwise distinct integers x_i with $1 \le x_i \le p^{\varepsilon}$, and such that*

$$\sum_{i=1}^{k} \frac{1}{x_i} \equiv c \pmod{p}.$$

(Here, the reciprocals are taken modulo p). This has been generalized by Croot [21] to the case when the denominators are all of the form x_i^k for a general positive integer k.

5. The Story of an Incorrect Conjecture

Naturally, not every conjecture of Erdős and the author in [28] was correct. Here is an example of one such conjecture and some of the subsequent developments. In [28], the following question was raised.

Suppose that a_k are positive integers satisfying

$$1 < a_1 < a_2 < \ldots < a_t. \tag{3}$$

Is it true that if $\sum_{k=1}^{t} \frac{1}{a_k} < 2$, then there exist $\varepsilon_k = 0$ or 1 so that

$$\sum_{k=1}^{t} \frac{\varepsilon_k}{a_k} < 1 \quad \text{and} \quad \sum_{k=1}^{t} \frac{1-\varepsilon_k}{a_k} < 1?$$

As noted in [28], this is not true if we just assume that

$$1 < a_1 \le a_2 \le \ldots \le a_t \tag{4}$$

as the sequence 2, 3, 3, 5, 5, 5, 5 shows. However, it was pointed out by Sándor [49] that our conjecture was too optimistic since the sequence consisting of the divisors of 120 with the exception of 1 and 120 provides a counterexample. In fact, Sándor proved the more general result that for every $n \ge 2$, there exist integers a_k satisfying (3) such that $\sum_{k=1}^{t} \frac{1}{a_k} < n$ and that this sum cannot be split into n parts so that all the partial sums are ≤ 1. However, he also shows that for such a sequence the sum cannot be too much less than n. Specifically, Sándor proves:

Theorem 4. *Suppose $n \ge 2$. If $1 < a_1 < a_2 < \ldots < a_t$ are integers and*

$$\sum_{k=1}^{t} \frac{1}{a_k} < n - \frac{n}{e^{n-1}}$$

then this sum can be decomposed into n parts so that all partial sums are ≤ 1.

It was however conjectured by Erdős, Spencer and the author that if the a_k satisfy (4), as well as the stronger condition

$$\sum_{k=1}^{t} \frac{1}{a_k} < n - \frac{1}{30}, \tag{5}$$

then the a_k can be split into n sequences $a_k^{(i)}, 1 \le i \le n$, so that

$$\sum_k \frac{1}{a_k^{(i)}} \le 1$$

for all i. The reason that the bound $n - \frac{1}{30}$ was chosen was because of the example $a_1 = 2$, $a_2 = a_3 = 3$, $a_4 = a_5 = \ldots = a_{5n-3} = 5$. Put another way, define $\alpha(n)$ to be the least real number so that if the a_k satisfy (4) and

$$\sum_{k=1}^{t} \frac{1}{a_k} < n - \alpha(n) \tag{6}$$

then the a_k can be split into n sequences $a_k^{(i)}, 1 \le i \le n$, so that

$$\sum_k \frac{1}{a_k^{(i)}} \le 1$$

for all i. Thus, the conjecture in [28] was that $\alpha(n) = \frac{1}{30}$. In [49] it was shown by Sándor that $\alpha(n) \le \frac{1}{2}$. This was improved by Chen [16] who shows that $\alpha(n) \le \frac{1}{3}$. This in turn was followed by the paper of Fang and Chen [31] who prove that $\alpha(n) \le \frac{2}{7}$. However, the original conjecture that $\alpha(n) = \frac{1}{30}$ was finally disproved by Guo [38] who showed that $\alpha(n) \ge \frac{5}{132} > \frac{1}{30}$. He shows that for the sequence $a_1 = 2$, $a_2 = 3$, $a_4 = 4$, $a_5 = \ldots = a_{11n-12} = 11$,

$$\sum_{k=1}^{11n-12} \frac{1}{a_k} = n - \frac{5}{132},$$

but for any partition of $\{1, 2, \ldots, 11n-12\} = \cup_{j=1}^{n} A_j$, there exists a j such that $\sum_{k \in A_j} \frac{1}{a_k} > 1$. At present, we have no guess as to what the truth is for this problem.

6. Concluding Remarks

We have tried to give a sample of the very many interesting questions and results that were inspired by Paul Erdős' interest in Egyptian fractions. Of course, this list is far from complete, and in fact the subject is still quite dynamic. For further references, the reader can consult [39], [28], [52] or [14], for example, and the references therein.

