



# Enumerating split-pair arrangements

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## Abstract

An arrangement of the multi-set  $\{1, 1, 2, 2, \dots, n, n\}$  is said to be “split-pair” if for all  $i < n$ , between the two occurrences of  $i$  there is exactly one  $i + 1$ . We enumerate the number of split-pair arrangements and in particular show that the number of such arrangements is  $(-1)^{n+1}2^n(2^{2n} - 1)B_{2n}$  where  $B_i$  is the  $i$ th Bernoulli number.

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## 1. Introduction

In robotic scheduling, a single robot arm moves identical product along through a predetermined sequence of machines labeled  $M_1, M_2, \dots, M_n, M_{n+1}$ . The first machine  $M_1$  is an unlimited supply of the raw product, the last machine  $M_{n+1}$  is unlimited storage for the finished product, while the remaining machines can only support one product at a time. In going from the raw product to the finished product, each machine must be visited in turn.

Associated to each machine there is a time for processing the product and in addition there is a time cost in moving the robotic arm between machines. A natural question to ask is what is the most efficient cyclic sequence of moves for the robotic arm to make to maximize the *average throughput* of the products. By a move we mean moving a product from machine  $M_i$  (after the product has been processed on that machine) to machine  $M_{i+1}$  (which is currently empty). Such a move will be denoted by  $a_i$ .

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Table 1  
Split-pair arrangements for  $D_1, D_2$  and  $D_3$

|         |   |
|---------|---|
| $n = 1$ | 11  |
| $n = 2$ | 1212, 2121  |
| $n = 3$ | 121323, 123123, 132132, 132312, 213213, 213231,<br>231213, 231231, 312132, 312312, 321321, 323121 |

For example, one possible sequence of moves is for the arm to move in the sequence  $a_1 a_2 \dots a_n$ , i.e., it takes a product from start to finish waiting at each machine for it to be processed before moving to the next one. However, instead of waiting for a product to process at a machine the robotic arm is allowed to move to other machines which may have product ready to move to the next machine and thus increase the average throughput.

The assumption of being cyclic means that each  $a_i, 1 \leq i \leq n$ , occurs equally often in the cycle (since otherwise we would have a “collision” of products on some machine). An  $r$ -unit cycle is one for which the number of times that each action occurs during the cycle is  $r$ . (For a comprehensive survey of robotic scheduling, the reader is referred to [1].)

It is easy to see that the number of possible 1-unit cycles is  $n!$  (i.e., any permutation of the actions gives a valid 1-cycle). It is not quite so easy to see how many valid 2-unit cycles there are. This is because not every arrangement of the actions  $\{a_1, a_1, a_2, a_2, \dots, a_n, a_n\}$  gives a valid cycle. The problem arises in that in order for action  $a_i$  to be completed, the machine  $M_{i+1}$  must currently be empty or  $i = n$ . What this translates into is that between the two occurrences of  $a_i$  in the arrangement of the action there must be exactly one occurrence of the action  $a_{i+1}$  (i.e., to empty machine  $M_{i+1}$ ) or  $i = n$ . This leads naturally to *split-pair* arrangements. A split-pair arrangement of the multi-set  $D_n := \{1, 1, 2, 2, 3, 3, \dots, n, n\}$  is an arrangement  $x_1 x_2 x_3 \dots x_{2n}$  of the elements of  $D_n$  such that for each  $i < n$ , the two occurrences of  $i$  are separated by exactly one occurrence of  $i + 1$ .

We denote by  $s_n$  the number of split-pair arrangements of  $D_n$ . In Table 1 we have listed all the split-pair arrangements for  $n = 1, 2, 3$  and thus we have  $s_1 = 1, s_2 = 2$ , and  $s_3 = 12$ .

Our main result is to derive a simple expression for  $s_n$ , namely we have the following:

**Theorem 1.** For  $n \geq 1, s_n = (-1)^{n+1} 2^n (2^{2n} - 1) B_{2n}$  where  $B_i$  is the  $i$ th Bernoulli number.

More information about Bernoulli numbers can be found in Appendix A as well as in many introductory texts (i.e., see [4]).

