Quasi-Random Graphs With Given Degree Sequences

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ABSTRACT: It is now known that many properties of the objects in certain combinatorial structures are equivalent, in the sense that any object possessing any of the properties must of necessity possess them all. These properties, termed quasirandom, have been described for a variety of structures such as graphs, hypergraphs, tournaments, Boolean functions, and subsets of $\mathbf{Z}_n$, and most recently, sparse graphs. In this article, we extend these ideas to the more complex case of graphs which have a given degree sequence. © 2007 Wiley Periodicals, Inc. Random Struct. Alg., 32, 1–19, 2008

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1. INTRODUCTION

During recent years there has been increasing interest in investigating the following phenomenon. For a given finite collection C of "objects," suppose we have some probability distribution given on C. Typically, there are many properties which are satisfied by most (or almost all) of the objects in C as seen in [4]. It turns out, however, that in many cases there is a large subclass Q of these properties which are strongly correlated, in the sense that any object in C which satisfies any of the properties in Q must in fact necessarily satisfy *all* the properties in Q. Such properties are called "quasi-random." Specific cases where this behavior is investigated can be found in [14, 15, 21] (for graphs), [12, 13, 16, 17] (for

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hypergraphs), [19] (for tournaments), [18] (for sequences), [25] (for permutations), and [20] (for sparse graphs), for example.

In this article we will take C to be the class $\mathcal{G}_n(\mathbf{d})$ of all graphs on n vertices having some given degree sequence $\mathbf{d}$. This is rather different from the classical model of a random graph, in which all vertices have the same expected degree. Special cases of such graph families include the so-called power law graphs in which the number of vertices of degree k is proportional to $k^{-\beta}$ for some positive real β. Such graphs arise in a variety of applications such as Web connectivity [2,5,6,9,24,26,28,29], communication networks [1,3], biological networks [22], collaboration graphs [27], etc.

In this article, we will introduce a class of quasi-random properties for $\mathcal{G}_n(\mathbf{d})$ and establish quantitative bounds on the strength of correlation between these properties. In particular, these results generalize and strengthen those in [20, 21].

2. NOTATION

We will consider graphs $G = (V, E)$ where V denotes the set of vertices of G and E denotes the set of edges of G. (For undefined graph theory terminology, see [33].) Our graphs will be undirected, having no loops or multiple edges. We will let $|V|$, the cardinality of V, be denoted by n.

If $\{x, y\} \in E$ is an edge of G, we say that x and y are adjacent, and write this as $x \sim y$. The neighborhood $\mathrm{nd}(x)$ of a vertex $x \in V$ is defined by

$$\mathrm{nd}(x) := \{y \in V : y \sim x \text{ in } G\}.$$

For $x \in V$, the degree d_x of x, denotes $|\mathrm{nd}(x)|$. The degree sequence $\mathbf{d} = \mathbf{d}_G$ of G is given by

$$\mathbf{d} = (d_x : x \in V),$$

or equivalently, $\mathbf{d}$ can be viewed as a mapping $\mathbf{d} : V \to \mathbb{Z}^+ \cup \{0\}$. For $X, Y \subseteq V$, define

$$e(X, Y) := |\{(x, y) : x \in X, y \in Y \text{ and } x \sim y\}|.$$

For $X \subseteq V$, define $\mathrm{vol}(X)$, the volume of X, by

$$\mathrm{vol}(X) = \sum_{x \in X} d_x.$$

A walk $P = P_t(x, y)$ from x to y is a sequence $P = (x_0, x_1, \ldots, x_t)$, where $x_0 = x, x_t = y$ and $x_i \sim x_{i+1}$ for $0 \le i < t$. Such a walk is said to have length t. Here we do not require all x_i's to be distinct. If all x_i's are different, we say the walk is a path.

In this article, we consider graphs with every vertex having positive degree. The weight $w(P)$ of such a walk P is defined to be

$$w(P) = \prod_{0<i<t} \frac{1}{d_{x_i}}$$

(thus, both endpoints are excluded in the product). If P has length 1 (and therefore is an edge of G), then $w(P)$ is defined to be 1.

A *circuit* C of length t is a sequence of t vertices $(x_1, x_2, \ldots, x_t)$ where $x_i \sim x_{i+1}$, $1 \le i < t$, and $x_t \sim x_1$. (We remark that in this definition, a circuit can be viewed as a rooted closed walk.) The weight $w(C)$ of such a circuit is defined by

$$w(C) = \prod_{1 \le i \le t} \frac{1}{d_{x_i}}.$$

The weighted adjacency matrix $M = M(G)$ is an $n \times n$ matrix with rows and columns indexed by V, defined by

$$M(x,y) = \begin{cases} \frac{1}{\sqrt{d_x d_y}} & \text{if } x \sim y, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Note that M can be written as $M = I - \mathcal{L}$ where I is the identity matrix and $\mathcal{L}$ denotes the (normalized) Laplacian (see [11]). The eigenvalues of M are denoted by $\rho_i, 0 \le i \le n-1$, indexed so that

$$1 = \rho_0 \ge |\rho_1| \ge |\rho_2| \ge \dots \ge |\rho_{n-1}|$$

using the Perron-Frobenius theorem. Note that $\rho_0 = 1$ has as its eigenvector $(\sqrt{d_x})_{x \in V}$.

Finally, define for $X, Y \subseteq V$, and $t \ge 1$,

$$e_t(X,Y) = \sum_{P \in P_t(X,Y)} w(P)$$

where $P_t(X,Y)$ denotes the set of all walks of length t between $x \in X$ and $y \in Y$. This is a weighted version of the number of walks of length t between X and Y. Note that $e_1(X,Y) = e(X,Y)$. In particular, $e_1(V,V) = \sum_x d_x = \text{vol}(G)$. It is not difficult to check that for $t \ge 1$, we have $e_t(V,V) = \text{vol}(G)$.

3. THE QUASI-RANDOM PROPERTIES

In this section we will state various properties that the $G \in \mathcal{G}_n(\mathbf{d})$ might satisfy. Each of these properties will depend on a parameter ε, which we will always assume to satisfy $0 < \varepsilon < 1$. The closer ε is to 0, the more the graph in question behaves like a random graph with respect to the property in question, that is, the more the value of the corresponding parameter is closer to its expected value for a random graph in $\mathcal{G}_n(\mathbf{d})$.

DISC(ε):

For all $X, Y \subseteq V$,

$$\left| e(X,Y) - \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)} \right| \le \varepsilon \text{ vol}(G).$$

DISC$_t(\varepsilon)$:

For all $X, Y \subseteq V$,

$$\left| e_t(X,Y) - \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)} \right| \le \varepsilon \text{ vol}(G).$$

Note that DISC$_1(\varepsilon)$ is just DISC(ε).