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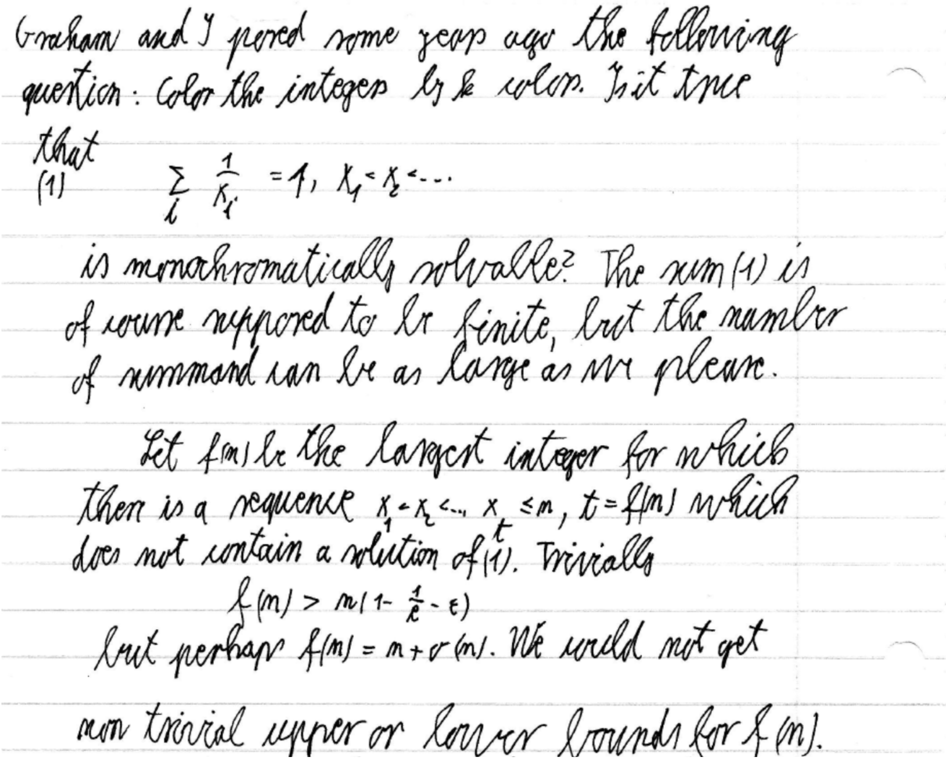
7. Appendix: Some (Undated) Notes of Erdős on Egyptian Fractions



Graham and I posed some years ago the following question: Color the integers by k colors. Is it true that

(1) $\sum_i \frac{1}{x_i} = 1, \quad x_1 < x_2 < \dots$

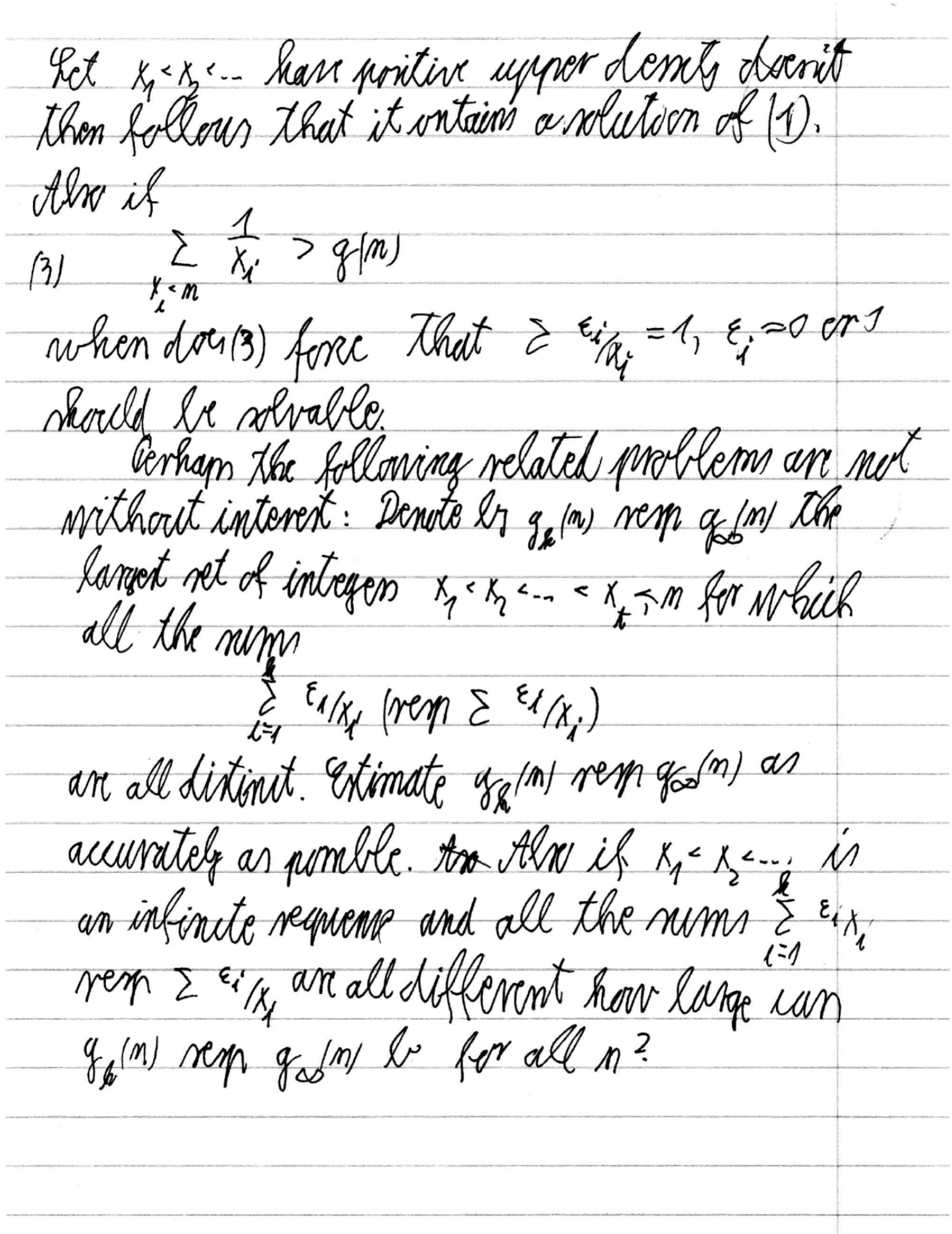
is monochromatically solvable? The sum (1) is of course supposed to be finite, but the number of summands can be as large as we please.