## 2. Preliminaries

Our proof of Theorem 1 will follow the proof of a related problem given in [3]. Namely, we derive recurrences for the quantities of interest, define a generating function, show that this generating function satisfies a certain partial differential equation (PDE), and then identify specific coefficients arising from this PDE to get the final result. Before we begin we will find it useful to refine our count by counting split-pair arrangements according to the location(s) of the  $n$ , as well as introduce some matrices.

Let  $s_n(i)$ , denote the number of split-pair arrangements  $x_1 x_2 x_3 \dots x_{2n}$  of  $D_n$  for which  $x_i = n$ . Similarly, let  $s_n(i, j)$  denote the number of split-pair arrangements of  $D_n$  for which  $x_i = x_j = n$ . Note that if  $x_1 x_2 x_3 \dots x_{2n}$  is a split-pair arrangement then so is the cyclic permutation  $x_2 x_3 \dots x_{2n} x_1$  (and by induction, this is true for any cyclic permutation  $x_k x_{k+1} \dots x_{k-1}$ ). In

addition, the reverse arrangement  $x_{2n}x_{2n-1} \dots x_1$  is also a split-pair arrangement. These observations imply the following results.

**Lemma 2.** For any  $i$ ,  $s_n = ns_n(i)$ .

**Lemma 3.** If  $|i - j| = |i' - j'|$  or  $|i - j| = 2n - |i' - j'|$  then  $s_n(i, j) = s_n(i', j')$ .

Let  $S_n$  denote the matrix with  $(i, j)$  entry equal to  $s_n(i, j)$ . Several examples are given below:

$$S_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{pmatrix},$$

$$S_4 = \begin{pmatrix} 0 & 0 & 4 & 8 & 10 & 8 & 4 & 0 \\ 0 & 0 & 0 & 4 & 8 & 10 & 8 & 4 \\ 4 & 0 & 0 & 0 & 4 & 8 & 10 & 8 \\ 8 & 4 & 0 & 0 & 0 & 4 & 8 & 10 \\ 10 & 8 & 4 & 0 & 0 & 0 & 4 & 8 \\ 8 & 10 & 8 & 4 & 0 & 0 & 0 & 4 \\ 4 & 8 & 10 & 8 & 4 & 0 & 0 & 0 \\ 0 & 4 & 8 & 10 & 8 & 4 & 0 & 0 \end{pmatrix}.$$

The structure of these matrices is a consequence of the symmetry of  $s_n(i, j)$  together with the result of Lemma 3. Note in particular that the rows are cyclic shifts of each other, so it suffices to know only the first row of  $S_n$ . We thus collect all the first rows of the  $S_n$  to form the (infinite) matrix  $A$  defined entrywise by

$$A(n, k) = s_n(1, k + 1),$$

for  $1 \leq k < 2n$ , and 0 otherwise. A portion of  $A$  is shown below, where the rows are indexed by  $n = 1, 2, \dots$  and the columns are indexed by  $k = 1, 2, \dots$ :

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 4 & 8 & 10 & 8 & 4 & 0 & 0 & 0 & \dots \\ 0 & 34 & 68 & 94 & 104 & 94 & 68 & 34 & 0 & \dots \\ 0 & 496 & 992 & 1420 & 1712 & 1816 & 1712 & 1420 & 992 & \dots \\ \dots & \dots \end{pmatrix}.$$

The matrix  $A$  will play a central role in the proof of Theorem 1. So we will now focus on establishing properties of  $A$ , most important among them a recurrence for the entries of  $A$  (see Theorem 4).

First, since by Lemma 3,  $s_n(1, k + 1) = s_n(1, 2n - k + 1)$ , we see that  $A(n, k) = A(n, 2n - k)$  for  $1 \leq k \leq 2n - 1$ .