EIG(ε):

With the matrix $M = M(G) = (M(x,y))_{x,y \in V}$ as defined in (1) and with eigenvalues satisfying

$$1 = \rho_0 \ge |\rho_1| \ge |\rho_2| \ge \dots \ge |\rho_{n-1}|,$$

we have

$$|\rho_i| < \varepsilon \quad \text{for all } i \geq 1.$$

TRACE$_{2t}(\varepsilon)$:

The eigenvalues of M satisfy

$$\sum_{i\geq 1} \rho_i^{2t} \leq \varepsilon.$$

CIRCUIT$_t(\varepsilon)$:

The weighted sum of the t-circuits C_t in G satisfies

$$\left| \sum_{C_t: t-\text{circuit}} w(C_t) - 1 \right| \leq \varepsilon.$$

We will prove the following implications in Section 4:

Theorem 1. *For $t \geq 2$, the following implications hold.*

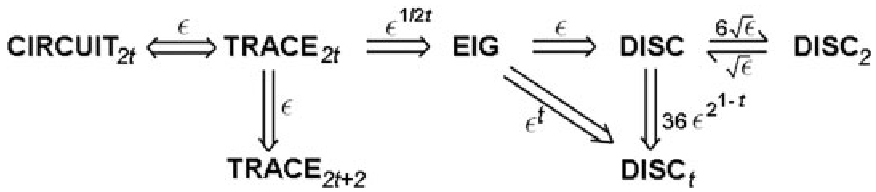

Fig. 1. Implications of several properties of $G_n(\mathbf{d})$.

Here the notation $A \overset{\delta}{\Rightarrow} B$ is shorthand for $A(\varepsilon) \Rightarrow B(\delta)$. We say A implies B, denoted by $A \Rightarrow B$, if for every $\beta > 0$, there exists $\alpha > 0$ such that $A(\alpha) \Rightarrow B(\beta)$.

There are several one-way implications in the above Fig. 1. A natural question is which, if any, of the reverse directions hold for any of these implications. In Section 5, we will give counterexamples which show that EIG $\not\Rightarrow$ Trace$_{2t}$ for any $t \geq 1$.

In Section 6, we introduce an additional property $\mathbf{U}_t$. Then we show that if a graph satisfies $\mathbf{U}_{t-1}$ for some $t \geq 2$, then DISC $\Rightarrow$ CIRCUIT$_{2t}$. Using property $\mathbf{U}_t$, we will prove the following result.

Theorem 2. *If G satisfies $\mathbf{U}_{t-1}$ for some $t \geq 2$, then* CIRCUIT$_{2t}$, TRACE$_{2t}$, EIG, DISC, DISC$_2$, DISC$_t$ *are all equivalent.*

4. THE IMPLICATIONS

Lemma 1. EIG$(\varepsilon) \Longrightarrow$ DISC(ε).

Proof. For $S \subseteq V$, define

$$f_S(x) = \begin{cases} \sqrt{d_x} & \text{if } x \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $X, Y \subseteq V$,

$$e(X, Y) = \langle f_X, M f_Y \rangle$$

where $\langle f, g \rangle = \sum_{x\in V} f(x)g(x)$ denotes the usual inner product.

Now, write

$$f_X = \sum_i a_i \phi_i$$

where the ϕ_i's form an orthonormal basis of eigenvectors with

$$\phi_0(v) = \sqrt{\frac{d_v}{\mathrm{vol}(G)}},$$

for all $v \in V$. Hence,

$$\begin{aligned} a_0 &= \langle f_X, \phi_0 \rangle \\ &= \sum_{x \in X} \frac{d_x}{\sqrt{\mathrm{vol}(G)}} \\ &= \frac{\mathrm{vol}(X)}{\sqrt{\mathrm{vol}(G)}}. \end{aligned}$$

Similarly, we write

$$f_Y = \sum_i b_i \phi_i.$$

Thus,

$$\begin{aligned} \langle f_X, M f_Y \rangle &= a_0 b_0 + \sum_{i \geq 1} \rho_i a_i b_i \\ &= \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} + \sum_{i \geq 1} \rho_i a_i b_i \end{aligned}$$

Therefore,

$$\begin{aligned} \left| e(X,Y) - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} \right| &= \left| \sum_{i \geq 1} \rho_i a_i b_i \right| \\ &\leq \max_{i \geq 1} |\rho_i| \left(\sum_{i \geq 1} |a_i|^2 \right)^{1/2} \left(\sum_{i \geq 1} |b_i|^2 \right)^{1/2} \\ &\leq \varepsilon \|f_X\| \, \|f_Y\| \\ &= \varepsilon \sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)} \\ &\leq \varepsilon \ \mathrm{vol}(G) \end{aligned}$$

by using EIG and the Cauchy-Schwarz inequality where $\|\cdot\|$ denotes the L_2-norm. Therefore, the proof is complete. ∎

In a similar way, we prove

Lemma 2. *EIG*$(\varepsilon) \Longrightarrow$ *DISC*$_t(\varepsilon^t)$ *for any* $t \geq 1$.

Proof. In this case we observe that for $X, Y \subseteq V$,

$$e_t(X,Y) = \langle f_X, M^t f_Y \rangle$$

(using the notation of Lemma 1). Thus, writing

$$f_X = \sum_i a_i \phi_i, f_Y = \sum_i b_i \phi_i,$$

we find

$$\begin{aligned} \left| \langle f_X, M^t f_Y \rangle - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} \right| &= \left| \langle f_X, M^t f_Y \rangle - \rho_0^t a_0 b_0 \right| \\ &\le \max_{i \ge 1} |\rho_i|^t \sum_{i \ge 1} |a_i b_i| \\ &\le \max_{i \ge 1} |\rho_i|^t \|f_X\| \ \|f_Y\| \\ &\le \varepsilon^t \sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)} \\ &\le \varepsilon^t \ \mathrm{vol}(G) \end{aligned}$$

and Lemma 2 is proved. ∎

Lemma 3. $\mathrm{CIRCUIT}_{2t}(\varepsilon) \Longleftrightarrow \mathrm{TRACE}_{2t}(\varepsilon)$.

Proof. Let $C_{2t}^*(u)$ denote a rooted $2t$-circuit with starting and ending point u. Then,

$$M^{2t}(u,u) = \sum_{C_{2t}^*(u)} w\big(C_{2t}^*(u)\big).$$

Thus, the trace of the matrix M^{2t} can be expressed as:

$$\begin{aligned} \mathrm{Tr}(M^{2t}) &= \sum_u \sum_{C_{2t}^*(u)} w\big(C_{2t}^*(u)\big) \\ &= \sum_{C_{2t}} w(C_{2t}). \end{aligned}$$

On the other hand, the same trace can be evaluated using eigenvalues:

$$\begin{aligned} \mathrm{Tr}(M^{2t}) &= \sum_i \rho_i^{2t} \\ &= 1 + \sum_{i \ge 1} \rho_i^{2t}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \left| \sum_{C_{2t}} w(C_{2t}) - 1 \right| &= |\mathrm{Tr}(M^{2t}) - 1| \\ &= \sum_{i \ge 1} \rho_i^{2t} \end{aligned}$$

and Lemma 3 is proved. ∎

Lemma 4. $\text{TRACE}_{2t}(\varepsilon) \Longrightarrow \text{EIG}(\varepsilon^{1/2t})$, *for any* $t \geq 1$.