Let $f(m)$ be the largest integer for which there is a sequence $x_1 < x_2 < \dots x_t \le m$, $t = f(m)$ which does not contain a solution of (1). Trivially

$$f(m) > m(1 - \tfrac{1}{e} - \varepsilon)$$

but perhaps $f(m) = m + o(m)$. We could not get non trivial upper or lower bounds for $f(m)$.

Fig. 2. Some notes of Erdős on Egyptian fractions



Let $x_1 < x_2 < \dots$ have positive upper density does it then follows that it contains a solution of (1).

Also if

(3) $\sum_{x_i < n} \frac{1}{x_i} > g(n)$

when does (3) force that $\sum \varepsilon_i/x_i = 1$, $\varepsilon_i = 0$ or 1 should be solvable.

Perhaps the following related problems are not without interest: Denote by $g_k(n)$ resp $g_\infty(n)$ the largest set of integers $x_1 < x_2 < \dots < x_k \leq n$ for which all the sums

$\sum_{i=1}^{k} \varepsilon_i/x_i$ (resp $\sum \varepsilon_i/x_i$)

are all distinct. Estimate $g_k(n)$ resp $g_\infty(n)$ as accurately as possible. Also if $x_1 < x_2 < \dots$ is an infinite sequence and all the sums $\sum_{i=1}^{k} \varepsilon_i x_i$ resp $\sum \varepsilon_i/x_i$ are all different how large can $g_k(n)$ resp $g_\infty(n)$ be for all n?

Fig. 3. More notes of Erdős on Egyptian fractions

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Graham and I conjectured several years ago
that if we color the integers by k colors then

(1) $\sum_i \frac{1}{x_i}$, $x_1 < x_2 < \dots$

is solvable monochromatically. (In (1) the sum is of course finite).
More generally we conjectured that if we divide the integers into k clas
~~there is~~ S_r, $1 \le r \le k$, there is an r so that every rational $\frac{a}{b}$ can
be written in the form

(2) $\frac{a}{b} = \sum \frac{1}{x_i}$, $x_1 < x_2 < \dots$, $x_i \in S_r$

and the sum in (2) is of course finite. The slightly weaker conjecture: (2) i
solvable monochromatically is also open. These attractive conjectures have
perhaps been undeservedly neglected. We formulated several related problems
which we feel are also interesting. Let $f(r,m)$ be the smallest integer for
which if we divide the integers $2 \le t \le f(r,m)$ into r classes then (1) is sol
monochromatically. Prove that $f(r,m)$ exists for every r and estimate $f(r,m)$
as well as possible. A further complication can be added: Denote by $f(r,t$
the smallest integer (if it exists) for which if we divide the integers $2 \le t \le f($
into r classes then