We next claim that  $A(n, 3) = 2A(n, 2)$  for  $n \geq 3$ . To see this, suppose that  $nxny \dots wz$  is a split-pair arrangement for  $D_n$  which is counted by  $s_n(1, 3)$ . Then we must have  $x = n - 1$  and  $y, z \neq n - 1$ . Hence, we can form two split-pair arrangements counted by  $s_n(1, 4)$ , namely,

$nz(n-1)ny \dots w$  and  $n(n-1)yn \dots wz$ . Furthermore, any split-pair arrangement counted by  $s_n(1, 4)$  must arise this way. Consequently,

$$A(n, 3) = s_n(1, 4) = 2s_n(1, 3) = 2A(n, 2), \quad n \geq 3. \tag{1}$$

Next note that if  $(n-1)x \dots$  is a split-pair arrangement for  $D_{n-1}$  then we can place two  $n$ 's on each side of the initial  $n-1$  to form  $n(n-1)nx \dots$ , which is a split-pair arrangement for  $D_n$  which is counted by  $s_n(1, 3)$  (and conversely, any such split-pair arrangement can be generated this way). This shows:

$$A(n, 2) = s_n(1, 3) = \sum_{k \geq 1} s_{n-1}(1, k) = \sum_{k \geq 1} A(n-1, k), \quad n \geq 2. \tag{2}$$

Finally, it is easy to see from the definitions of  $A(n, k)$  that

$$A(1, 1) = 1, \quad A(k, 1) = A(1, k) = 0, \quad k \geq 2. \tag{3}$$

Observe that if  $x_1x_2 \dots x_{2n}$  is a split-pair arrangement for  $D_n$ ,  $n \geq 2$ , then by removing the two occurrences of  $n$ , we are left with a split-pair arrangement for  $D_{n-1}$ . We are interested in going the other way. The question is this: for a given split-pair arrangement  $X = x_1x_2 \dots x_{2n-2}$  for  $D_{n-1}$ , in how many ways can we insert two  $n$ 's in order to form a split-pair arrangement for  $D_n$ ? Of course, this will depend on the location of the  $(n-1)$ 's in  $X$ .

We will count the number of ways we can form a split-pair arrangement for  $D_n$  from  $X$  which has its first entry equal to  $n$  (and so, is counted by  $s_n(1)$ ). So suppose we insert  $n$ 's as follows into  $X$  to form  $X' = x'_1x'_2 \dots x'_{2n} = nx_1x_2 \dots x_{j-2}nx_{j-1} \dots x_{2n-2}$ . Thus,  $x'_1 = x'_j = n$ . If the indices in  $X'$  where the two  $(n-1)$ 's occur are  $u$  and  $v$  with  $u < v$ , so that  $x'_u = x'_v = n-1$  then in order for  $X'$  to a split-pair arrangement for  $D_n$  it is necessary and sufficient that

$$u + 1 < j < v + 2$$

or equivalently,

$$u \leq j - 2, \quad v \geq j - 1.$$

From the above, it follows that

$$s_n(1, j) = \sum_{\substack{u \leq j-2 \\ v \geq j-1}} s_{n-1}(u, v) = \sum_{\substack{1 \leq u \leq j-2 \\ 1 \leq v \leq 2n-2}} s_{n-1}(u, v) - \sum_{\substack{1 \leq u \leq j-2 \\ 1 \leq v \leq j-2}} s_{n-1}(u, v).$$

If we now define  $t_m(j)$  by

$$t_m(j) = \sum_{k=1}^j s_m(1, k)$$

then it follows from the structure of  $S_{n-1}$  that

$$\sum_{\substack{1 \leq u \leq j-2 \\ 1 \leq v \leq j-2}} s_{n-1}(u, v) = 2 \sum_{k=1}^{j-2} t_{n-1}(k).$$

Consequently we have

$$s_n(1, j) = (j - 2)t_{n-1}(2n - 2) - 2 \sum_{k=1}^{j-2} t_{n-1}(k), \quad j \geq 2, \quad n \geq 2. \tag{4}$$

**Theorem 4.** We have

$$A(n, k) = 2A(n, k - 1) - A(n, k - 2) - 2A(n - 1, k - 2),$$

for  $n \geq 2$ , and  $3 \leq k \leq 2n + 1$ .