Proof. By hypothesis, we have

$$\left|\sum_{i} \rho_i^{2t} - 1\right| = \sum_{i\geq 1} \rho_i^{2t} \leq \varepsilon.$$

Therefore,

$$\max_{i\geq 1} |\rho_i| \leq \varepsilon^{1/2t}.$$

∎

Lemma 5. $\text{TRACE}_{2t}(\varepsilon) \Longrightarrow \text{TRACE}_{2t+2}(\varepsilon)$.

Proof. Since $|\rho_i| \leq 1$ for all i, we have

$$\sum_{i\geq 1} \rho_i^{2t+2} \leq \sum_{i\geq 1} \rho_i^{2t} \leq \varepsilon$$

by hypothesis. ∎

Lemma 6. *For* $t \geq 1$, $\text{DISC}_{2t}(\varepsilon) \Longrightarrow \text{DISC}_t(\sqrt{\varepsilon})$.

Proof. For $X \subseteq V$,

$$\begin{aligned} e_{2t}(X,X) &= \sum_{x,x'\in X} \sum_{y} \frac{e_t(x,y)e_t(y,x')}{d_y} \qquad (2)\\ &= \sum_{y} \frac{e_t(y,X)^2}{d_y}. \end{aligned}$$

By applying $\text{DISC}_{2t}(\varepsilon)$ to $e_{2t}(X,X)$, we have

$$\sum_{y} \frac{e_t(y,X)^2}{d_y} \leq \frac{\text{vol}(X)^2}{\text{vol}(G)} + \varepsilon\ \text{vol}(G).$$

Note that

$$\sum_{y} e_t(y,X) = e_t(V,X) = \sum_{x\in V} d_x = \text{vol}(X).$$

Therefore,

$$\begin{aligned} &\sum_{y} \left(e_t(y,X) - \frac{d_y \text{vol}(X)}{\text{vol}(G)}\right)^2 \frac{1}{d_y}\\ &= \sum_{y} \frac{e_t(y,X)^2}{d_y} - 2e_t(V,X)\frac{\text{vol}(X)}{\text{vol}(G)} + \frac{\text{vol}(X)^2}{\text{vol}(G)}\\ &= \sum_{y} \frac{e_t(y,X)^2}{d_y} - \frac{\text{vol}(X)^2}{\text{vol}(G)}\\ &\leq \varepsilon\ \text{vol}(G). \end{aligned}$$

by (2) and $\mathrm{DISC}_{2t}(\varepsilon)$. But

$$\sum_y \left(e_t(y,X) - \frac{d_y \mathrm{vol}(X)}{\mathrm{vol}(G)}\right)^2 \frac{1}{d_y}$$
$$\geq \sum_{y\in Y} \left(e_t(y,X) - \frac{d_y \mathrm{vol}(X)}{\mathrm{vol}(G)}\right)^2 \frac{1}{d_y}$$
$$\geq \left(\sum_{y\in Y} \left(e_t(y,X) - d_y \frac{\mathrm{vol}(X)}{\mathrm{vol}(G)}\right)\right)^2 \frac{1}{\sum_{y\in Y} d_y}$$
$$= \left(e_t(Y,X) - \frac{\mathrm{vol}(Y)\mathrm{vol}(X)}{\mathrm{vol}(G)}\right)^2 /\mathrm{vol}(Y)$$

by applying the Cauchy-Schwarz inequality. Thus,

$$\left|e_t(Y,X) - \frac{\mathrm{vol}(Y)\mathrm{vol}(X)}{\mathrm{vol}(G)}\right| \leq \sqrt{\varepsilon\ \mathrm{vol}(Y)\mathrm{vol}(G)}$$
$$\leq \sqrt{\varepsilon}\ \mathrm{vol}(G).$$

This is exactly $\mathrm{DISC}_t(\sqrt{\varepsilon})$. ∎

Lemma 7. *For any* $t \geq 1$, $\mathrm{DISC}_t(\varepsilon) \Longrightarrow \mathrm{DISC}_{t+1}(6\sqrt{\varepsilon})$.

Proof. For $X, Y \subseteq V$,

$$\left|e_t(X,Y) - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)}\right| \leq \varepsilon\ \mathrm{vol}(G). \tag{3}$$

Consider

$$e_{t+1}(X,Y) = \sum_v \frac{e(X,v)e_t(v,Y)}{d_v}.$$

Define

$$S_1 := \left\{z \in V : e_t(z,Y) > \frac{d_z}{\mathrm{vol}(G)}(\mathrm{vol}(Y) + \sqrt{\varepsilon}\ \mathrm{vol}(G))\right\}.$$

Thus,

$$\sum_{z\in S_1} e_t(z,Y) = e_t(S_1,Y) > \frac{\mathrm{vol}(S_1)\mathrm{vol}(Y)}{\mathrm{vol}(G)} + \sqrt{\varepsilon}\ \mathrm{vol}(S_1).$$

Hence, by (3) applied to $X = S_1$ and Y,

$$\mathrm{vol}(S_1) < \sqrt{\varepsilon}\ \mathrm{vol}(G). \tag{4}$$

In the same way, if we define $S_2 := \{z \in V : e_t(z,Y) < \frac{d_z}{\mathrm{vol}(G)}(\mathrm{vol}(Y) - \sqrt{\varepsilon}\ \mathrm{vol}(G))\}$, then

$$\mathrm{vol}(S_2) < \sqrt{\varepsilon}\ \mathrm{vol}(G). \tag{5}$$

Now,

$$e_{t+1}(X,Y) = \left(\sum_{v\notin S_1\cup S_2} + \sum_{v\in S_1\cup S_2}\right) \frac{e(X,v)e_t(v,Y)}{d_v}.$$

