Old and New Problems and Results in Ramsey Theory

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Abstract

In this note, I will describe a variety of problems from Ramsey theory on which I would like to see progress made. I will also discuss several recent results which do indeed make progress on some of these problems.

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1 Introduction

Ramsey theory has sometimes been described as the study of unavoidable regularity in large structures. That is, one would like to know when it is the case that whenever the elements of some (sufficiently large) object are partitioned into a finite number of classes (i.e., colored with a finite number of colors), there is always at least one (color) class which contains all the elements of some regular structure. When this is the case, one additionally would like to have quantitative estimates of what "sufficiently large" means. In this sense, the guiding philosophy of Ramsey theory can be described by the phrase: *"Complete disorder is impossible"*. The roots of Ramsey theory go back to the work of Ramsey [37], Schur [40], van der Waerden [53], Rado [35, 36], Erdős [9], Szekeres [14, 15], Turán [16] and even Hilbert [27] (and even further, if you count the Pigeon-hole principle). A fuller account of this field can be found in the books [24], [23] and [31]. In this paper, I will focus on a number of both classical and new problems in this subject, and on some of the recent progress which has been made. Following the tradition popularized by Erdős, I am offering small monetary rewards for some of these problems, in the hope that these might stimulate further progress on them.

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2 The Growth of $R(n)$

Define $R(n)$ to be the least integer such that any graph on $R(n)$ vertices contains **either** a clique of size n **or** an independent set of size n. It was shown in the classical paper of Ramsey [37] that $R(n)$ always exists. One of the oldest open problems in Ramsey theory is to determine or at least estimate, the rate of growth of $R(n)$. Some of the earliest estimates (due to Erdős [9]) are:

$$\frac{1}{e\sqrt{2}} n 2^{n/2}(1+o(1)) < R(n) \le \binom{2n-2}{n-1}.$$

Unfortunately, very little progress has been made in the past half century on these bounds (and this is not for lack of trying!). The best known improvements of these bounds are:

$$\frac{2}{e\sqrt{2}} n 2^{n/2}(1+o(1)) < R(n) < n^{-1/2+c\sqrt{\log n}}\binom{2n-2}{n-1}$$

for a suitable $c > 0$. The lower bound is due to Spencer [45] while the upper bound is due to Thomason [49].

However, very recently a significant improvement on the upper bound was established by Conlon [7]. He showed that for a suitable $c > 0$,

$$R(n+1) < n^{c\frac{\log n}{\log\log n}}\binom{2n}{n}.$$

This implies that

$$R(n+1) = O\left(\frac{1}{n^s}\binom{2n}{n}\right)$$

for every $s > 0$.

Problem 1. ($100) Prove that $\lim_{n\to\infty} R(n)^{1/n}$ exists.

Problem 2. ($250) Assuming this limit exists, what is it?

Of course, the limit would have to lie between $\sqrt{2}$ and 4. One popular guess is that it is 2. (Well, why not!)

Both of these problems (and the associated prizes) were frequently mentioned by Erdős in his uncountably many talks and problems papers. More complete descriptions of these and many other related problems in this vein can be found in the monograph **Erdős on Graphs: His Legacy of Unsolved Problems** by Fan Chung and the author [5].

Problem 3. ($100) Give a **constructive** proof that for some $c > 0$, $R(n) > (1+c)^n$.

As is well known, the best lower bounds currently available for $R(n)$ have been obtained using the probabilistic method [2].

One generalization of the Ramsey number $R(n)$ is the so-called off-diagonal Ramsey number $R(k, l)$, which is defined to be the least integer R such that in any red/blue coloring of the edges of the complete graph on R vertices, either there is a red complete graph on k vertices, or a blue complete graph on l vertices formed.

Problem 4. (Erdős – $250) Prove or disprove that

$$R(4, n) > \frac{n^3}{\log^c n}$$

for some absolute constant c, provided that n is sufficiently large.

More generally, is it true that for k fixed,

$$R(k, n) > \frac{n^{k-1}}{\log^c n}$$

for some absolute constant c, provided that n is sufficiently large?

We close this section with three nice questions concerning off-diagonal Ramsey numbers. Specific references to the literature for these problems can be found in [5].

Problem 5. (Erdős/Burr) Prove that

$$R(n+1, n) > (1+c)R(n, n)$$

for some fixed $c > 0$.

Conjecture 1. (Erdős/Sós)

$$R(3, n+1) - R(3, n) \to \infty.$$

Conjecture 2. (Erdős/Sós)

$$R(3, n+1) - R(3, n) = o(n).$$

3 The Erdős/Szekeres Problem

For each positive integer n, let $f(n)$ denote the least integer such that any set of $f(n)$ points in the plane in general position always contains a subset of size n which form a **convex** n-gon. The general problem here is to determine or at least estimate the function $f(n)$.