**Proof.** Using (4) we have for  $3 \leq k \leq 2n$ ,

$$\begin{aligned} & A(n, k) - 2A(n, k - 1) + A(n, k - 2) \\ &= s_n(1, k + 1) - 2s_n(1, k) + s_n(1, k - 1) \\ &= (k - 1)t_{n-1}(2n - 2) - 2 \sum_{i=1}^{k-1} t_{n-1}(i) - 2(k - 2)t_{n-1}(2n - 2) \\ &\quad + 4 \sum_{i=1}^{k-2} t_{n-1}(i) + (k - 3)t_{n-1}(2n - 2) - 2 \sum_{i=1}^{k-3} t_{n-1}(i) \\ &= -2t_{n-1}(k - 1) + 2t_{n-1}(k - 2) = -2 \sum_{j=1}^{k-1} s_{n-1}(1, j) + 2 \sum_{j=1}^{k-2} s_{n-1}(1, j) \\ &= -2s_{n-1}(1, k - 1) = -2A(n - 1, k - 2), \end{aligned}$$

as claimed. For  $k = 2n + 1$  the result follows from (1) by symmetry.  $\square$

The final matrix we will need is the (infinite) matrix  $A'$  defined as follows:

$$A'(n, k) = \begin{cases} \frac{A(n/2+1, k)}{2^{n/2}} & \text{for } n \text{ even, } 1 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

A part of  $A'$  is shown below:

$$A' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1/2 & 1 & 5/4 & 1 & 1/2 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & \dots \end{bmatrix}.$$

It follows from (1)–(3), Lemma 3 and Theorem 4 that  $A'$  satisfies the following:

- (i)  $A'(n, 1) = 0$ , for  $n \geq 1$ ,
- (ii)  $A'(n, 2) = A'(n, n)$ , for  $n \geq 2$ ,
- (iii)  $A'(n, 3) = A'(n, n - 1) = 2A'(n, 2)$ , for  $n \geq 3$ ,
- (iv)  $A'(n + 2, 2) = \frac{1}{2} \sum_{k=1}^n A'(n, k)$ , for  $n \geq 2$ ,
- (v)  $A'(n, k) = 2A'(n, k - 1) - A'(n, k - 2) - A'(n - 2, k - 2)$ , for  $n \geq 3$  and  $3 \leq k \leq n + 1$ .

### 3. Proof of Theorem 1

From the definitions above along with Lemma 3 and (2), we note that

$$s_n = ns_n(1) = n \sum_{k \geq 1} s_n(1, k) = nA(n + 1, 2) = n2^n A'(2n, 2).$$

To finish the proof of Theorem 1 it suffices to show that for  $n \geq 1$

$$A'(2n, 2) = (-1)^{n-1} \frac{2^{2n} - 1}{n} B_{2n}. \tag{6}$$

It is this equality which we shall establish.

**Proof of Theorem 1.** The proof will proceed by a number of steps. First, we define the (mixed) generating function  $G(x, y)$  by

$$G(x, y) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} A'(n, k) \frac{x^n}{n!} y^k.$$

We now rewrite  $G(x, y)$  using the recurrence for  $A'(n, k)$ :

$$\begin{aligned} G(x, y) &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} A'(n, k) \frac{x^n y^k}{n!} = \sum_{n=1}^{\infty} \sum_{k=2}^{n+1} A'(n, k) \frac{x^n y^k}{n!} \\ &= y^2 \sum_{n=1}^{\infty} A'(n, 2) \frac{x^n}{n!} + \sum_{n=4}^{\infty} \sum_{k=3}^{n+1} A'(n, k) \frac{x^n y^k}{n!} \\ &= y^2 \sum_{n=1}^{\infty} A'(n, 2) \frac{x^n}{n!} \\ &\quad + \sum_{n=4}^{\infty} \sum_{k=3}^{n+1} (2A'(n, k-1) - A'(n, k-2) - A'(n-2, k-2)) \frac{x^n y^k}{n!} \\ &= y^2 \sum_{n=1}^{\infty} A'(n, 2) \frac{x^n}{n!} + 2y \sum_{n=4}^{\infty} \sum_{k=2}^n A'(n, k) \frac{x^n y^k}{n!} \\ &\quad - y^2 \sum_{n=4}^{\infty} \sum_{k=1}^{n-1} A'(n, k) \frac{x^n y^k}{n!} - y^2 \int_0^x \int_0^y G(u, y) du dv \\ &= A'(2, 2) \frac{x^2 y^2}{2!} + y^2 \sum_{n=4}^{\infty} A'(n, 2) \frac{x^n}{n!} \\ &\quad + 2y \left( \sum_{n=1}^{\infty} \sum_{k=1}^n A'(n, k) \frac{x^n y^k}{n!} - A'(2, 2) \frac{x^2 y^2}{2!} \right) \\ &\quad - y^2 \left( \sum_{n=1}^{\infty} \sum_{k=1}^n A'(n, k) \frac{x^n y^k}{n!} - \sum_{n=1}^{\infty} A'(n, n) \frac{x^n y^n}{n!} \right) - y^2 \int_0^x \int_0^y G(u, y) du dv. \end{aligned}$$