For the first sum, we have

$$\begin{aligned}\sum_{v\notin S_1\cup S_2}\frac{e(X,v)e_t(v,Y)}{d_v} &\le \sum_{v\notin S_1\cup S_2}\frac{e(X,v)}{d_v}\frac{d_v}{\mathrm{vol}(G)}(\mathrm{vol}(Y)+\sqrt{\varepsilon}\,\mathrm{vol}(G))\\ &\le \sum_{v}\frac{e(X,v)}{\mathrm{vol}(G)}(\mathrm{vol}(Y)+\sqrt{\varepsilon}\,\mathrm{vol}(G))\\ &\le \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)}+\sqrt{\varepsilon}\,\mathrm{vol}(G)\end{aligned}$$

and

$$\begin{aligned}\sum_{v\notin S_1\cup S_2}\frac{e(X,v)e_t(v,Y)}{d_v} &\ge \sum_{v\notin S_1\cup S_2}\frac{e(X,v)}{d_v}\frac{d_v}{\mathrm{vol}(G)}(\mathrm{vol}(Y)-\sqrt{\varepsilon}\,\mathrm{vol}(G))\\ &\ge \sum_{v\notin S_1\cup S_2}\frac{e(X,v)}{\mathrm{vol}(G)}(\mathrm{vol}(Y)-\sqrt{\varepsilon}\,\mathrm{vol}(G))\\ &\ge \frac{(\mathrm{vol}(X)-\mathrm{vol}(S_1)-\mathrm{vol}(S_2))}{\mathrm{vol}(G)}(\mathrm{vol}(Y)-\sqrt{\varepsilon}\,\mathrm{vol}(G))\\ &\ge \frac{(\mathrm{vol}(X)-2\sqrt{\varepsilon}\,\mathrm{vol}(G))}{\mathrm{vol}(G)}(\mathrm{vol}(Y)-\sqrt{\varepsilon}\,\mathrm{vol}(G))\\ &\ge \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)}-3\sqrt{\varepsilon}\,\mathrm{vol}(G),\end{aligned}$$

by using (4) and (5). Thus,

$$\left|\sum_{v\notin S_1\cup S_2}\frac{e(X,v)e_t(v,Y)}{d_v}-\frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)}\right|\le 3\sqrt{\varepsilon}\,\mathrm{vol}(G).$$

For the second sum, we have

$$\begin{aligned}\sum_{v\in S_1\cup S_2}\frac{e(X,v)e_t(v,Y)}{d_v} &\le \sum_{v\in S_1\cup S_2}\frac{d_v e_t(v,Y)}{d_v}\\ &= e_t(S_1\cup S_2,Y)\\ &\le \frac{(\mathrm{vol}(S_1)+\mathrm{vol}(S_2))\mathrm{vol}(Y)}{\mathrm{vol}(G)}+\varepsilon\,\mathrm{vol}(G) \quad \text{by DISC}_t(\varepsilon)\\ &\le \frac{2\sqrt{\varepsilon}\,\mathrm{vol}(G)\mathrm{vol}(Y)}{\mathrm{vol}(G)}+\varepsilon\,\mathrm{vol}(G)\\ &= 2\sqrt{\varepsilon}\,\mathrm{vol}(Y)+\varepsilon\,\mathrm{vol}(G)\\ &\le 3\sqrt{\varepsilon}\,\mathrm{vol}(G).\end{aligned}$$

Putting these two estimates together, we obtain

$$\left|e_{t+1}(X,Y)-\frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)}\right|\le 6\sqrt{\varepsilon}\,\mathrm{vol}(G)$$

which is $\mathrm{DISC}_{t+1}(6\sqrt{\varepsilon})$. ∎

Lemma 8. *For any integers s and t, $DISC_s$ and $DISC_t$ are related as follows:*

i. *If $s < t$, then* $\mathrm{DISC}_s(\varepsilon) \Longrightarrow \mathrm{DISC}_t(36\varepsilon^{1/2^{t-s}})$.
As a special case, $\mathrm{DISC}(\varepsilon) \Longrightarrow \mathrm{DISC}_t(36\varepsilon^{1/2^{t-1}})$.

ii. *If $t < s \le 2^k t$ for some k, then* $\mathrm{DISC}_s(\varepsilon) \Longrightarrow \mathrm{DISC}_t\big(36^{1/2^k}\varepsilon^{1/2^{2^k t+k-s}}\big)$.

Proof. (i) follows from Lemma 7, i.e.,

$$\begin{aligned}\mathrm{DISC}_s(\varepsilon) &\Rightarrow \mathrm{DISC}_s(36\varepsilon)\\ &\Rightarrow \mathrm{DISC}_{s+1}(36\varepsilon^{1/2})\\ &\Rightarrow \dots\\ &\Rightarrow \mathrm{DISC}_t(36\varepsilon^{1/2^{t-s}})\end{aligned}$$

To prove (ii), we have, from (i) that

$$\mathrm{DISC}_s(\varepsilon) \Rightarrow \mathrm{DISC}_{2^k t}\big(36\varepsilon^{1/2^{2^k t-s}}\big)$$

Now apply Lemma 6 k times to get the desired implication. ∎

By combining Lemmas 1 to 8, we have proved all the implications in Theorem 1.

5. SEPARATION OF PROPERTIES

In this section, we give an example showing that at least one of the implications in Theorem 1 cannot be reversed. Whether this is true of the others is not known at this point.

Fact 1. For any $t \ge 1$,

$$\mathrm{EIG}(\varepsilon) \not\Longleftarrow \mathrm{TRACE}_{2t}(\delta)$$

for any $\delta = \delta(\varepsilon)$.

Proof. Choose $t \ge 1$ and let $G = G(n)$ be a random regular graph with n vertices and vertex degree $n^{1/t}$. Thus, $M = M(G)$ has

$$M(u,v) = \begin{cases} 1/n^{1/t} & \text{if } u \sim v,\\ 0 & \text{otherwise.}\end{cases}$$

It was shown in [23] that the eigenvalue distribution of $M(G)$ for a random graph G with a given expected degree distribution satisfies the semi-circle law if the minimum degree is greater than a power of $\log n$. As a consequence, if $1 = \rho_0 \ge |\rho_1| \ge |\rho_2| \ge \dots \ge |\rho_{n-1}|$ are the eigenvalues of M, then

1. $\rho_1 = (1 + o(1))2/n^{1/2t}$,
2. If $N(x)$ denotes the number of ρ_i with $\rho_i \le 2x/n^{1/2t}$, then

$$\frac{N(x)}{n} = (1+o(1))\frac{2}{\pi}\int_{-1}^{x}\sqrt{1-u^2}du.$$

In particular, for $x = 1/2$, we have

$$\frac{N(1/2)}{n} = (1 + o(1)) \left(\frac{2}{3} + \frac{\sqrt{3}}{4\pi} \right) \approx 0.8045 \ldots.$$

Thus,

$$\sum_{i \geq 1} \rho_i^{2t} \geq 2(n - N(1/2)) \left(\frac{1}{n^{1/2t}} \right)^{2t} \geq 0.391.$$

Hence, for any $\varepsilon > 0$, G satisfies $\mathrm{EIG}(\varepsilon)$, provided $n \geq n_0$, but G does *not* satisfy $\mathrm{TRACE}_{2t}(0.39)$.

It would be interesting to know if some of the other possible implications hold. For example, does DISC $\Rightarrow$ EIG?

