MATHEMATICS

A Panoramic View

Ronald Graham

THE PRINCETON COMPANION TO MATHEMATICS. Edited by Timothy Gowers and associate editors June Barrow-Green and Imre Leader. xxii + 1,034 pp. Princeton University Press, 2008. $99.

This impressive book represents an extremely ambitious and, I might add, highly successful attempt by Timothy Gowers and his coeditors, June Barrow-Green and Imre Leader, to give a current account of the subject of mathematics. It has something for nearly everyone, from beginning students of mathematics who would like to get some sense of what the subject is all about, all the way to professional mathematicians who would like to get a better idea of what their colleagues are doing.

Monochromatic paths through a lattice of randomly colored hexagons illustrate the concept of percolation in an essay by Gordon Slade included in *The Princeton Companion to Mathematics*. Both all-gray and all-white paths extend from north to south, but on the east-west axis no monochromatic path crosses the heavy black line. Slade describes notable recent progress in understanding such random processes.

The 75-page introduction, which was written by Gowers, gives a very readable account of the basic branches of mathematics (algebra, geometry, analysis) and how these overlap and relate to one another, how they have developed and are continuing to do so, and how they are driven in large part by the types of questions mathematicians ask. This section should be mandatory reading for any prospective mathematics student.

Most of the articles that make up the rest of the book were written by leading experts. For example, Carl Pomerance has contributed a stimulating essay on computational number theory, Cliff Taubes provides a wonderful overview of differential topology, and Jordan Ellenberg gives a thoughtful summary of arithmetic geometry. In a few cases, the choice of contributor was more creative. In the preface, the editors say that expository skill was as important a criterion as expertise in the selection of contributors. Even so, they found it necessary to take what they refer to as an "interventionist" approach, editing most of the articles substantially and in some cases asking the contributor for a complete rewrite. A great deal of tact must have been required, but the result is that the book is highly readable throughout. Martin Gardner, the most successful recreational mathematics expositor in history, has said that he operates under the principle that if something isn't completely clear to him, it would be a disservice to his readers to include it, and apparently in that same spirit Gowers and his colleagues decided not to allow any material in the book that they didn't themselves understand. The level of difficulty is not uniform; they accepted that some of the articles will be too hard for some readers, but their goal was for ideas to be discussed at the most accessible level practical.

The introduction is followed by a collection of informative articles on the origins of modern mathematics. Topics covered include number systems, geometry, abstract algebra, algorithms, the development of rigor in mathematical analysis, and the development of the idea of proof. The section concludes with a discussion of the crisis in the foundations of mathematics that occurred during the early part of the 20th century and resulted in the formulation of axiomatic set principles, the rise of intuitionism and the emergence of modern mathematical logic as a refinement of axiomatics.

Next is a section consisting of 99 shorter essays on major concepts in modern mathematics, such as the axiom of choice, categories, curvature, the Fourier transform, homology and cohomology, *L*-functions, manifolds, von Neumann algebras and much more. Some of the essay titles may sound a bit mysterious to nonmathematicians, but the articles themselves are almost without exception very accessible and informative.

In the section that follows—the longest in the book—the major branches of mathematics are treated in more depth, in 26 articles, each 10 to 30 pages long. These essays vary in style, but they have been made as reader-friendly as possible given the topics covered, which include algebraic geometry, differential topology, moduli spaces, operator algebras and set theory. I particularly enjoyed the piece by Barry Mazur on algebraic numbers and Sergiu Klainerman's discussion of partial differential equations. These articles help flesh out the basic content of the major subdisciplines of modern mathematics and show how they interrelate. It is interesting to note that one of the longest articles is on computational complexity, a topic that didn't even exist 50 years ago. It is hard to imagine a mathematician who wouldn't benefit from reading at least some of these essays.

The next section contains pieces one to three pages long on a (somewhat subjective) selection of various major theorems and open problems, such as the ABC conjecture, the Banach-Tarski paradox, Gödel's theorem, the Riemann hypothesis, the Poincaré conjecture, the *P* versus *NP* problem, Fermat's Last Theorem and the Weil conjectures. These essays are a very good source of information about the history and current status of many of the classic problems. Given their nature, that status is dynamic, or so we hope at least; as the great German mathematician David Hilbert asserted, "Wir müssen wissen. Wir werden wissen." (We must know. We will know.) As an example of major progress, we have the recent resolution of the famous Poincaré conjecture. It's actually discussed in greater detail in the preceding section—in the essay on differential topology, which is cross-referenced here.

Next come very brief biographical sketches of 96 famous mathematicians, from Pythagoras to Bourbaki, almost all of them known for work carried out before 1950. After that is a section containing longer articles about the influence of mathematics on such fields as chemistry, biology, cryptography, network design, economics, statistics, music and art. In the final section, "Perspectives," we find an intriguing assortment of short essays on such topics as the art of problem solving (a subject that continues to elicit strong opinions), mathematics as an experimental science (certainly an ever-increasing trend nowadays) and advice for young people who might want to become mathematicians.

Overall, Gowers, Barrow-Green and Leader must be congratulated on a magnificent achievement. Of course, even in a book this massive (there are more than 1,000 pages), it's inevitable that some important topics have been omitted—transcendence theory, for example. No doubt many of these omissions will be rectified in the next edition. It would be nice if the book were available in a lighter-weight multivolume paperback or electronic version; the cloth *Companion* took up far too much space in my suitcase!

If I had to choose just one book in the world to give an interested reader some idea of the scope, goals and achievements of modern mathematics, without a doubt this would be the one. So try it. I guarantee you'll like it!

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