It now follows (with some computation) that:

$$G(x, y) = \frac{1}{4}x^2y^2(1 - y)^2 + y^2 \sum_{n=4}^{\infty} A'(n, 2) \frac{x^n}{n!} + y^2 \sum_{n=4}^{\infty} A'(n, n) \frac{x^n y^n}{n!} + (2y - y^2)G(x, y) - y^2 \int_0^x \int_0^y G(u, y) du dv.$$

Differentiating the above expression for  $G(x, y)$  twice with respect to  $x$  (and using (ii)), we have the following differential equation for  $G(x, y)$ :

$$(1 - y)^2 \frac{\partial^2}{\partial x^2} G(x, y) + y^2 G(x, y) = \frac{1}{2}y^2(1 - y)^2 + y^2 \sum_{n=2}^{\infty} A'(n + 2, 2) \frac{x^n}{n!} + y^4 \sum_{n=2}^{\infty} A'(n + 2, 2) \frac{x^n y^n}{n!} \tag{7}$$

with initial conditions  $G(0, y) = 0$  and  $\frac{\partial G}{\partial x}(0, y) = 0$ .

Now, set

$$c_n(y) = \sum_{k=1}^n A'(n, k) y^k$$

so that

$$G(x, y) = \sum_{n=1}^{\infty} c_n(y) \frac{x^n}{n!}.$$

By identifying coefficients of powers of  $x$  in (7), we obtain the recurrences

$$(1 - y)^2 c_2(y) = \frac{1}{2}y^2(1 - y)^2, \\ (1 - y)^2 c_{n+2}(y) + y^2 c_n(y) = y^2 A'(n + 2, 2) + y^{n+4} A'(n + 2, 2), \quad n \geq 2.$$

It now follows by induction on  $n$  that for  $n \geq 1$

$$c_{2n}(y) = \frac{1}{2}(-1)^{n-1} \frac{y^{2n}}{(1 - y)^{2n-2}} + \sum_{k=2}^n (-1)^{n-k} A'(2k, 2) \left(\frac{y}{1 - y}\right)^{2n-2k+2} + y^{2n+2} \sum_{k=2}^n (-1)^{n-k} A'(2k, 2) \frac{1}{(1 - y)^{2n-2k+2}}. \tag{8}$$

Expanding powers of  $1 - y$  using the binomial theorem, we can rewrite (8) as

$$c_{2n}(y) = \sum_{k=2}^{2n} A'(2n, k) y^k = \frac{1}{2} \sum_{i=0}^{\infty} (-1)^{n-1} y^{2n+i} \binom{2n + i - 3}{i} + \sum_{j=2}^n \sum_{i=0}^{\infty} (-1)^{n-j} A'(2j, 2) y^{2n-2j+2+i} \binom{2n - 2j + i + 1}{i} + \sum_{j=2}^n \sum_{i=0}^{\infty} (-1)^{n-j} A'(2j, 2) y^{2n+2+i} \binom{2n - 2j + i + 1}{i}.$$