Recently, Bilu and Linial [7] proved the following partial implication for regular graphs:

For a d-regular graph G on n vertices, if for all $X, Y \subset V$,

$$\left| e(X, Y) - \frac{d}{n} |X|\,|Y| \right| \leq \alpha d \sqrt{|X|\,|Y|}, \tag{6}$$

then $|\rho_1| = O(\alpha(\log(1/\alpha) + 1))$. ∎

The above property in (6) was introduced by Thomason [31] in the context of what he called (p, α)-jumbled graphs. Of course, this property is quite a bit stronger than DISC. Properties of random graphs based on this concept (without the equivalence relations) are often referred as the pseudo-random properties. The reader is referred to [30, 32] for discussions on pseudo-random graphs.

Butler [10] combines the methods in [7] and [8] to prove the following:

For a graph G with no isolated vertices, if for all $X, Y \subset V$,

$$\left| e_t(X, Y) - \frac{\mathrm{vol}(X)\,\mathrm{vol}(Y)}{\mathrm{vol}(G)} \right| \leq \alpha \sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)},$$

then $|\rho_1|^t \leq 18\alpha(1 - \frac{5}{2}\log\alpha)$.

For $t = 1$, this is the best possible (up to a constant) by considering a class of regular graphs constructed by Bollobás and Nikiforov [8]. In their example, the graphs have $\alpha = Cn^{-1/6}$ and $|\rho_1| \geq c\alpha \log n$ for some constants c and C.

6. REVERSING THE IMPLICATIONS

It is clear from the examples in the preceding section that in order to establish some of the reverse implications, e.g., DISC $\Rightarrow \mathrm{CIRCUIT}_{2t}$, we will have to make further assumptions for the $G \in \mathcal{G}_n(\mathbf{d})$. One such condition is the following:

For $t \geq 1$, a graph satisfies $U_t(C)$ if for all $x, y \in V$, $e_t(x, y) \leq C\frac{d_x d_y}{\mathrm{vol}(G)}$.

We will think of C as a large positive real. We note that for $t = 1$ and for G with minimum degree αn, the property $U_1(C)$ is automatically satisfied for $C \geq 1/\alpha^2$.

Note that for a d-regular graph, U_t implies that $n \leq Cd^t$ or, equivalently, the volume of the graph is of order at least $n^{1+1/t}$.

Lemma 9. *For any $t \geq 1$,*

$$U_t(C) \Longrightarrow U_{t+1}(C).$$

Proof. Observe that

$$\begin{aligned} e_{t+1}(x,y) &= \sum_z \frac{e(x,z)e_t(z,y)}{d_z} \\ &\leq \sum_z \frac{e(x,z)}{d_z} \cdot C\frac{d_z d_y}{\mathrm{vol}(G)} \\ &= C\frac{d_y}{\mathrm{vol}(G)} \sum_z e(x,z) \\ &= C\frac{d_x d_y}{\mathrm{vol}(G)}. \end{aligned}$$

The lemma is proved. ∎

Theorem 3. *If G satisfies $U_{t-1}(C)$ for some $t \geq 2$, then*

$$\mathrm{DISC}(\varepsilon) \Longrightarrow \mathrm{CIRCUIT}_{2t}(\eta)$$

where $\eta = 2C'C^2\varepsilon/\delta + 2C^2(C'+1)^2\delta + 20\sqrt{\delta} + 12\delta + 16C'^4\delta^{3/2} + 8C'^2\delta^{3/2}$, with $C' = \lceil C/\delta^{1/4}\rceil$, and $\delta = \max\{\sqrt{\varepsilon}, 36\varepsilon^{1/2^{t-2}}\}$. (Note that $\eta \to 0$ as $\varepsilon \to 0$.)

Proof. We are going to consider the sum

$$\sum_{u,v\in V} \frac{1}{d_u d_v}\left(e_t(u,v) - \frac{d_u d_v}{\mathrm{vol}(G)}\right)^2$$

where, as usual, $V = V(G)$.

Since G satisfies $\mathrm{DISC}(\varepsilon)$ by hypothesis, then by Lemma 8, G also satisfies $\mathrm{DISC}_{t-1}(\delta)$ where $\delta \geq 36\varepsilon^{1/2^{t-2}}$, i.e.,

$$\left| e_{t-1}(X,Y) - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} \right| \leq \delta\ \mathrm{vol}(G)$$

for all $X, Y \subseteq V$.

We here choose $\delta = \max\{\sqrt{\varepsilon}, 36\varepsilon^{1/2^{t-2}}\}$. For a fixed vertex u, we partition the vertex set V into the sets $W_i = W_i(u), 0 \leq i < C'$, as follows. (To simplify the notation, we use W_i instead of $W_i(u)$ below.)

$$\begin{aligned} W_0 &= \left\{ v : 0 \leq e_{t-1}(u,v) < \delta^{1/4}\frac{d_u d_v}{\mathrm{vol}(G)} \right\}, \\ W_1 &= \left\{ v : \delta^{1/4}\frac{d_u d_v}{\mathrm{vol}(G)} \leq e_{t-1}(u,v) < 2\delta^{1/4}\frac{d_u d_v}{\mathrm{vol}(G)} \right\}, \\ W_2 &= \left\{ v : 2\delta^{1/4}\frac{d_u d_v}{\mathrm{vol}(G)} \leq e_{t-1}(u,v) < 3\delta^{1/4}\frac{d_u d_v}{\mathrm{vol}(G)} \right\}, \end{aligned}$$

and, in general,

$$W_i = \left\{ v : i\delta^{1/4} \frac{d_u d_v}{\mathrm{vol}(G)} \le e_{t-1}(u,v) < (i+1)\delta^{1/4} \frac{d_u d_v}{\mathrm{vol}(G)} \right\}$$

for $0 \le i < C' = \lceil C/\delta^{1/4} \rceil$. Since $e_{t-1}(u,v) \le C d_u d_v/\mathrm{vol}(G)$ by $\mathrm{U}_{t-1}(C)$, the W_i form a partition of V.