This problem has a long and very interesting history which can be found, for example, in [25]. In particular, the original paper of Erdős and Szekeres [14] treating this problem contained an independent proof of Ramsey's theorem, among with other now classical results. Their original 1935 estimates

$$2^{n-2} + 1 \leq f(n) \leq \binom{2n-4}{n-2} + 1$$

remained unchanged until 1997, at which time the upper bound was improved by Fan Chung and the author [6] to:

$$f(n) \leq \binom{2n-4}{n-2},$$

a modest improvement, to be sure! However, this was followed by a rapid series of further improvements [50], the current best being that of Tóth and Valtr [51]

$$f(n) \leq \binom{2n-5}{n-2} + 1$$

which is about half as large as the original upper bound. It is suspected by many people that the lower bound is the truth. It has been known for some time that $f(3) = 3, f(4) = 5$, and $f(5) = 9$, all agreeing with the lower bound above. Very recently, it has been shown by Peters and Szekeres [47] that the lower bound is also the truth for $n = 6$, namely that $f(6) = 17$. Their proof required the use of some 1500 hours of computing on a computer with a 2 GHz processor. The lower bound of $2^{n-2}+1$ results from an explicit construction in [14] and [15], where it is shown that there are sets of 2^{n-2} points in the plane in general position which contain no convex subset of size n. (It is a nice exercise to construct such sets if you haven't already seen the construction). This prompts the following:

Problem 6. ($1000) Prove or disprove that

$$f(n) = 2^{n-2} + 1.$$

for all $n \geq 2$.

As a warm-up, one might like to first tackle

Problem 7. ($100) Show that

$$f(n) = o\left(\frac{4^n}{\sqrt{n}}\right).$$

A related question also raised by Erdős is whether the analogous results hold for **empty** convex n-gons. That is, if we define $g(n)$ to be the least integer g such any set of g points in the plane in general position contains the vertices of a convex n-gon which contains none of the other points in its interior, then does $g(n)$ always exist, and if so, what is its value? It is known that $g(3) = 3, g(4) = 5$ and $g(5) = 10 > 9 = f(5)$. Somewhat unexpectedly, it was shown by Horton [29] in 1983 that $g(n)$ does not exist for $n \geq 7$. The remaining open problem since then was the existence of an empty convex hexagon in a sufficiently large planar set in general position. This has just now been resolved in a very nice paper of Gerken [20]. He shows that $g(6)$ exists and in fact satisfies $30 \leq g(6) \leq 1717$. A simpler presentation (with a weaker upper bound) can be found in the paper of Valtr [52]. The reader is referred to the survey paper of Morris and Soltan [32] for a complete collection of results on the Erdős-Szekeres problem and its many variants.

4 Partition Regular Equations

We say that an equation $f(x, y, z, \ldots) = 0$ is **partition regular** if for any partition of the set of nonnegative numbers $\mathbb{N}$ into finitely many classes $C_1, C_2, \ldots, C_r$, some C_i contains a nontrivial solution to the equation. (Nontrivial means not all the variables are equal). Often we think of the C_i as **colors**, and the solution in a single class as **monochromatic.** A rather complete theory of partition regularity for (systems of) linear equations was developed by Rado [35]. For example, $x + y = z$ and $x + y = 2z$ are partition regular, but $x + y = 3z$ is not. In fact, a single homogenous linear equation over $\mathbb{N}$ is partition regular if and only if it has a nontrivial solution in 0's and 1's, (i.e., not all 0). However, for **nonlinear** equations, the situation is much less clear. For example, it was shown by Rödl [38] that the equation $1/x + 1/y = 1/z$ is partition regular. Another recent result with a partition regularity flavor for nonlinear equations is the striking result of Croot [8] who showed that for any r-coloring of the integers greater than 1, the equation $\sum_{i \in I} 1/x_i = 1,$ has a monochromatic solution for some finite set I. In fact, he proved that an appropriate I can always be found in the interval $[2, e^{167000r}]$. Very recently, Hippler [28] proved in his doctoral thesis that for $r = 2$, the exact bound for this problem is 208. That is, any 2-coloring of the set $\{2, 3, \ldots, 208\}$ must contain a monochromatic subset whose reciprocals sum to 1, and this is not true if 208 is replaced by 207.

The following problem of Erdős and the author has been open for over 30 years [10]:

Problem 7. ($250) Determine whether the equation

$$x^2 + y^2 = z^2$$

is partition regular.

There is actually very little data (in either direction) to know which way to guess.