Equating coefficients of  $y^k$  on the right- and left-hand sides of the above now yields

$$A'(2n, k) = \sum_{j=2}^n (-1)^{n-j} \binom{k-1}{2n-2j+1} A'(2j, 2), \quad \text{for } 2 \leq k \leq 2n-1, \tag{9}$$

and

$$A'(2n, 2n) = \frac{1}{2}(-1)^{n-1} + \sum_{j=2}^n (-1)^{n-j} \binom{2n-1}{2n-2j+1} A'(2j, 2). \tag{10}$$

Finally, we use induction on  $n$  to prove that (6) holds. Since  $A'(2, 2) = \frac{1}{2} = \frac{2^{2*1}-1}{1}(-1)^{1-1} B_2$  then (6) holds for  $n = 1$ . We now assume that (6) holds for  $n \leq N$ , and show it also holds for  $n = N + 1$ :

$$\begin{aligned} A'(2N+2, 2) &= \frac{1}{2} \sum_{k=1}^{2N} A'(2N, k) \\ &= \frac{1}{2} \sum_{k=1}^{2N} \sum_{j=2}^N (-1)^{N-j} \binom{k-1}{2N-2j+1} A'(2j, 2) + \frac{1}{2}(-1)^{N-1} \\ &= \frac{1}{2} \sum_{k=1}^{2N} \sum_{j=1}^N (-1)^{N-j} \binom{k-1}{2N-2j+1} A'(2j, 2) \\ &= \frac{1}{2} \sum_{k=1}^{2N} \sum_{j=1}^N (-1)^{N-j} \binom{k-1}{2N-2j+1} \frac{2^{2j}-1}{j} (-1)^{j-1} B_{2j} \\ &= (-1)^{N-1} \sum_{j=1}^N \binom{2N}{2j-2} \frac{2^{2j}-1}{2j} B_{2j} \\ &= (-1)^{(N+1)-1} \frac{2^{2(N+1)}-1}{(N+1)} B_{2N+2}. \end{aligned}$$

The second line follows from (9) and (10) with some simplification. Line three uses the induction hypothesis while line four involves a binomial coefficient sum. Line five follows from Lemma 5 found in Appendix A. This completes the induction step and the proof of (6) is complete, which concludes the proof of Theorem 1.  $\square$

#### 4. Concluding remarks

The sequence  $(s_n(1): n \geq 1) = (1, 1, 4, 34, 496, \dots)$  has occurred several times previously in the literature (the reader is referred to [7] for a comprehensive collection of integer sequences). For example, it arose in the study of strictly ordered binary trees in [6] and in the enumeration of regular unimodular triangulations in [2]. It also occurs in [5] where it counts the number of so-called 0–1–2 increasing trees on  $2n - 1$  vertices with  $n$  end-vertices. So far, no one has managed to establish explicit bijections between any pair of these classes of objects. It would certainly be of interest to do so.

One might also look at variations of our original problem of enumerating split-pair arrangements of  $\{1, 1, 2, 2, \dots, n, n\}$ . For example, one might require in addition that between the two

occurrences of  $n$  there is exactly one occurrence of 1 (which makes the original problem more symmetric). In terms of robotic scheduling this could correspond to a product having to go through a cycle of machines several times before it is completed. We have not attempted to enumerate such arrangements in this note (the corresponding sequence did not previously appear in [7]).

Another natural generalization coming out from the original robotic scheduling problem (for 3-unit cycles) would be to consider the same enumeration problem but with  $D_n = \{1, 1, 1, 2, 2, 2, \dots, n, n, n\}$  (or more generally with  $k$  copies of each integer). We hope to return to some of these problems at a later time.

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**Appendix A**

In this appendix we present an identity involving Bernoulli numbers which was needed in the preceding arguments (where we include a few extra details for the reader’s benefit). Recall that the Bernoulli numbers  $B_n$  can be defined by the following exponential generating function:

$$\frac{z}{e^z - 1} = \sum_{n \geq 0} B_n \frac{z^n}{n!}.$$

Thus, we have  $B_0 = 1, B_1 = -1/2, B_2 = 1/6$ , etc., and in particular,  $B_{2k+1} = 0$  for all  $k \geq 1$  (for more information, see [4]).