Since

$$W_i \subseteq \left\{ v : \left| e_{t-1}(u,v) - i\delta^{1/4} \frac{d_u d_v}{\mathrm{vol}(G)} \right| < \delta^{1/4} \frac{d_u d_v}{\mathrm{vol}(G)} \right\},$$

then

$$\left| \sum_{v \in W_i} \left(e_{t-1}(u,v) - i\delta^{1/4} \frac{d_u d_v}{\mathrm{vol}(G)} \right) \right| < \sum_{v \in W_i} \delta^{1/4} \frac{d_u d_v}{\mathrm{vol}(G)}$$

$$\text{and} \quad \left| e_{t-1}(u,W_i) - i\delta^{1/4} \frac{d_u \mathrm{vol}(W_i)}{\mathrm{vol}(G)} \right| < \delta^{1/4} \frac{d_u \mathrm{vol}(W_i)}{\mathrm{vol}(G)}. \tag{7}$$

Since $\sum_i e_{t-1}(u,W_i) = e_{t-1}(u,V) = d_u$, then

$$\begin{aligned} &\left| \sum_i i\delta^{1/4} \frac{d_u \mathrm{vol}(W_i)}{\mathrm{vol}(G)} - d_u \right| \\ &= \left| \sum_i i\delta^{1/4} \frac{d_u \mathrm{vol}(W_i)}{\mathrm{vol}(G)} - \sum_i e_{t-1}(u,W_i) \right| \\ &\le \delta^{1/4} \sum_i \frac{d_u \mathrm{vol}(W_i)}{\mathrm{vol}(G)} \\ &= \delta^{1/4} d_u. \end{aligned} \tag{8}$$

Now, for each i, if $\mathrm{vol}(W_i) \ge \sqrt{\delta}\,\mathrm{vol}(G)$, then define $X_i = X_i(u)$ and $X_i' = X_i'(u)$ as follows:

$$X_i = \left\{ v : e(v,W_i) > \frac{d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)}(1+\sqrt{\delta}) \right\},$$

$$X_i' = \left\{ v : e(v,W_i) < \frac{d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)}(1-\sqrt{\delta}) \right\}.$$

If $\mathrm{vol}(W_i) < \sqrt{\delta}\,\mathrm{vol}(G)$ then define $X_i = X_i' = \emptyset$. Also define

$$W_u^* = \bigcup \{W_i : \mathrm{vol}(W_i) < \sqrt{\delta}\,\mathrm{vol}(G)\}.$$

Thus,

$$\mathrm{vol}\big(W_u^*\big) \le C'\sqrt{\delta}\mathrm{vol}(G)$$

since there are just C' possible values of i.

By $\mathrm{DISC}(\varepsilon)$, we have

$$\left| e(W_i, X_i) - \frac{\mathrm{vol}(W_i)\mathrm{vol}(X_i)}{\mathrm{vol}(G)} \right| \le \varepsilon\ \mathrm{vol}(G),$$

but from the definition of X_i, we have

$$\begin{aligned}\left| e(W_i, X_i) - \frac{\mathrm{vol}(W_i)\mathrm{vol}(X_i)}{\mathrm{vol}(G)} \right| &\geq \sqrt{\delta}\ \frac{\mathrm{vol}(X_i)\mathrm{vol}(W_i)}{\mathrm{vol}(G)} \\ &\geq \sqrt{\delta}\ \frac{\sqrt{\delta}\mathrm{vol}(G)\mathrm{vol}(X_i)}{\mathrm{vol}(G)} \\ &= \delta\ \mathrm{vol}(X_i).\end{aligned}$$

Therefore,

$$\mathrm{vol}(X_i) \leq \varepsilon/\delta\ \mathrm{vol}(G).$$

A similar argument shows that

$$\mathrm{vol}\big(X_i'\big) \leq \varepsilon/\delta\ \mathrm{vol}(G)$$

as well. Consequently, for each u, we consider

$$X_u := \bigcup \big\{X_i \cup X_i' : W_i \not\subseteq W_u^*\big\}$$

and we have

$$\mathrm{vol}(X_u) \leq 2C'\varepsilon/\delta\ \mathrm{vol}(G). \tag{9}$$

For $v \notin X_u$, we have, from the definition of X_u,

$$\begin{aligned} e\big(W_u^*, v\big) &= d_v - \sum_{W_i \not\subseteq W_u^*} e(W_i, v) \\ &\leq d_v - \sum_{W_i \not\subseteq W_u^*} (1 - \sqrt{\delta})\ \frac{d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)} \\ &= d_v - (1 - \sqrt{\delta})\ \frac{d_v\big(\mathrm{vol}(G) - \mathrm{vol}\big(W_u^*\big)\big)}{\mathrm{vol}(G)} \\ &= \sqrt{\delta} d_v + (1 - \sqrt{\delta})\ \frac{\mathrm{vol}\big(W_u^*\big)}{\mathrm{vol}(G)} d_v \\ &\leq \sqrt{\delta} d_v + (1 - \sqrt{\delta}) C'\ \frac{\sqrt{\delta}\mathrm{vol}(G)}{\mathrm{vol}(G)} d_v \\ &\leq (C' + 1)\sqrt{\delta} d_v. \end{aligned} \tag{10}$$

We now begin considering the sum,

$$\sum_u \sum_v \frac{1}{d_u d_v} \left(e_t(u,v) - \frac{d_u d_v}{\mathrm{vol}(G)}\right)^2 = \sum_u \left(\sum_{v \in X_u} + \sum_{v \notin X_u}\right) \frac{1}{d_u d_v} \left(e_t(u,v) - \frac{d_u d_v}{\mathrm{vol}(G)}\right)^2.$$

For the first sum, we use property $\mathbf{U}_t(C)$ and Lemma 9 to obtain the following estimate:

$$\begin{aligned} \sum_u \sum_{v \in X_u} \frac{1}{d_u d_v} \left(e_t(u,v) - \frac{d_u d_v}{\mathrm{vol}(G)}\right)^2 &\leq \sum_u \sum_{v \in X_u} \frac{1}{d_u d_v} C^2 \left(\frac{d_u d_v}{\mathrm{vol}(G)}\right)^2 \\ &= C^2 \frac{\mathrm{vol}(X_u)}{\mathrm{vol}(G)} \\ &\leq 2C'C^2\varepsilon/\delta \end{aligned} \tag{11}$$

by (9). For the second sum we have

$$\sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(e_t(u,v) - \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2$$
$$= \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(\sum_z \frac{e_{t-1}(u,z)e(z,v)}{d_z} - \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2$$
$$= \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(\left(\sum_{z \in W_u^*} + \sum_{z \notin W_u^*} \right) \frac{e_{t-1}(u,z)e(z,v)}{d_z} - \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2$$
$$\leq \sum_u \sum_{v \notin X_u} \frac{2}{d_u d_v} \left(\left(\sum_{z \in W_u^*} \frac{e_{t-1}(u,z)e(z,v)}{d_z} \right)^2 + \left(\sum_{z \notin W_u^*} \frac{e_{t-1}(u,z)e(z,v)}{d_z} - \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2 \right).$$

For the first sum just above (without a factor of 2), we have

$$\sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(\sum_{z \in W_u^*} \frac{e_{t-1}(u,z)e(z,v)}{d_z} \right)^2$$
$$\leq \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(\sum_{z \in W_u^*} \frac{C d_u d_z e(z,v)}{d_z \mathrm{vol}(G)} \right)^2 \quad \text{by } \mathrm{U}_{t-1}(C),$$
$$= \sum_u \sum_{v \notin X_u} \frac{C^2 d_u^2 \big(e(W_u^*, v)\big)^2}{d_u d_v \mathrm{vol}(G)^2}$$
$$\leq \sum_u \sum_{v \notin X_u} \frac{C^2 (C'+1)^2 \delta d_u d_v}{\mathrm{vol}(G)^2} \quad \text{by (10)}$$
$$\leq C^2 (C'+1)^2 \delta. \tag{12}$$