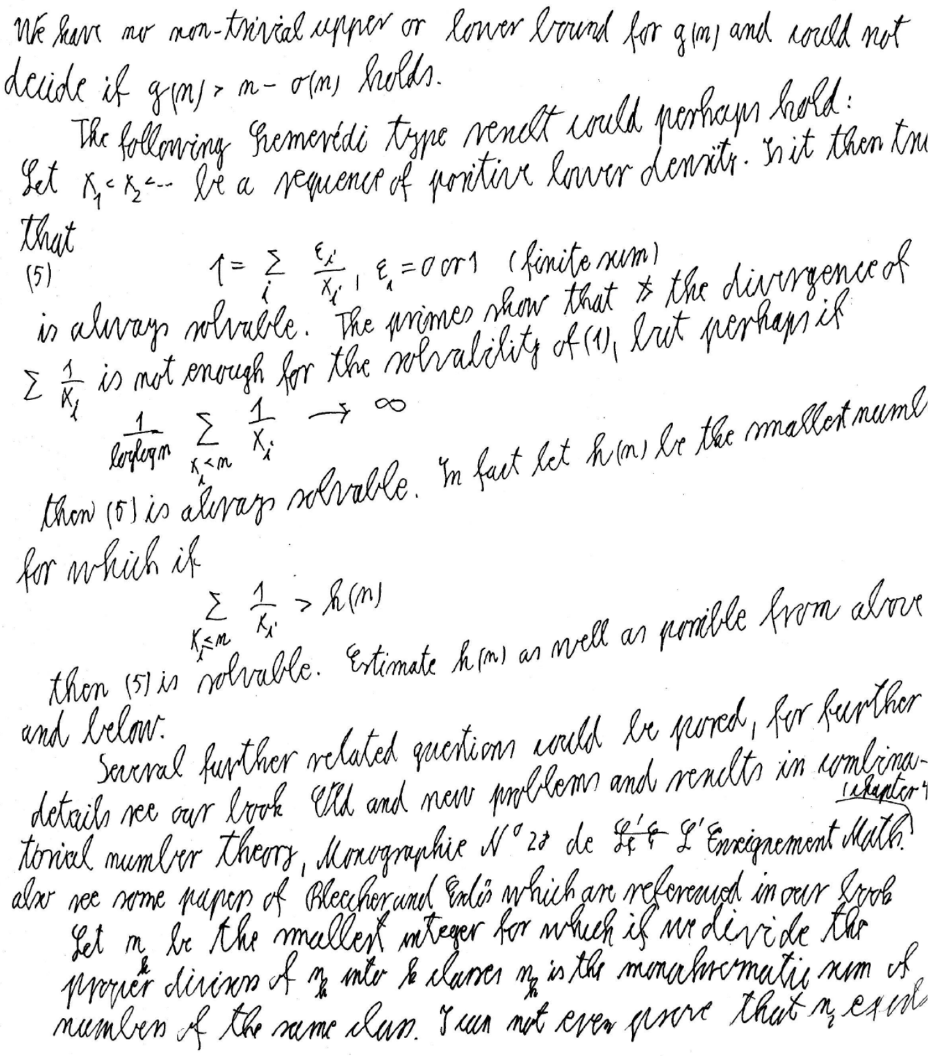
(3) $1 = \sum \frac{1}{x_r}$, $2 \le x_1 < x_2 < \dots < x_n \le f(r,t,m)$, $3 \le n \le t$

is solvable monochromatically. Clearly for small t $f(r,t,m)$ does not exist e
for $r=2$. Perhaps one could try to determine the smallest t for which $f(2,t$
exists.

Determine or estimate the smallest $g(m)$ for which in every set of $g($
integers ~~2≤t~~ $2 \le x_1 < \dots < x_t \le m$, $t = g(m)$ (1) is solvable. Trivially

(4) $g(m) > m(1-\frac{1}{e}) + O(1)$

Fig. 4. Notes of Erdős on Egyptian fractions (while visiting Bell Labs)



We have no non-trivial upper or lower bound for $g(n)$ and could not decide if $g(n) > n - \sigma(n)$ holds.

The following Szemerédi type result could perhaps hold:
Let $x_1 < x_2 < \dots$ be a sequence of positive lower density. Is it then true that

(5) $$1 = \sum_i \frac{\varepsilon_i}{x_i}, \quad \varepsilon_i = 0 \text{ or } 1 \text{ (finite sum)}$$

is always solvable. The primes show that ~~x~~ the divergence of $\sum \frac{1}{x_i}$ is not enough for the solvability of (1), but perhaps if

$$\frac{1}{\log\log n} \sum_{x_i < n} \frac{1}{x_i} \to \infty$$

then (5) is always solvable. In fact let $h(n)$ be the smallest numb[er] for which if

$$\sum_{x_i \le n} \frac{1}{x_i} > h(n)$$

then (5) is solvable. Estimate $h(n)$ as well as possible from above and below.

Several further related questions could be posed, for further details see our book Old and new problems and results in combinatorial number theory, Monographie N° 28 de ~~L'E~~ L'Enseignement Math. (chapter 4) also see some papers of Bleicher and Erdős which are referenced in our book.

Let n_k be the smallest integer for which if we divide the proper divisors of n_k into k classes n_k is the monochromatic sum of numbers of the same class. I can not even prove that n_2 exists.

Fig. 5. Notes of Erdős on Egyptian fractions (while visiting Bell Labs)