Let us say that an equation $f(x_1, x_2, \ldots, x_n) = 0$ is r-**partition regular** if for any partition of the integers in r color classes, there is a nontrivial solution to this equation in a single color.

Conjecture (Rado) For each n, there is a least integer $M = M(n)$ so that if the linear homogeneous equation $f(x_1, x_2, \ldots, x_n) = 0$ is M-partition regular, then in fact the equation is partition regular.

Rado showed that this conjecture holds for $n = 1$ and $n = 2$. Very recently, Fox and Kleitman [17] have now shown that this conjecture also holds for $n = 3$, and in fact, that $M(3) \leq 36$.

Challenge. Prove that $M(n)$ always exists and determine (or estimate) its value.

An interesting phenomenon has been recently observed by Fox, Radoičić, Alexeev and the author [1, 18] which shows how the axioms of set theory can affect the outcome of some of these questions. For example, consider the linear equation $E : x + y + z - 4w = 0$. This is certainly not partition regular, and in fact, there is a 4-coloring of the integers which prevents E from having any (nontrivial) monochromatic solution. However, suppose we change the question and asked whether E has monochromatic solutions in **reals** for every 4-coloring of the reals. It can be shown that in ZFC, there exist 4-colorings of the reals for which E has no monochromatic solution. However, if we replace the Axiom of Choice (the "C" in ZFC) by LM (which is the axiom asserting that every set of reals is Lebesgue measurable), then in the system ZF + LM (which is consistent if ZFC is), the answer is yes. In other words, in this system every 4-coloring of the reals always contains a nontrivial monochromatic solution to E. On the other hand, this distinction does not occur for the equation $x + y - z = 0$, for example.

(Wide-open) Question. For which (systems of) equations does this distinction occur?

5 The Chromatic Number of the Plane

In this problem, which goes back to Nelson in 1950 (and perhaps even to Hadwiger in 1944; see [4, 44]), we are asked for the minimum number r of colors needed so that the points of the Euclidean plane $\mathbb{E}^2$ can be r-colored in such a way that any two points separated by distance 1 have **different** colors. (Strictly speaking, we might think of this as an "anti-Ramsey" question, since in this case we are looking for the smallest number of colors needed to prevent the occurrence of a particular monochromatic configuration). This number, called the chromatic number of the plane, and denoted by $\chi(\mathbb{E}^2)$, is known to satisfy

$$4 \leq \chi(\mathbb{E}^2) \leq 7.$$

Both of these inequalities are quite easy to see, and have been known since the time the problem was proposed. If we operate in Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC), then it follows by compactness that if $\chi(\mathbb{E}^2) = r$, then in fact there is a **finite** set which also requires r colors to legally color it. However, as pointed out recently by Shelah and Soifer, if we replace the Axiom of Choice by the two axioms Dependent Choice (= DC) and the assertion that every set of reals is Lebesgue measurable (= LM), then ZF + DC + LM is just as consistent as ZFC but now we no longer have compactness and the answer can change (see [42] and [43] for details).

An interesting related result of O'Donnell [33, 34] shows that for every integer g, there is a unit distance graph in $\mathbb{E}^2$ with girth greater than g which has chromatic number 4. Perhaps, this is evidence that $\chi(\mathbb{E}^2)$ is at least 5?

Problem 8. ($100) Show that $\chi(\mathbb{E}^2) \geq 5$.

Problem 9. ($250) Show that $\chi(\mathbb{E}^2) \leq 6$.

In higher dimensions, it is known that (see [4]):

$$6 \leq \chi(\mathbb{E}^3) \leq 15, \quad (1.239 + o(1))^n < \chi(\mathbb{E}^n) < (3 + o(1))^n.$$

6 Euclidean Ramsey Sets

Let us say that a finite subset X of Euclidean space $\mathbb{E}^n$ is **Ramsey** if for any number of colors r, there is an integer $N = N(X, r)$ such that in any r-coloring of the points of $\mathbb{E}^N$, there is always a monochromatic "copy" X'

of X. In other words, X' can be obtained from X by some Euclidean motion (rotation and translation). This subject had its genesis in a series of papers by Erdős et al. [11, 12, 13]. In particular, it was shown that the Cartesian product of Ramsey sets is Ramsey, so that since a 2-point set is obviously Ramsey, then so is any subset of the vertices of a rectangular parallelepiped. On the other hand, it was also shown that any Ramsey set X must lie on the surface of some **sphere**. In such a case, we say that X is **spherical**.

Problem 10. ($1000) Prove that all spherical sets are Ramsey.

As a warm-up to this problem, one might work on the simpler:

Problem 11. ($100) Prove that any 4-point subset of a circle is Ramsey.