For the second sum above, we have

$$\sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(\sum_{z \notin W_u^*} \frac{e_{t-1}(u,z)e(z,v)}{d_z} - \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2$$
$$= \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(\sum_{W_i \not\subseteq W_u^*} \sum_{z \in W_i} \frac{e_{t-1}(u,z)e(z,v)}{d_z} - \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2$$
$$\leq \sum_u \sum_{v \notin X_u} \frac{2}{d_u d_v} \left(\left(\sum_{W_i \not\subseteq W_u^*} \sum_{z \in W_i} \frac{i\delta^{1/4} d_u d_z e(z,v)}{d_z \mathrm{vol}(G)} - \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2 \right.$$
$$\left. + \left(\sum_{W_i \not\subseteq W_u^*} \sum_{z \in W_i} \frac{\delta^{1/4} d_u d_z e(z,v)}{d_z \mathrm{vol}(G)} \right)^2 \right)$$

since $|A - a| \leq b \Rightarrow (A-B)^2 \leq 2((a-B)^2 + b^2)$ and inequalities in (7).

For the second sum we have

$$\sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(\sum_{W_i \not\subseteq W_u^*} \sum_{z \in W_i} \frac{\delta^{1/4} d_u e(z,v)}{\mathrm{vol}(G)} \right)^2$$

$$= \sqrt{\delta} \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(\sum_{W_i \not\subseteq W_u^*} \frac{d_u e(W_i, v)}{\mathrm{vol}(G)} \right)^2$$

$$\leq \sqrt{\delta} \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(\sum_{W_i \not\subseteq W_u^*} \frac{d_u}{\mathrm{vol}(G)} (1+\sqrt{\delta}) \frac{d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)} \right)^2 \quad \text{by def. of } X_u$$

$$\leq \sqrt{\delta} \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left((1+\sqrt{\delta}) \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2$$

$$\leq \sqrt{\delta} + 3\delta \tag{13}$$

(upper bounded by the sum over all u and v).

Finally, for the first sum we have

$$\sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(\sum_{W_i \not\subseteq W_u^*} \sum_{z \in W_i} \frac{i\delta^{1/4} d_u e(z,v)}{\mathrm{vol}(G)} - \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2$$

$$= \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(\sum_{W_i \not\subseteq W_u^*} \frac{i\delta^{1/4} d_u e(W_i,v)}{\mathrm{vol}(G)} - \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2$$

$$\leq \sum_u \sum_{v \notin X_u} \frac{2}{d_u d_v} \left(\left(\sum_{W_i \not\subseteq W_u^*} \frac{i\delta^{1/4} d_u d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)^2} - \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2 \right.$$

$$\left. + \left(\sum_{W_i \not\subseteq W_u^*} \frac{i\delta^{1/4} d_u d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)^2} \sqrt{\delta} \right)^2 \right) \quad \text{by the def. of } X_u,$$

$$\leq \sum_u \sum_{v \notin X_u} \frac{2}{d_u d_v} \left(2 \left(\sum_i \frac{i\delta^{1/4} d_u d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)^2} - \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2 \right.$$

$$\left. + 2 \left(\frac{\sum_{W_i \subseteq W_u^*} i\delta^{1/4} d_u d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)^2} \right)^2 + \left(\sum_{W_i \not\subseteq W_u^*} \frac{i\delta^{1/4} d_u d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)^2} \sqrt{\delta} \right)^2 \right)$$

$$\leq \sum_{u,v} \frac{1}{d_u d_v} \left(4 \left(\frac{d_u d_v \delta^{1/4}}{\mathrm{vol}(G)} \right)^2 + 4 \left(\frac{C'^2 \delta^{1/4} d_u d_v \delta^{1/2}}{\mathrm{vol}(G)} \right)^2 + 2 \left(\frac{C' d_u d_v \delta^{3/4}}{\mathrm{vol}(G)} \right)^2 \right)$$

by (8), def. of W_u^* and the fact that $i < C'$,

$$\leq 4\sqrt{\delta} + 4C'^4 \delta^{3/2} + 2C'^2 \delta^{3/2}. \tag{14}$$

Now, we have to put everything together.

First observe that

$$\begin{aligned}
&\sum_{u,v\in V} \frac{1}{d_u d_v}\left(e_t(u,v) - \frac{d_u d_v}{\mathrm{vol}(G)}\right)^2 \\
&= \sum_{u,v\in V} \frac{1}{d_u d_v} e_t(u,v)^2 - 2\sum_{u,v\in V} \frac{e_t(u,v)}{\mathrm{vol}(G)} + \sum_{u,v\in V} \frac{d_u d_v}{\mathrm{vol}(G)^2} \\
&= \sum_{C_{2t}\, 2t-\text{circuit}} w(C_{2t}) - 2\frac{e_t(V,V)}{\mathrm{vol}(G)} + 1 \\
&= \sum_{C_{2t}\, 2t-\text{circuit}} w(C_{2t}) - 1
\end{aligned}$$

(using $e_t(V,V) = \mathrm{vol}(G)$) so that the preceding results, including inequalities (11), (12), (13) and (14), give

$$\begin{aligned}
&\left| \sum_{C_{2t}\, 2t-\text{circuit}} w(C_{2t}) - 1 \right| \\
&= \sum_{u,v\in V} \frac{1}{d_u d_v}\left(e_t(u,v) - \frac{d_u d_v}{\mathrm{vol}(G)}\right)^2 \\
&\le 2C'C^2\varepsilon/\delta + 2C^2(C'+1)^2\delta + 4(\sqrt{\delta} + 3\delta) + 4(4\sqrt{\delta} + 4C'^4\delta^{3/2} + 2C'^2\delta^{3/2}) \\
&\le 2C'C^2\varepsilon/\delta + 2C^2(C'+1)^2\delta + 20\sqrt{\delta} + 12\delta + 16C'^4\delta^{3/2} + 8C'2\delta^{3/2}.
\end{aligned}$$

This proves Theorem 3. ∎

Corollary 1. *If G has minimum degree αn, then*

$$\mathrm{DISC}(\varepsilon) \Longrightarrow \mathrm{CIRCUIT}_{2t}(\eta)$$

where η depends only on ε, α and t.