**Lemma 5.** For  $n \geq 2$ , we have

$$\sum_{j=1}^{n-1} \binom{2n-2}{2j-2} \frac{4^j - 1}{2^j} B_{2j} = -\frac{4^n - 1}{n} B_{2n}.$$

**Proof.**

$$\begin{aligned} \sum_{n \geq 0} 4^n (4^n - 1) B_{2n} \frac{z^{2n-1}}{(2n)!} &= \frac{1}{z} \sum_{n \geq 0} B_{2n} \frac{(4z)^{2n}}{(2n)!} - \frac{1}{z} \sum_{n \geq 0} B_{2n} \frac{(2z)^{2n}}{(2n)!} \\ &= \frac{1}{z} \left( \sum_{n \geq 0} B_n \frac{(4z)^n}{(n)!} - 4z B_1 \right) - \frac{1}{z} \left( \sum_{n \geq 0} B_n \frac{(2z)^n}{(n)!} - 2z B_1 \right) \\ &= \frac{1}{z} \left( \frac{4z}{e^{4z} - 1} + \frac{1}{2} 4z \right) - \frac{1}{z} \left( \frac{2z}{e^{2z} - 1} + \frac{1}{2} 2z \right) \\ &= \frac{4}{e^{4z} - 1} - \frac{2}{e^{2z} - 1} + 1 \\ &= \frac{e^{4z} - 2e^{2z} + 1}{e^{4z} - 1} = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \tanh(z). \end{aligned}$$

Therefore, we have

$$\tanh(z/2) = \frac{e^z - 1}{e^z + 1} = 2 \sum_{n \geq 0} (4^n - 1) B_{2n} \frac{z^{2n-1}}{(2n)!}$$

and

$$\frac{d}{dz} \tanh(z/2) = \frac{2e^z}{(e^z + 1)^2} = 2 \sum_{n \geq 0} \frac{4^{n+1} - 1}{2(n+1)} B_{2n+2} \frac{z^{2n}}{(2n)!}.$$

Also, since

$$\cosh(z) = \frac{e^z + e^{-z}}{2} = \sum_{n \geq 0} \frac{z^{2n}}{(2n)!}$$

then

$$\begin{aligned} (1 + \cosh(z)) \frac{d}{dz} \tanh(z/2) &= \left(1 + \frac{e^z + e^{-z}}{2}\right) \left(\frac{2e^z}{(e^z + 1)^2}\right) \\ &= \frac{(e^z + 1)^2}{2e^z} \frac{2e^z}{(e^z + 1)^2} = 1. \end{aligned} \tag{11}$$

Finally, we form the binomial convolution of the generating functions for  $\cosh(z)$  and  $\frac{d}{dz} \tanh(z/2)$ :

$$\begin{aligned} \cosh(z) \frac{d}{dz} \tanh(z/2) &= 2 \sum_{n \geq 0} \sum_{k=0}^n \frac{1}{(2n - 2k)!} \frac{4^{k+1} - 1}{2k + 2} \frac{1}{(2k)!} (2n)! B_{2k+2} \frac{z^{2n}}{(2n)!} \\ &= 2 \sum_{n \geq 0} \sum_{k=0}^n \binom{2n}{2k} \frac{4^{k+1} - 1}{2k + 2} B_{2k+2} \frac{z^{2n}}{(2n)!} \\ &= -\frac{d}{dz} \tanh(z/2) = -2 \sum_{n \geq 0} \frac{4^{n+1} - 1}{2n + 2} B_{2n+2} \frac{z^{2n}}{(2n)!} \end{aligned}$$

by (11). Consequently, we have

$$\sum_{k=0}^n \binom{2n}{2k} \frac{4^{k+1} - 1}{2k + 2} B_{2k+2} = -\frac{4^{n+1} - 1}{2n + 2} B_{2n+2}$$

which implies

$$\sum_{j=1}^{n-1} \binom{2n-2}{2j-2} \frac{4^j - 1}{2j} B_{2j} = -\frac{4^n - 1}{n} B_{2n}$$

as desired.  $\square$

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