In order not to be too discouraging, we mention several more (presumably easier) Euclidean Ramsey problems.

Conjecture 1. ($25) For any 3-point set T, there is a 3-coloring of $\mathbb{E}^2$ which has no monochromatic copy of T.

Conjecture 2. ($50) In any 2-coloring of $\mathbb{E}^2$, a monochromatic copy of **every** 3-point set occurs, except possibly for a single equilateral triangle.

Very recently, Jelínek, Kynčl, Stolař and Valla [30] have shown that Conjecture 2 is true if one of the color classes is a closed set and the other color class is an open set. In fact, in this case, *every* 3-point set T occurs monochromatically.

Fact. [46] For any set L of 3 collinear points, there is a 16-coloring of $\mathbb{E}^n$ which contains no monochromatic copy of L.

Question. Is 16 the best possible constant here?
Perhaps the right answer is 4 (or even 3 !).

7 Van der Waerden's Theorem

The classical theorem of van der Waerden [53] on arithmetic progressions asserts that for any finite coloring of $\mathbb{N}$, there always exists arbitrarily long monochromatic arithmetic progressions. The finite version (for two colors) guarantees the existence of a least number $W(n)$ such that if the integers $\{1, 2, \ldots, W(n)\}$ are 2-colored, then a monochromatic n-term arithmetic progression (n-AP) must always be formed. The estimation of the function $W(n)$ has challenged mathematicians ever since van der Waerden proved this result in 1927. The first upper bound, due to van der Waerden, grew like the Ackermann function, and was not even primitive recursive (his proof was a double induction on n and the number of colors). This was finally remedied by a new proof by Shelah [41] who reduced it to a function residing in the 5th level of the Gregorchik hierarchy (basically towers of towers). The current champion is based on a striking result of Gowers [21] concerning the upper density of subsets of $[1, N]$ which contain no n-AP. From this result, one can deduce the estimate that for all n, we have:

$$W(n) < 2^{2^{2^{2^{2^{n+9}}}}}$$

In particular, this settled a long-standing conjecture I had made on the size of $W(n)$ (which asserted that $W(n)$ was upper-bounded by an exponential tower of 2's of height n), and as a result, left me \$1000 poorer (but much happier). Undaunted, I now propose the following:

Conjecture. (\$1000) For all n,

$$W(n) < 2^{n^2}.$$

I might point out that the best lower bound (due to Berlekamp [3]) has been around for almost 40 years:

$$W(n+1) \geq n2^n$$

for n prime (the proof uses finite fields).

Isn't it about time for some improvement here?

8 Combinatorial Lines

For a finite set $A = \{a_1, a_2, \ldots, a_t\}$, let A^N denote the set of N-tuples from A. A **combinatorial line** in A^N is a set of t N-tuples $X_1, X_2, \ldots, X_t$ where

$$X_k = (X_k(1), X_k(2), \ldots, X_k(N))$$

and for $1 \le j \le N$, either all $X_k(j)$ are equal, or $X_k(j) = a_k$, $1 \le k \le t$. This concept was first introduced in the seminal paper of Hales and Jewett [26].

In 1990, Furstenberg and Katznelson [19] proved the following beautiful theorem, generalizing Szemerédi's great density theorem for arithmetic progressions [48]:

Theorem. For every $\epsilon > 0$, there exists a least $N = N(\epsilon, t)$ such if $R \subseteq A^N$ with $|R| > \epsilon t^N$ then R must contain a combinatorial line.

Unfortunately, the ergodic theory tools used by Furstenberg and Katznelson do not allow us to conclude anything about the growth rate of $N(\epsilon, t)$ as $\epsilon \to 0$.

Problem. Establish **any** upper bound on $N(\epsilon, t)$.

The case of $t = 2$ is instructive (and is the only case we can handle!). In this case, using $A = \{0, 1\}$, we see that a combinatorial line in A^N is equivalent to having two subsets $X, Y \subseteq \{1, 2, \ldots, N\}$ with $X \subset Y$. By the well known result of Sperner, this must happen as soon as $|R| > \binom{N}{\lfloor N/2 \rfloor}$. This implies that $N(\epsilon, 2) < c\epsilon^{-2}$ for a suitable $c > 0$.

Warm-up Problem. Establish an upper bound on $N(\epsilon, 3)$.

In particular, it would be of great interest to obtain Gowers' type bounds on these quantities, that is, bounded towers of exponents (which might be called "Gowers towers"!).

9 Concluding Remarks

Of course, in this brief note I have only been able to touch on a few of the problems in this area that are most attractive to me, and for which I feel the time is ripe for making further progress. Much richer collections of problems and results in this subject can be found in a variety of sources, such as [4], [5], [22], [23], and [24].

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