Theorem 4. *If G has minimum degree αn for some constant α, then* $\mathrm{CIRCUIT}_{2t}$, TRACE_{2t}, EIG, DISC, DISC_2, DISC_t *are all equivalent for $t \ge 2$.*

7. CONCLUDING REMARKS

We can summarize the main theorems in the following:

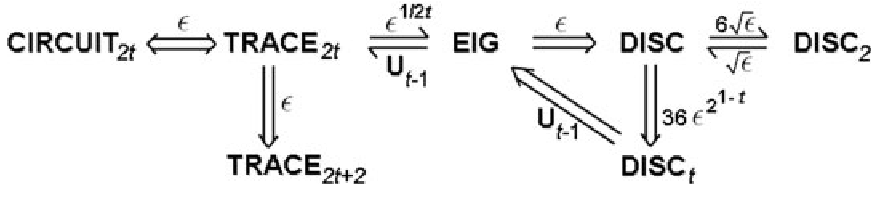

Fig. 2. Quasi-random properties for $G_n(\mathbf{d})$.

We should note that if for our degree sequence **d**, we choose all d_i to be (approximately) equal, so that the $G \in \mathcal{G}(\mathbf{d})$ are (approximately) regular, then our results specialize to the case of sparse random graphs considered in [20], except that here we get explicit functions of ε (as opposed to the expressions with $o(1)$ terms occurring in [20]). What are other properties which might be included in Theorem 1? Can condition U_{t-1} be replaced by a weaker condition to allow DISC $\Rightarrow$ $\mathrm{CIRCUIT}_{2t}$ to be proved (Fig. 2)? We hope to return to this in the future.

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REFERENCES

[1] W. Aiello, F. Chung, and L. Lu, "A random graph model for massive graphs," In Proceedings of the 32nd Annual ACM Symposium on Theory of Computing, New York, 2000, pp. 171–180.

[2] W. Aiello, F. Chung, and L. Lu, A random graph model for power law graphs, Exp Math 10 (2001), 53–66.

[3] W. Aiello, F. Chung, and L. Lu, "Random evolution in massive graphs," In Eds. J. Abello, et al., Massive Comput. 4, Kluwer Academic Publishers, 2002, pp. 97–122.

[4] N. Alon and J. H. Spencer, The probabilistic method, Wiley, New York, 1992.

[5] A.-L. Barabási and R. Albert, Emergence of scaling in random networks, Science 286 (1999), 509–512.

[6] A.-L. Barabási, R. Albert, and H. Jeong, Scale-free characteristics of random networks: The topology of the world wide web, Physica a 281 (2000), 69–77.

[7] Y. Bilu and N. Linial, Lifts, discrepancy and nearly optimal spectral gap (in press).

[8] B. Bollobás and V. Nikiforov, Hermitian matrices and graphs: Singular values and discrepancy, Discrete Math 285 (2004), 17–32.

[9] A. Broder, R. Kumar, F. Maghoul, P. Raghavan, S. Rajagopalan, R. Stata, A. Tompkins, and J. Wiener, "Graph Structure in the Web," In Proceedings of the WWW9 Conference, May, Amsterdam, 2000.

[10] S. Butler, Using descrepancy to control singular values for nonnegative matrices, Linear Algebra Appl 419 (2006), 486–493.

[11] F. Chung, Spectral graph theory, AMS Publications, 1997, xii+207 pp.

[12] F. R. K. Chung, The regularity lemma for hypergraphs and quasi-randomness, Random Struct Algorithms 2 (1991), 241–252.

[13] F. R. K. Chung and R. L. Graham, Cohomological aspects of hypergraphs, Trans Am Math Soc 334 (1992), 365–388.

[14] F. R. K. Chung and R. L. Graham, "Maximum cuts and quasi-random graphs," In Eds. Alan Frieze and Tomasz Luczak, Random graphs, Wiley, New York, 1992, pp. 23–34.

[15] F. R. K. Chung and R. L. Graham, On graphs not containing prescribed induced subgraphs, In Eds. A. Baker, et al., A tribute to Paul Erdö, Cambridge University Press, 1990, pp. 111–120.

[16] F. R. K. Chung and R. L. Graham, Quasi-random hypergraphs, Random Struct Algorithms 1 (1990), 105–124.

[17] F. R. K. Chung and R. L. Graham, Quasi-random set systems, J Am Math Soc 4 (1991), 151–196.

[18] F. R. K. Chung and R. L. Graham, Quasi-random subsets of Z_n, J Combin Theory (A) 61 (1992), 64–86.

[19] F. R. K. Chung and R. L. Graham, Quasi-random tournaments, J Graph Theory 15 (1991), 173–198.

[20] F. Chung and R. L. Graham, Sparse quasi-random graphs, Combinatorica 22 (2002), 217–244.

[21] F. R. K. Chung, R. L. Graham, and R. M. Wilson, Quasi-random graphs, Combinatorica 9 (1989), 345–362.

[22] F. Chung, L. Lu, T. G. Dewey, and D. J. Galas, Duplication models for biological networks, J Comput Biol 10 (2003), 677–688.

[23] F. Chung, L. Lu, and V. Vu, The spectra of random graphs with given expected degrees, Proc Nat Acad Sci USA 100 (2003), 6313–6318.

[24] C. Cooper and A. Frieze, A general model of undirected Web graphs, Random Struct Algorithms 22 (2003), 311–335.

[25] J. Cooper, Quasirandom permutations, J Comb Theory A 106 (2004), 123–143.

[26] M. Faloutsos, P. Faloutsos, and C. Faloutsos, On power-law relationships of the Internet topology, SIGCOMM'99 29 (1999), 251–262.

[27] J. Grossman, P. Ion, and R. de Castro, Facts about Erdös numbers and the collaboration graph, http://www.oakland.edu/ grossman/trivia.html.

[28] H. Jeong, B. Tomber, R. Albert, Z. Oltvai, and A. L. Babárasi, The large-scale organization of metabolic networks, Nature 407 (2000), 378–382.

[29] J. Kleinberg, S. R. Kumar, P. Raphavan, S. Rajagopalan, and A. Tomkins, The web as a graph: Measurements, models and methods, In Proceedings of the International Conference on Combinatorics and Computing, Tokyo, 1999.

[30] M. Krivelevich and B. Sudakov, "Pseudo-random graphs," More sets, graphs and numbers, E. Gyori, G. O. H. Katona, and L. Lovász (Editors), Bolyai Soc. Math. Stud. 15, Springer, Berlin, 2006, pp. 199–266.

[31] A. Thomason, "Pseudo-random graphs," In Ed. M. Karónski, Ann Discrete Math 33 (1987), 307–331. Proceedings of Random Graphs, Poznán.

[32] A. Thomason, Random graphs, strongly regular graphs and pseudo-random graphs, In Ed. C. Whitehead, Survey in combinatorics, 1987; London Math Soc Lecture Note Ser 123 (1987), 173–195.

[33] D. West, Introduction to graph theory, Prentice Hall, Upper Saddle River, NJ, 1996, xvi+512 p.